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The Hyperbolic Darboux Image and Rectifying Gaussian Surface of Nonlightlike Curve in Minkowski 3-Space

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Abstract In this paper, we give the classification of the singularities of hyperbolic Darboux image and rectifying Gaussian surface of nonlightlike curve in Minkowski 3-space. We establish the relationship between the singularities and the geometric invariants of curves which are deeply related to its order of contact with helices.

Keywords Minkowski 3-space; nonlightlike curve; hyperbolic Darboux image; rectifying Gaussian surface; singularity.

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1. The basic concepts and the main results

Bruce, Izumiya etc. have studied the singularities of curves and surfaces and also the curves and the surfaces induced by the curves in Euclidean 3-space by constructing height functions and distance-squared functions[1−5]. In this paper, using the similar methods, we construct the binormal height functions and the related functions of the curves in Minkowski 3-space. We study the classifications of singularities of the hyperbolic Darboux image and the rectifying Gaussian surface of nonlightlike curve in Minkowski 3-space. We also give the relationship between the singularities and the geometric invariants of curves. The other results of the space curves in Minkowski 3-space can refer to [6, 7] by the third author of this paper.

Now we introduce the basic concepts concerned in this paper and give the main results in this paper.

Let $R^3=\{(x_1,x_2,x_3)|x_1,x_2,x_3\in R\}$ be the 3-dimensional vector space. For any two vectors $x=(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3)$ in R^3, their pseudo scalar product is defined by: $\langle x,y\rangle=-x_1y_1+x_2y_2+x_3y_3$. $(R^3,\langle,\rangle)$ is called 3-dimensional pseudo Euclidean space or Minkowski 3-space, denoted as R^3_1. For any two vectors $x=(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3)\in R^3_1$, their

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pseudo vector product is defined by:

$$x \wedge y = \begin{vmatrix} -e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

Let $x \in R_1^3$ be a non-zero vector. If $\langle x, x\rangle > 0, \langle x, x\rangle = 0$ or $\langle x, x\rangle < 0$, then x is called a spacelike vector, a lightlike vector or a timelike vector, respectively. The norm of $x \in R_1^3$ is defined by $\| x \| = \sqrt{\mathrm{sign}(x)\langle x, x\rangle}$, where $\mathrm{sign}(x)$ is given by:

$$\mathrm{sign}(x) = \begin{cases} 1, & \text{x: spacelike vector,} \\ 0, & \text{x: lightlike vector,} \\ -1, & \text{x: timelike vector.} \end{cases}$$

x is called a unit vector, if $\| x \| = 1$.

Let $\gamma : I \to R_1^3$; $\gamma(t) = (x_1(t), x_2(t), x_3(t))$ be a regular curve in R_1^3 (i.e., $\dot{\gamma}(t) \neq 0$ for any $t \in I$), where I is an open interval. For any $t \in I$, the curve γ is called a spacelike curve, a lightlike curve or a timelike curve if $\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle > 0, \langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle = 0$ or $\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle < 0$, respectively. γ is called a nonlightlike curve if γ is a spacelike curve or a timelike curve. The arc-length of a nonlightlike curve γ, measured from $\gamma(t_0), t_0 \in I$ is

$$s(t) = \int_{t_0}^{t} \| \dot{\gamma}(t) \| \, \mathrm{d}t.$$

Then the parameter s is determined such that $\| \gamma'(s) \| = 1$ for the nonlightlike curve, where $\gamma'(s) = \frac{\mathrm{d}\gamma}{\mathrm{d}s}(s)$. We denote $t(s) = \gamma'(s)$, and $t(s)$ is called a unit tangent vector of γ at s. We define the curvature of γ at s by $k(s) = \sqrt{|\langle \gamma''(s), \gamma''(s)\rangle|}$. If $k(s) \neq 0$, then the unit principal normal vector $n(s)$ of the curve γ at s is given by $\gamma''(s) = k(s) \cdot n(s)$. We denote $\varepsilon(\gamma) = \mathrm{sign}(t(s))$, and $\delta(\gamma) = \mathrm{sign}(n(s))$. The unit vector $b(s) = t(s) \wedge n(s)$ is called a unit binormal vector of the curve γ at s and $\mathrm{sign}(b(s)) = -\varepsilon(\gamma)\delta(\gamma)$. Then the following Frenet-Serret type formula holds[7]:

$$\begin{cases} t'(s) = k(s) \cdot n(s), \\ n'(s) = -\varepsilon(\gamma) \cdot \delta(\gamma) k(s) \cdot t(s) + \varepsilon(\gamma) \cdot \tau(s) \cdot b(s), \\ b'(s) = \tau(s) \cdot n(s), \end{cases}$$

where $\tau(s)$ is the torsion of the curve γ at s.

