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A Remark on Wallman Compactification of Locales

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Abstract In this note, we prove that the Banaschewski-Mulvey's compact regular reflection construction of locales is isomorphic to the Johnstone Wallman compcactification of locales. We show that a subfit semi-normal locale is normal, but the converse is not true in general. Furthermore, we generalize the main result in [4].

Keywords normal locale; Wallman compactification; regular ideal.

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1. Introduction and preliminaries

The majority work on the Wallman compactification comes after [1], [2], in which the Wallman compactification $\omega_B X$ is homeomorphic to the Stone-Cech compactification βX. In [3], Banaschewski and Mulevy gave the construction of the Stone-Cech compactification of an arbitrary locale A by means of the sublattice of the lattice $\mathrm{Idl}(A)$ of all ideals of A. But in the absence of the axiom of choice, the compact regular reflection does not coincide with the completely compact regular reflection. An alternative method of construction compactification of semi-normal subfit locale was described in [4]. In [5], an explicit description of the compact regular reflection of a locale was given by introducing a two-element relation on it. In this note, we show that the locale $\mathrm{CR}(A)$ of all regular ideals of locale A is isomorphic to the locale $\mathrm{Idl}(A)_J$ introduced in [6]. Moreover, we get the Wallman compactification of normal locales, which builds up the relation between the constructions in [3], [4]. Furthermore, it generalizes the result in [4].

We recall some basic definitions and results of locales that are needed in this paper. More terminologies and notations which are not explained here are taken from [7].

In [6], Johnstone introduced the nuleus J on the locale $\mathrm{Idl}(A)$ of ideals of a distributive lattice A, where, for any $I \in \mathrm{Idl}(A)$,

$$J(I) = \{a \in A \mid (\forall b \in A)(a \vee b = 1 \Rightarrow \exists (c \in I)(b \vee c = 1))\}.$$

The sublocale of J-fixed ideals is denoted by $\mathrm{Idl}(A)_J$. Locale A is said to be semi-normal if, whenever $a \vee b = 1$, there exist c and d with $a \vee c = b \vee d = 1$ and $c \wedge d = J(\{0\})$ while we call

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locale A subfit if, whenever $a \not\leq b$, there exists c with $a \vee c = 1$, but $b \vee c \neq 1$. By CR(A) we mean the set of regular ideas of locale A, $\Downarrow a = \{x \in A \mid x \prec a\}$. For a general background on category theory, we refer to [8].

2. Main results

Lemma 1 *Let A be normal locale. We have:*

(1) If $a \prec b$, then $\neg b \prec \neg a$;

(2) If $a \prec b$, then there exists $c \in A$ such that $a \prec c \prec b$.

Lemma 2 CR(A) *forms a subframe of* Idl(A).

Lemma 3 *If A is a normal locale, then* CR(A) *is the compact regular reflection of A.*

Proof By Lemma 2, the compaction of CR(A) is trivial.

In the following we show that CR(A) is regular. Assume $a \prec b$. By Lemma 1(2), there exist x, y and z with $a \prec x \prec y \prec z \prec b$, such that $a \wedge \neg x \leq a \wedge \neg a = 0$. Moreover, $\Downarrow a \wedge \Downarrow (\neg x) = \Downarrow a \bigcap \Downarrow (\neg x) = \{0\}$. By Lemma 1(1), $\neg b \prec \neg z \prec \neg y \prec \neg x \prec \neg a$, so that $\neg y \in \Downarrow (\neg x)$. Also $z \in \Downarrow b$, and $\neg y \vee z = 1$, we have $\Downarrow (\neg x) \vee \Downarrow b = \downarrow \{m \vee n \mid m \in \Downarrow (\neg c),\ n \in \Downarrow b\} = A$, i.e., $\Downarrow a \prec \Downarrow b$. For any $I \in$ CR(A) and $a \in A$, by the definition of regular ideals, we can find $b \in A$ with $a \prec b$ so that $\Downarrow a \prec \Downarrow b \leq I$, i.e., $\Downarrow a \prec I$, then $I = \bigcup\{\downarrow a \mid a \in I\} = \bigvee\{\Downarrow a \mid a \in I\}$, i.e., CR($A$) is regular.

Lastly, we show the reflection. For any $I \in$ CR(A), let $f^*(I) = \bigvee I$. It is easy to verify that f^* is a frame homomorphism. The right adjoint of f^* is denoted by f_*, such that for any $a \in A$, $f_*(a) = \Downarrow a$; by f_A we mean the local continuous map. Suppose that we are given a compact regular locale B and a local continuous map $h : A \to B$. For any $I \in$ CR(B), let

$$\mathrm{CR}(h)^*(I) = \{a \in A \mid (\exists b \in I)(a \leq h^*(b))\}.$$

Since h^* is a frame homomorphism, it preserves the $\prec$ relations. So CR$(h)^*(I)$ is a regular ideal, and the following diagram commutes

$$\begin{array}{rcl} A & \xrightarrow{f_A} & \mathrm{CR}(A) \\ h \downarrow & & \downarrow \mathrm{CR}(h) \\ B & \xrightarrow{f_B} & \mathrm{CR}(B) \end{array}$$

Since B is a compact regular locale, f_B is a isomorphism. So $h = f_B^{-1} \cdot \mathrm{CR}(h) \cdot f_A$, i.e., h can be factored by f_A. The uniqueness of this factorization can be obtained by the density of f and [7] (III Corollary 1.3).

Proposition 4 *If A is a normal locale, then* CR(A) *is the regular reflection of* Idl(A).

