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Empirical Likelihood of Density Function for Dependent Series

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Abstract With the application of the special properties of strongly stationary $m-$dependent series, this paper is concerned with the empirical likelihood confidence intervals of density function under $m-$dependent series. The limit distribution of empirical likelihood ratio statistics is given out, and the empirical likelihood confidence intervals of parameters can be constructed. A simulation study is conducted to show the finite sample performance of the empirical likelihood based method.

Keywords $m-$dependent series; density function; empirical likelihood.

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Empirical likelihood (EL) method is a nonparametric method of inference introduced by Owen[1−2]. It has been studied extensively because of its generality and effectiveness. The EL method has been successfully applied in many areas such as linear regression models[3−5], generalized linear models[6], quantile[7], general estimation equation[8], semiparametric model[9], dependent process[10], survival analysis[11], amongst others, mixture models[12] and so on. More references on the EL can be found in the recent monograph of Owen[13]. It should be noted that the above work seems to focus on independent data, but the actual data are usually dependent. Chen[14] discussed the empirical likelihood confidence intervals of density function under i.i.d. samples. As far as the dependent samples are concerned, the situations are more complex. Jiang and Qin[15] discussed the empirical likelihood confidence intervals of density function under φ-mixing random variables. In this paper, we will use the special properties of strongly stationary m-dependent series to construct the empirical likelihood confidences intervals of density function under m-dependent series.

1. Introduction and main result

Let $X_1, X_2, \ldots$ be strongly stationary $m-$dependent random samples from an unknown density function $f(x)$. For a fixed $x \in R$, let $f(x) = \theta_1$. We refer to [16] for the definitions of

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strongly stationary random series and $m-$dependent.

To state the main results of this paper, we make the following assumptions:

(i) For $i=1,2,\ldots,m-1$, $g_i(x,y)$ is the joint density function of (X_1,X_{i+1}), and $g_i(x,y)$ and $f(x)$ admit continuous rth $(r\geq 2)$ derivatives in neighbourhoods of the points (x,y) and x, respectively.

(ii) $K(\cdot)$ is a kernel, $K=K(-u)$, and $K(\cdot)$ is a kernel of order r, that is:

$$\int K(y)\mathrm{d}y=1,\quad \int y^iK(y)\mathrm{d}y=0,\quad i=1,2,\ldots,r-1,\quad \int y^rK(y)\mathrm{d}y=k,\quad k\neq 0.$$

For instance, the Gaussian kernel, $K(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ is a kernel of order 2.

(iii) h is a bandwidth parameter, which satisfies $h=\mathrm{o}(n^{-\frac{1}{5}}), nh\longrightarrow\infty$.

Let

$$\omega_i=\frac{1}{h}K(\frac{x-X_i}{h})-\theta_1,\quad i=1,2,\ldots,n.$$

The empirical log-likelihood ratio of θ_1 is

$$l(\theta_1)=2\sum_{i=1}^{n}\log(1+\lambda\omega_i),\tag{1}$$

where λ is the solution of

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\omega_i}{1+\lambda\omega_i}=0.\tag{2}$$

Theorem *Suppose that Assumptions (i)–(iii) hold, then*

$$l(\theta_1)\longrightarrow\chi^2(1),\quad \text{as}\quad n\longrightarrow\infty.$$

2. Lemmas

Lemma 1 *Let $X_1,X_2,\ldots,X_n$ be strongly stationary m-dependent series with $E|X_1|^r<\infty$. For some $r\geq 2$, then $E|\sum_{i=1}^n X_i|^r\leq Cn^{r/2}E|X_i|^r$.*

The proof can be found in [17].

Lemma 2 *Let $X_1,X_2,\ldots,X_n$ be univariate m-dependent r.v.s., $EX_i=0, E|X_i|^3<\infty$ $(i=1,2,\ldots,n)$. Then*

$$\sup_x|P(\frac{\sum_{i=1}^n X_i}{B_n}<x)-\Phi(x)|\leq\frac{C(m+1)^3\sum_{i=1}^n E|X_i|^3}{B_n^3},$$

where $B_n=\sqrt{E(\sum_{i=1}^n X_i)^2}$ and $\Phi(x)$ is the distribution function of standard normal random variable.

The proof can be found in [18].

3. Simulation study

In this section, we carry out some simulations to show the finite sample performance of the proposed method.