For any unit speed curve $\gamma : I \longrightarrow R_1^3$, $D(s) = \tau(s)t(s) - \kappa(s)b(s)$ is called a Darboux vector of γ. We define a vector $\tilde{D} = (\tau/\kappa)(s)t(s) - b(s)$ and call it a modified Darboux vector along γ.

We define the hyperbolic Darboux image of the curve γ as: $d : I \longrightarrow H^2$, $d(s) = \frac{D(s)}{\|D(s)\|}$, where $H^2 = \{(x_1, x_2, x_3) | x_2^2 + x_3^2 - x_1^2 = -1\}$ is a 2-dimensional hyperboloid in R_1^3.

The rectifying Gaussian surface of γ is defined by $RG_\gamma(s, u) = ut(s) - b(s)$, $u \in R$.

The curves concerned in this paper are all unit speed nonlightlike curves. The curves and the mappings in this paper are all C^∞ if there is not special announcement. Our main results are formulated as follows:

Theorem 1.1 *Let* $\mathrm{Im}(S^1, R_1^3)$ *be the space of regular curves* $\gamma : S^1 \longrightarrow R_1^3$ *with* $\tau(s) \neq 0, \kappa(s) \neq$

0 equipped with C^∞-Whitney topology. Then there exists a residual set $\Theta \subset \mathrm{Im}(S^1, R^3_1)$, such that for any $\gamma \in \Theta$, the following properties hold.

(1) The number of the points $s \in S^1$ with $(\tau/\kappa)'(s) = 0$ is finite.

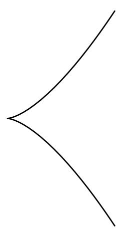
(2) There is no point $s \in S^1$ with $(\tau/\kappa)'(s) = (\tau/\kappa)''(s) = 0$.

Theorem 1.2 *Let $\gamma : I \longrightarrow R^3_1$ be a regular curve with $\tau(s) \neq 0, \kappa(s) \neq 0$ and $\varepsilon(\gamma)\tau^2(s) - \varepsilon(\gamma)\delta(\gamma)\kappa^2(s) < 0$. Then we have the following:*

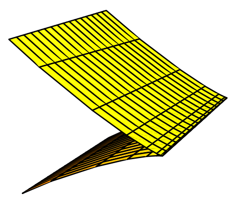
(1) The hyperbolic Darboux image is locally diffeomorphic to the cusp C (Fig.1) at $d(s_0)$ if and only if $(\tau/\kappa)'(s_0) = 0$ and $(\tau/\kappa)''(s_0) \neq 0$.

(2) (a) The rectifying Gaussian surface RG_γ is locally diffeomorphic to the cuspidaledge $C \times R$ (fig.2) at $u_0 t(s_0) - b(s_0)$ if and only if $(\tau/\kappa)'(s_0) \neq 0$ and $u_0 = (\tau/\kappa)(s_0)$.

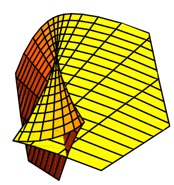
(b) The rectifying Gaussian surface RG_γ is locally diffeomorphic to SW at $u_0 t(s_0) - b(s_0)$ if and only if $(\tau/\kappa)'(s_0) = 0$, $(\tau/\kappa)''(s_0) \neq 0$ and $u_0 = (\tau/\kappa)(s_0)$, where $C = \{(x_1, x_2) | x_1^2 = x_2^3\}$ is the ordinary cusp and $SW = \{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is the swallowtail.