Proof By the proof of Lemma 3, CR(A) is regular. It is obvious that the inclusion map f : CR(A) $\to$ Idl(A) is a frame homomorphism. For any regular locale B and frame homomorphism $g : B \to$ Idl(A), we only need to show that g can be uniquely factored by f. Since g is a frame homomorphism, it preserves the "$\prec$" relations. So $g(B)$ is a regular subframe of $\mathrm{Idl}(A)$. A is normal, by Lemma 3, $\mathrm{CR}(A)$ is the compact regular reflection of A. By [9], we know that $\mathrm{CR}(A)$ is the maximal regular subframe of $\mathrm{Idl}(A)$, so that $g(B) \subseteq \mathrm{CR}(A)$. The existence and the uniqueness of the factorization is obvious.

Remark Since $\mathrm{CR}(A)$ is compact, the regular reflection of $\mathrm{Idl}(A)$ coincides with the compact regular reflection.

Proposition 5 *If A is a normal lattice, then* $\mathrm{CR}(A) \cong \mathrm{Idl}(A)_J$.

Proof Given an ideal I of A, define

$$K(I) = \{a \in A \mid (\exists b \in I)(a \prec b)\}.$$

Since the relation "$\prec$" is stable under finite joins and meets, it is easy to verify that $K(I)$ is an ideal and K preserves orders. Moreover, since A is normal, $K(I)$ is a regular ideal.

Let I be a regular ideal. By the definition of $J : \mathrm{Idl}(A) \to \mathrm{Idl}(A)$, $I \subseteq J(I)$ and $I \subseteq K(J(I))$. For any $a \in K(J(I))$, by the definition of K, there exists $b \in J(I)$ with $a \prec b$, so $b \vee \neg a = 1$. Since A is normal, we can find c and d in A such that $\neg a \vee c = b \vee d = 1$, $c \wedge d = 0$ and $a \prec c$. Since $b \vee d = 1$ and $b \in J(I)$, by the definition of J, there exists $e \in I$ with $e \vee d = 1$, also $c \prec e$, so $a \prec c \prec e$ and $a \in I$, i.e., $K(J(I)) \subseteq I$. We have $K(J(I)) = I$.

Suppose $I \in \mathrm{Idl}(A)_J$. By the definition of K and J, we have $K(I) \subseteq I$ and $J(K(I)) \subseteq I$. For any $a \in I$, assume that we have an element $b \in A$ with $a \vee b = 1$. By the normality of A, we can find c and $d \in A$ such that $a \vee c = b \vee d = 1$ and $c \wedge d = 0$. Then $d \prec a$, so $d \in K(I)$. Moreover, $a \in J(K(I))$, i.e., $J(K(I)) = I$. $\mathrm{CR}(A) \cong \mathrm{Idl}(A)_J$.

Proposition 6 *A semi-normal subfit locale is normal.*

Proof By the definition of semi-normality and subfitness, we only need to show that $J(\{0\}) = \{0\}$. By the definition of J, $J(\{0\}) = \{a \in A \mid (\forall b \in A)(a \vee b = 1 \Rightarrow b = 1)\} = \{a \in A \mid (\forall b \in A)(b \neq 1 \Rightarrow a \vee b \neq 1\}$. Suppose that we have an element $a \in J(\{0\})$ with $a \neq 0$. So $a \not\leq 0$. By the definition of subfitness, there exists an element $c \in A$ such that $a \vee c = 1$, but $c \vee 0 = c \neq 1$.

Remark The converse of proposition 6 is not true in general, for the normality does not imply the subfitness. In [4], we know that a locale is normal and subfit if and only if it is embeddable as a flat sublocale of a compact locale. The regularity is inherited by arbitrary sublocales, so if the normality implies the subfitness, then the normality implies the regularity. In the following we give an example to show that the normality does not imply the regularity.

Example Suppose (X, σ) is normal space, and σ is a normal locale. Let $X' = X \bigcup \{\top\}$, for a certain element $a \in X$, $\sigma' = \{U \subseteq X' \| (U \in \sigma) \text{ or } (a \in U \Rightarrow U \bigcap X \in \sigma)\}$. It is easy to verify that σ' is a topology on X', $\{\top\}^- = \{\top\}$ and $\top \in \{a\}^-$ in X'. For any two closed sets F_1 and F_2 of X', we know that $F_1 \bigcap X$ and $F_2 \bigcap X$ are closed in X. Since X is normal, we can find two open sets U and V in X, which are also open in X', such that $F_1 \subseteq U$, $F_2 \subseteq V$ and $U \bigcap V = \emptyset$.

Suppose that $\top \in F_1$. Let $U' = U \bigcup \{\top\}$. Then U' is open in X', such that $F_1 \subseteq U'$, $F_2 \subseteq V$ and $U' \bigcap V = \emptyset$, i.e., X' is normal. But we cannot find an open set U in X' such that $a \in X' - U$ and $\top \in U$, so X' is not regular. The locale of open sets of X' is normal, but is not regular.

In [4], Johnstone proved that $\mathrm{Idl}(A)_J$ is the compact regular reflection of a semi-normal subfit locale A. By Proposition 5, we have

Theorem 7 (Wallman compactification of normal locales) *If A is a normal locale, then $\mathrm{Idl}(A)_J$ is the compact regular reflection of A.*

For any distributive lattice, we have $\mathrm{Idl}(A)_J \cong \max(A)$ with the axiom of choice. We have

Corollary 8 *If X is a normal Sober space, then $\max(\Omega(X))$ (the Wallman compactification of X) is the compact Hausdorff reflection.*

Corollary 9 *A compact regular locale is spatial.*

Proof Since A is compact regular, A is normal and $\mathrm{CR}(A) \cong A$. By Proposition 5, $\mathrm{CR}(A) \cong \mathrm{Idl}(A)_J$, so $A \cong \max(A)$, i.e., A is spatial.

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