In our simulation studies, we generate $e_i's$ i.i.d. from the standard normal distribution $N(0,1)$, and $X_i = e_i + 0.5e_{i-1} - 0.3e_{i-2}$, so $X_i's$ are $2-$dependent random series. The density function of X_i is

$$f(x) = \frac{1}{\sqrt{2\pi} \times \sqrt{1.34}} \exp\{-\frac{x^2}{2 \times 1.34}\}.$$

The kernel function $K(t)$ is the Gaussian kernel

$$K(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}.$$

Also, three different bandwidths of h are selected to be $(n \log n)^{-1/2}$, $(n \log n)^{-1/3}$ and $(n \log n)^{-1/5}$, respectively. It is easy to check all the conditions (i)–(iii) in the paper are satisfied. The sample size has been chosen to be 10, 20 and 50, respectively. The coverage probabilities are calculated for the empirical likelihood method based on 500 simulated data. The nominal levels are taken to be $\alpha = 0.10$ and 0.05, respectively. The results are presented in Tables 1 and 2.

From Tables 1 and 2, we see that the empirical likelihood method performs well and the coverage accuracies increase as the size n increases. We choose three bandwidths $h = (n \log n)^{-1/2}$, $(n \log n)^{-1/3}$ and $(n \log n)^{-1/5}$. Clearly, the bandwidth plays an important role here. However, we shall not address the problem of how to find the optimal bandwidth.

n	$h = (n \log n)^{-1/2}$	$h = (n \log n)^{-1/3}$	$h = (n \log n)^{-1/5}$
10	0.6220	0.7500	0.8400
20	0.6560	0.8220	0.8680
50	0.7400	0.8400	0.9040

Table 1 Coverage probabilities for θ_1,$\alpha = 0.10$

n	$h = (n \log n)^{-1/2}$	$h = (n \log n)^{-1/3}$	$h = (n \log n)^{-1/5}$
10	0.6300	0.7900	0.8880
20	0.7080	0.8800	0.9220
50	0.7760	0.8960	0.9420

Table 2 Coverage probabilities for θ_1,$\alpha = 0.05$

4. Proof of main result

Proof of Theorem Put

$$\bar{\omega} = \frac{1}{n}\sum_{i=1}^{n} \omega_i, \quad S = \frac{h}{n}\sum_{i=1}^{n} \omega_i^2, \quad Z_n = \max_{1 \le i \le n} |\omega_i|.$$

Noting that

$$|\omega_i| = |\frac{1}{h}K(\frac{x - X_i}{h}) - \theta_1| \le \frac{K(\frac{x-X_i}{h})}{h} + \theta_1 \le \frac{C}{h} + \theta_1 = \mathrm{O}(h^{-1}),$$

it is clear to see that

$$Z_n = \max_{1\le i\le n} |\omega_i| = \mathrm{O}(h^{-1}) \quad \text{a.s.} \tag{3}$$

Let $\rho_2(K) = \int K^2(t)\mathrm{d}t$. Since $S - ES = \frac{h}{n}\sum_{i=1}^{n}(\omega_i^2 - E\omega_i^2)$, from Lemma 1, we have

$$E(S-ES)^2 \le \frac{Ch^2}{n^2} n \max_{1\le i\le n} E(\omega_i^2 - E\omega_i^2)^2 = \frac{Ch^2}{n} E(\omega_1^2 - E\omega_1^2).$$

Noting that

$$\begin{aligned}
E(\omega_1^2 - E\omega_1^2)^2 \le E\omega_1^4 &= E(\frac{1}{h}K(\frac{x-X_1}{h}) - \theta_1)^4 \le CE(\frac{1}{h^4}K^4(\frac{x-X_1}{h})) + C \\
&\le \frac{C}{h^3}E(\frac{1}{h}K^4(\frac{x-X_1}{h})) + C \le \frac{C}{h^3} + C,
\end{aligned}$$

we have $E(S-ES)^2 \le \frac{C}{nh} + \frac{Ch^2}{n} \longrightarrow 0, n \longrightarrow \infty$. It follows that $S = ES + \mathrm{o}_p(1)$, and $ES = \theta_1\rho_2(K) + \mathrm{o}(1)$, so we have