Cusp
Figure 1



Cuspidaledge
Figure 2



Swallowtail
Figure 3

2. Families of smooth functions on a space curve.

In this section, we introduce two different families of functions on a space curve which are effective for the study of singularities of hyperbolic Darboux image and the rectifying Gaussian surface.

Let $\gamma : I \longrightarrow R^3_1$ be a unit speed nonlightlike space curve with $\kappa \neq 0$, $\tau \neq 0$ and $\varepsilon(\gamma)\tau^2(s) - \varepsilon(\gamma)\delta(\gamma)\kappa^2(s) < 0$. Now we define a two-parameter family of smooth functions on I.

$$G : I \times H^2 \longrightarrow R, \quad G(s, v) = \langle b(s), v \rangle.$$

We call it binormal directed height function. For any $v \in H^2$, we denote $g_v(s) = G(s, v)$.

Proposition 2.1 *Let $\gamma : I \longrightarrow R^3_1$ be a unit speed nonlightlike space curve with $\kappa \neq 0$, $\tau \neq 0$ and $\varepsilon(\gamma)\tau^2(s) - \varepsilon(\gamma)\delta(\gamma)\kappa^2(s) < 0$. Then we have*

(1) $g'_v(s) = 0$ if and only if there are real numbers λ and μ such that

$$v = \lambda t(s) + \mu b(s), \ \varepsilon(\gamma)\lambda^2 - \varepsilon(\gamma)\delta(\gamma)\mu^2 = -1.$$

(2) $g'_v(s) = g''_v(s) = 0$ *if and only if*

$$v = \pm\frac{\kappa(s)}{\sqrt{\varepsilon(\gamma)\delta(\gamma)\kappa^2(s) - \varepsilon(\gamma)\tau^2(s)}}((\tau/\kappa)(s)t(s) - b(s)).$$

(3) $g'_v(s) = g''_v(s) = g'''_v(s) = 0$ *if and only if*

$$v = \pm\frac{\kappa(s)}{\sqrt{\varepsilon(\gamma)\delta(\gamma)\kappa^2(s) - \varepsilon(\gamma)\tau^2(s)}}((\tau/\kappa)(s)t(s) - b(s)) \text{ and } (\tau/\kappa)'(s) = 0.$$

(4) $g'_v(s) = g''_v(s) = g'''_v(s) = g^{(4)}_v(s) = 0$ *if and only if*

$$v = \pm\frac{\kappa(s)}{\sqrt{\varepsilon(\gamma)\delta(\gamma)\kappa^2(s) - \varepsilon(\gamma)\tau^2(s)}}((\tau/\kappa)(s)t(s) - b(s)), (\tau/\kappa)'(s) = 0, \text{ and } (\tau/\kappa)''(s) = 0.$$

Proof (1) $g'_v(s) = \langle \tau(s)n(s), v\rangle$, then $g'_v(s) = 0$ if and only if

$$v = \lambda t(s) + \mu b(s), \varepsilon(\gamma)\lambda^2 - \varepsilon(\gamma)\delta(\gamma)\mu^2 = -1.$$

(2) When $g'_v(s) = 0$, $g''_v(s) = \langle \tau'(s)n(s) + \tau(s)(-\varepsilon(\gamma)\delta(\gamma)\kappa(s)t(s) + \varepsilon(\gamma)\tau(s)b(s)), v\rangle = \langle -\varepsilon(\gamma)\delta(\gamma)\kappa(s)\tau(s)t(s) + \varepsilon(\gamma)\tau^2(s)b(s), \lambda t(s) + \mu b(s)\rangle = -\lambda\delta(\gamma)\kappa(s)\tau(s) - \mu\delta(\gamma)\tau^2(s)$, then $g'_v(s) = g''_v(s) = 0$ if and only if

$$v = \pm\frac{\kappa(s)}{\sqrt{\varepsilon(\gamma)\delta(\gamma)\kappa^2(s) - \varepsilon(\gamma)\tau^2(s)}}((\tau/\kappa)(s)t(s) - b(s)),$$

where $(\tau/\kappa)(s) = -\frac{\lambda}{\mu}$, $\lambda = \mp\frac{\tau(s)}{\sqrt{\varepsilon(\gamma)\delta(\gamma)\kappa^2(s) - \varepsilon(\gamma)\tau^2(s)}}$, $\mu = \pm\frac{\kappa(s)}{\sqrt{\varepsilon(\gamma)\delta(\gamma)\kappa^2(s) - \varepsilon(\gamma)\tau^2(s)}}$.

By similar calculation we can get (3) and (4).

Now we define a 3-parameter family of smooth functions on the basis of the binormal directed height function.

$$\tilde{G} : I \times H^2 \times R \longrightarrow R, \quad \tilde{G}(s, v, w) = G(s, v) - \varepsilon(\gamma)\delta(\gamma)w.$$

For any given $(v, w) \in H^2 \times R$, we denote $\tilde{g}_{v,w}(s) = \tilde{G}(s, v, w)$.

By Proposition 2.1, we have the following propositions.

Proposition 2.2 *Let* $\gamma : I \longrightarrow R^3_1$ *be a unit speed curve with* $\varepsilon(\gamma)\tau^2(s) - \varepsilon(\gamma)\delta(\gamma)\kappa^2(s) < 0, \kappa \neq 0$ *and* $\tau \neq 0$. *Then*

(1) $\tilde{g}_{v,w}(s) = 0$ *if and only if* $\varepsilon(\gamma)\delta(\gamma)w = \langle b(s), v\rangle$.

(2) $\tilde{g}_{v,w}(s) = \tilde{g}'_{v,w}(s) = 0$ *if and only if* $v = \lambda t(s) + \mu b(s), \varepsilon(\gamma)\lambda^2 - \varepsilon(\gamma)\delta(\gamma)\mu^2 = -1$, *where* λ, μ *are any real numbers.*

(3) $\tilde{g}_{v,w}(s) = \tilde{g}'_{v,w}(s) = \tilde{g}''_{v,w}(s) = 0$ *if and only if*

$$v = \pm\frac{\kappa(s)}{\sqrt{\varepsilon(\gamma)\delta(\gamma)\kappa^2(s) - \varepsilon(\gamma)\tau^2(s)}}((\tau/\kappa)(s)t(s) - b(s)),$$

$$\varepsilon(\gamma)\delta(\gamma)w = \pm\frac{\varepsilon(\gamma)\delta(\gamma)\kappa(s)}{\sqrt{\varepsilon(\gamma)\delta(\gamma)\kappa^2(s) - \varepsilon(\gamma)\tau^2(s)}}.$$

(4) $\tilde{g}_{v,w}(s) = \tilde{g}'_{v,w}(s) = \tilde{g}''_{v,w}(s) = \tilde{g}'''_{v,w}(s) = 0$ *if and only if*

$$v = \pm\frac{\kappa(s)}{\sqrt{\varepsilon(\gamma)\delta(\gamma)\kappa^2(s) - \varepsilon(\gamma)\tau^2(s)}}((\tau/\kappa)(s)t(s) - b(s)),$$

$$\varepsilon(\gamma)\delta(\gamma)w = \pm\frac{\varepsilon(\gamma)\delta(\gamma)\kappa(s)}{\sqrt{\varepsilon(\gamma)\delta(\gamma)\kappa^2(s) - \varepsilon(\gamma)\tau^2(s)}}, (\tau/\kappa)'(s) = 0.$$

(5) $\tilde{g}_{v,w}(s) = \tilde{g}'_{v,w}(s) = \tilde{g}''_{v,w}(s) = \tilde{g}'''_{v,w}(s) = \tilde{g}^{(4)}_{v,w}(s) = 0$ *if and only if*

$$v = \pm\frac{\kappa(s)}{\sqrt{\varepsilon(\gamma)\delta(\gamma)\kappa^2(s) - \varepsilon(\gamma)\tau^2(s)}}((\tau/\kappa)(s)t(s) - b(s)),$$

$$\varepsilon(\gamma)\delta(\gamma)w = \pm\frac{\varepsilon(\gamma)\delta(\gamma)\kappa(s)}{\sqrt{\varepsilon(\gamma)\delta(\gamma)\kappa^2(s) - \varepsilon(\gamma)\tau^2(s)}}, \ (\tau/\kappa)'(s) = 0, \ \text{ and } \ (\tau/\kappa)''(s) = 0.$$

3. Proof of the main theorems.

Since the proof of Theorem 1.1 is completely similar to the [5], we do not give it again. Now we give the proof of Theorem 1.2 by the singularity theory of function germs.