$$S = \theta_1\rho_2(K) + \mathrm{o}_p(1). \tag{4}$$

Let $X_{ni} = h/n^{\frac{1}{3}}(\omega_i - E\omega_i)$. Then $EX_{ni} = 0$, and $E\omega_i = \mathrm{O}(h^r)$. So we get

$$\begin{aligned}
E|X_{ni}|^3 &= \frac{h^3}{n}E|(\omega_i - E\omega_i)^3| \\
&\le \frac{h^3}{n}(\mathrm{O}(h^{-3}) + \mathrm{O}(h^{-2})\mathrm{O}(h^r) + \mathrm{O}(h^{-1})\mathrm{O}(h^{2r}) + \mathrm{O}(h^{3r})) \\
&= \mathrm{O}(n^{-1}) + \mathrm{O}(n^{-1}h^{r+1}) + \mathrm{O}(n^{-1}h^{2r+2}) + \mathrm{O}(n^{-1}h^{3r+3}) \\
&= \mathrm{O}(n^{-1}) < +\infty.
\end{aligned}$$

Let $B_n^2 = \mathrm{Var}(\sum_{i=1}^{n} X_{ni}) = \sum_{i=1}^{n} EX_{ni}^2 + 2\sum_{i=1}^{m-1} EX_{n1}X_{ni+1}$. Noting that

$$\begin{aligned}
EX_{ni}^2 &= \frac{h^2}{n^{\frac{2}{3}}}E(\omega_i - E\omega_i)^2 = \frac{h^2}{n^{\frac{2}{3}}}[E\omega_1^2 - (E\omega_1)^2] \\
&= \frac{h^2}{n^{\frac{2}{3}}}E\omega_1^2 - \frac{h^2}{n^{\frac{2}{3}}}\mathrm{O}(h^{2r}) \\
&= \frac{h^2}{n^{\frac{2}{3}}}(\frac{1}{h}\theta_1\int K^2(v)\mathrm{d}v - \theta_1^2 + \mathrm{O}(h^r)) - \frac{h^2}{n^{\frac{2}{3}}}\mathrm{O}(h^{2r}) \\
&= \frac{h}{n^{\frac{2}{3}}}\theta_1\int K^2(v)\mathrm{d}v + \mathrm{O}(n^{-\frac{2}{3}}h^2) + \mathrm{O}(n^{-\frac{2}{3}}h^{r+2}) + \mathrm{O}(n^{-\frac{2}{3}}h^{2r+2}) \\
&= \frac{h}{n^{\frac{2}{3}}}\theta_1\int K^2(v)\mathrm{d}v + \mathrm{O}(n^{-\frac{2}{3}}h^2),
\end{aligned}$$

and

$$\begin{aligned}
EX_{n1}X_{ni+1} &= \frac{h^2}{n^{\frac{2}{3}}}E(\omega_1 - E\omega_1)(\omega_{i+1} - E\omega_{i+1}) \\
&= \frac{h^2}{n^{\frac{2}{3}}}E(\frac{1}{h}K(\frac{x-X_1}{h}) - E\frac{1}{h}K(\frac{x-X_1}{h}))(\frac{1}{h}K(\frac{x-X_{i+1}}{h}) - E\frac{1}{h}K(\frac{x-X_{i+1}}{h})),
\end{aligned}$$

we write

$$I = E(\frac{1}{h}K(\frac{x-X_1}{h}))(\frac{1}{h}K(\frac{x-X_{i+1}}{h})) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{h}K(\frac{x-u}{h})\frac{1}{h}K(\frac{x-v}{h})g_i(u,v)\mathrm{d}u\mathrm{d}v.$$

Let

$$\frac{x-u}{h}=u',\frac{x-v}{h}=v'.$$

Then

$$I=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}K(u')K(v')g_i(x-hu',x-hv')\mathrm{d}u'\mathrm{d}v'.$$

From Assumption (i), we know

$$I=g_i(x,x)+\mathrm{O}(h^r).$$

It follows that

$$E(\omega_1-E\omega_1)(\omega_{i+1}-E\omega_{i+1})=g_i(x,x)-\theta_1^2+\mathrm{O}(h^r).$$

Therefore

$$\sum_{i=1}^{n}EX_{n1}X_{ni+1}=\mathrm{O}(n^{-\frac{2}{3}}h^2).$$

Thus we have

$$B_n^2=\sum_{i=1}^{n}EX_{ni}^2+2\sum_{i=1}^{m-1}EX_{n1}X_{ni+1}=n^{\frac{1}{3}}h\theta_1\int K^2(v)\mathrm{d}v+\mathrm{O}(n^{\frac{1}{3}}h^2)+\mathrm{O}(n^{-\frac{2}{3}}h^2),$$

where $n^{\frac{1}{3}}h^2=\mathrm{o}(1)$ by the Assumption (iii).