Let $F : (R \times R^r, (s_0, x_0)) \longrightarrow R$ be a function germ. We call F an r-parameter unfolding of f, where $f(s) = F_{x_0}(s, x_0)$. If $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$ and $f^{(k+1)}(s_0) \neq 0$, then we call that $f(s)$ has A_k-singularity at s_0. We also call that $f(s)$ has $A_{\geq k}$-singulatity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$. Let F be the r-parameter unfolding of f, and $f(s)$ have A_k-singularity ($k \geq 1$) at s_0. We denote the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 as $j^{(k-1)}(\frac{\partial F}{\partial x_i}(s, x_0))(s_0) = \sum_{j=1}^{k-1} \alpha_{ji} s^j$, $i = 1, \ldots, r$. If the rank of the $(k-1) \times r$ matrix (α_{ji}) made up of the coefficients of the $(k-1)$-jet is $k-1$ $(k-1 \leq r)$, then F is called a (p) versal unfolding of f. Under the same condition, if the rank of $k \times r$ matrix $(\alpha_{0i}, \alpha_{ji})$ is k $(k \leq r)$, then F is called a versal unfolding of f, where $\alpha_{0i} = \frac{\partial F}{\partial x_i}(s_0, x_0)$. Now there are some important sets concerning with the unfolding related to the above notions.

The bifurcation set of F is the set: $B_F = \{x \in R^r | \frac{\partial F}{\partial s}(s, x) = \frac{\partial^2 F}{\partial s^2}(s, x) = 0\}$.

The discriminant set of F is the set: $D_F = \{x \in R^r | F(s, x) = \frac{\partial F}{\partial s}(s, x) = 0\}$.

Then we have the following propositions[5, p 150]

Theorem 3.1 *Let $F : (R \times R^r, (s_0, x_0)) \longrightarrow R$ be an r-parameter unfolding of $f(s)$ which has A_k-singularity at s_0.*

(1) Suppose F is a (p) versal unfolding of f.

(a) If $k = 2$, then B_F is locally diffeomorphic to $\{0\} \times R^{r-1}$.

(b) If $k = 3$, then B_F is locally diffeomorphic to $C \times R^{r-2}$.

(2) Suppose F is a versal unfolding of f.

(a) If $k = 1$, then D_F is locally diffeomorphic to $\{0\} \times R^{r-1}$.

(b) If $k = 2$, then D_F is locally diffeomorphic to $C \times R^{r-2}$.

(c) If $k = 3$, then D_F is locally diffeomorphic to $SW \times R^{r-3}$.

For the proof of Theorem 1.2, we have the following key propositions about the binormal directed function G.

Proposition 3.2 *If $g_{v_0}(s)$ has A_k-singularity ($k = 2, 3$) at s_0, then $G(s, v)$ is a (p) versal unfolding of $g_{v_0}(s)$.*

Proof Let $b(s)=(b_1(s),b_2(s),b_3(s)), v=(\sqrt{y_2^2+y_3^2+1},y_2,y_3)$. Then we have

$$G(s,v)=\langle b(s),v\rangle=-b_1(s)\sqrt{y_2^2+y_3^2+1}+b_2(s)y_2+b_3(s)y_3.$$

$$\partial G/\partial y_2=\frac{-y_2b_1(s)}{\sqrt{y_2^2+y_3^2+1}}+b_2(s), \partial G/\partial y_3=\frac{-y_3b_1(s)}{\sqrt{y_2^2+y_3^2+1}}+b_3(s).$$

$$j^2(\frac{\partial G}{\partial y_i}(s,y_0))(s_0)=(\frac{-y_{0i}b_1'(s_0)}{\sqrt{y_{02}^2+y_{03}^2+1}}+b_i'(s_0))s+\frac{1}{2}(\frac{-y_{0i}b_1''(s_0)}{\sqrt{y_{02}^2+y_{03}^2+1}}+b_i''(s_0))s^2, \quad i=2,3.$$

(i) When $g_{v_0}(s)$ has A_2-singularity at s_0, we need to prove the rank of 1×2 matrix

$$A=(\frac{-y_{02}b_1'(s_0)}{\sqrt{y_{02}^2+y_{03}^2+1}}+b_2'(s_0),\frac{-y_{03}b_1'(s_0)}{\sqrt{y_{02}^2+y_{03}^2+1}}+b_3'(s_0))$$

is 1. This conclusion can be gotten from the following conclusion.

(ii) When $g_{v_0}(s)$ has A_3-singularity at s_0 if and only if

$$v=\pm\frac{\kappa(s)}{\sqrt{\varepsilon(\gamma)\delta(\gamma)\kappa^2(s)-\varepsilon(\gamma)\tau^2(s)}}((\tau/\kappa(s))t(s)-b(s)), \quad (\tau/\kappa)'=0.$$

We need to prove the rank of the following 2×2 matrix

$$B=\begin{pmatrix} \frac{-y_{02}b_1'(s_0)}{\sqrt{y_{02}^2+y_{03}^2+1}}+b_2'(s_0) & \frac{-y_{03}b_1'(s_0)}{\sqrt{y_{02}^2+y_{03}^2+1}}+b_3'(s_0) \\ \frac{-y_{02}b_1''(s_0)}{2\sqrt{y_{02}^2+y_{03}^2+1}}+\frac{1}{2}b_2''(s_0) & \frac{-y_{03}b_1''(s_0)}{2\sqrt{y_{02}^2+y_{03}^2+1}}+\frac{1}{2}b_3''(s_0) \end{pmatrix}$$

is 2.

As $\varepsilon(\gamma)\delta(\gamma)\kappa^2(s)-\varepsilon(\gamma)\tau^2(s)>0$, we have

$$\begin{aligned}\det(B)&=\frac{\langle b'(s)\wedge b''(s),v\rangle}{2\sqrt{y_{02}^2+y_{03}^2+1}}=\pm\frac{\kappa^2(s)\tau^2(s)-\delta(\gamma)\tau^4}{2\sqrt{y_{02}^2+y_{03}^2+1}\sqrt{\varepsilon(\gamma)\delta(\gamma)\kappa^2(s)-\varepsilon(\gamma)\tau^2(s)}}\\ &=\pm\frac{\varepsilon(\gamma)\delta(\gamma)\tau^2(s)(\varepsilon(\gamma)\delta(\gamma)\kappa^2(s)-\varepsilon(\gamma)\tau^2(s))}{2\sqrt{y_{02}^2+y_{03}^2+1}\sqrt{\varepsilon(\gamma)\delta(\gamma)\kappa^2(s)-\varepsilon(\gamma)\tau^2(s)}}\\ &=\pm\frac{\varepsilon(\gamma)\delta(\gamma)\tau^2(s)}{2\sqrt{y_{02}^2+y_{03}^2+1}}\sqrt{\varepsilon(\gamma)\delta(\gamma)\kappa^2(s)-\varepsilon(\gamma)\tau^2(s)}\neq 0.\end{aligned}$$

Namely, the rank of B is 2. So when $k=2,3$, $G(s,v)$ are all (p) versal unfoldings of $g_{v_0}(s)$.

For the function $\tilde{G}$, the following propositions hold.

Proposition 3.3 *If $\tilde{g}_{v_0,w_0}(s)$ has A_k-singularity $(k=1,2,3)$ at s_0, then $\tilde{G}(s,v,w)$ is a versal unfolding of $\tilde{g}_{v_0,w_0}(s)$.*

Proof By Proposition 3.2 and the expression of $\tilde{G}$, we have

(i) When $\tilde{g}_{v_0,w_0}(s)$ has A_1-singularity at s_0, it follows from $\varepsilon(\gamma)\delta(\gamma)\neq 0$ that the rank of 1×3 matrix

$$C=(\frac{-y_{02}b_1(s_0)}{\sqrt{y_{02}^2+y_{03}^2+1}}+b_2(s_0),\frac{-y_{03}b_1(s_0)}{\sqrt{y_{02}^2+y_{03}^2+1}}+b_3(s_0)),-\varepsilon(\gamma)\delta(\gamma))$$

is 1.