From Lemma 2, we can see

$$\frac{\sum_{i=1}^{n}X_{ni}}{B_n}\xrightarrow{\mathcal{D}}N(0,1),\tag{5}$$

where " $\xrightarrow{\mathcal{D}}$ " denotes convergence in distribution. Obviously,

$$\sum_{i=1}^{n}X_{ni}=\frac{h}{n^{\frac{1}{3}}}\sum_{i=1}^{n}(\omega_i-E\omega_i)=n^{\frac{2}{3}}h\bar{\omega}+\mathrm{O}(n^{\frac{2}{3}}h^{r+1}),$$

which implies that

$$\frac{\sqrt{nh}\bar{\omega}}{\sqrt{\theta_1\int K^2(v)\mathrm{d}v}}+\mathrm{O}((nh^{2r+1})^{\frac{1}{2}})\xrightarrow{\mathcal{D}}N(0,1).$$

From Assumption(iii), we see that $(nh^{2r+1})^{\frac{1}{2}}=\mathrm{o}(1)$, so we have

$$\bar{\omega}=\mathrm{O}_p((nh)^{-\frac{1}{2}}).\tag{6}$$

By (2) we have

$$|\lambda|\leq\frac{\bar{\omega}}{\frac{S}{h}-Z_n\bar{\omega}}=\frac{h\bar{\omega}}{S-hZ_n\bar{\omega}}=\frac{\mathrm{O}_p(n^{-\frac{1}{2}}h^{\frac{1}{2}})}{\theta_1\int K^2(v)\mathrm{d}v+\mathrm{o}_p(1)}=\mathrm{O}_p(n^{-\frac{1}{2}}h^{\frac{1}{2}}).\tag{7}$$

Let $\gamma_i=\lambda\omega_i$. Then we get

$$\max_{1\leq i\leq n}|\gamma_i|=|\lambda|Z_n=\mathrm{O}_p(n^{-\frac{1}{2}}h^{\frac{1}{2}})\mathrm{O}_p(h^{-1})=\mathrm{O}_p((nh)^{-\frac{1}{2}}).$$

From (2), we also know

$$0=h\bar{\omega}-\lambda S+\frac{h}{n}\sum_{i=1}^{n}\omega_i\gamma_i^2/(1+\gamma_i).$$

Therefore, we may write $\lambda = hs^{-1}\bar{\omega} + \tau$, where

$$|\tau| \le \frac{h}{n}\sum_{i=1}^{n}|\omega_i|^3|\lambda|^2|1+\gamma_i|^{-1} = \mathrm{O}_p(h^{-1})\mathrm{O}_p(1)\mathrm{O}_p(n^{-1}h)\mathrm{O}_p(1) = \mathrm{O}_p(n^{-1}).$$

We may expand $\log(1+\gamma_i) = \gamma_i - \frac{\gamma_i^2}{2} + \eta_i$, where

$$\sum_{i=1}^{n}|\eta_i| \le B|\lambda|^3\sum_{i=1}^{n}|\omega_i|^3 = B\mathrm{O}_p(n^{-\frac{3}{2}}h^{\frac{3}{2}})\mathrm{O}_p(h^{-2})n\mathrm{O}_p(1) = \mathrm{o}_p(1).$$

So we have

$$l(\theta_1) = nhS^{-1}\bar{\omega}^2 - \frac{n\tau^2S}{h} + 2\sum_{i=1}^{n}\eta_i.$$

Since

$$\frac{n\tau^2S}{h} = n\mathrm{O}_p(n^{-2})\mathrm{O}_p(1)h^{-1} = \mathrm{O}_p((nh)^{-1}) = \mathrm{o}_p(1),$$

we get $l(\theta_1) = nhS^{-1}\bar{\omega}^2 + \mathrm{o}_p(1)$. This completes the proof. □

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