(ii) When $\tilde{g}_{v_0,w_0}(s)$ has A_2-singularity, by the proof of Proposition 3.2, we know that the matrix A is nonsingular, and that the 2×3 matrix

$$D=\begin{pmatrix} \frac{-y_{02}b_1(s_0)}{\sqrt{y_{02}^2+y_{03}^2+1}}+b_2(s_0) & \frac{-y_{03}b_1(s_0)}{\sqrt{y_{02}^2+y_{03}^2+1}}+b_3(s_0) & -\varepsilon(\gamma)\delta(\gamma) \\ \frac{-y_{02}b_1'(s_0)}{\sqrt{y_{02}^2+y_{03}^2+1}}+b_2'(s_0) & \frac{-y_{03}b_1'(s_0)}{\sqrt{y_{02}^2+y_{03}^2+1}}+b_3'(s_0) & 0 \end{pmatrix}$$

is nonsingular is equivalent to that the matrix A is nonsingular, so the rank of matrix D is 2.

(iii) When $\tilde{g}_{v_0,w_0}(s)$ has A_3-singularity at s_0, by the proof of Proposition 3.2 we know that matrix B is nonsingular, and that 3×3 matrix

$$E=\begin{pmatrix} \frac{-y_{02}b_1(s_0)}{\sqrt{y_{02}^2+y_{03}^2+1}}+b_2(s_0) & \frac{-y_{03}b_1(s_0)}{\sqrt{y_{02}^2+y_{03}^2+1}}+b_3(s_0) & -\varepsilon(\gamma)\delta(\gamma) \\ \frac{-y_{02}b_1'(s_0)}{\sqrt{y_{02}^2+y_{03}^2+1}}+b_2'(s_0) & \frac{-y_{03}b_1'(s_0)}{\sqrt{y_{02}^2+y_{03}^2+1}}+b_3'(s_0) & 0 \\ \frac{-y_{02}b_1''(s_0)}{2\sqrt{y_{02}^2+y_{03}^2+1}}+\frac{1}{2}b_2''(s_0) & \frac{-y_{03}b_1''(s_0)}{2\sqrt{y_{02}^2+y_{03}^2+1}}+\frac{1}{2}b_3''(s_0) & 0 \end{pmatrix}$$

is nonsingular is equivalent to that matrix B is nonsingular, so the rank of matrix E is 3. Hence $\tilde{G}$ is a versal unfolding of $\tilde{g}_{v_0,w_0}$.

In order to study the singularities of $RG_{(\gamma)}$, we now give a mapping on the space of parameters.

$$\Phi: H^2\times R\longrightarrow R_1^3,\ \Phi(\sqrt{y_{02}^2+y_{03}^2+1},y_{02},y_{03},w)=(\frac{\sqrt{y_{02}^2+y_{03}^2+1}}{w},\frac{y_{02}}{w},\frac{y_{03}}{w}).$$

Then Φ is a diffeomorphic mapping.

By Proposition 2.2, we can get the following propositions.

Proposition 3.4 *The discriminant set of $\tilde{G}$, $D_{\tilde{G}}$ is diffeomorphic to $(\tau/\kappa)(s)t(s)-b(s)$, that is, $\Phi(D_{\tilde{G}})=(\tau/\kappa)(s)t(s)-b(s)$, if $\tilde{g}_{v_0,w_0}(s)$ has A_k-singularity ($k\geq 2$) at s_0.*

The proof of Theorem 1.2 By Proposition 2.1, we know that the bifurcation set of $G(s,v)$ is hyperbolic Darboux image, by Theorem 3.1 and Proposition 3.2, Theorem 1.2(1) holds. By Propositions 2.2 and 3.4, we know that the discriminant set of $\tilde{G}(s,v,w)$ is diffeomorphic to the rectifying Gaussian surface, and by Theorem 3.1 and Proposition 3.3, Theorem 1.2(2) holds.

4. Helix and hyperbolic (pseudo spherical) tangent image

In this section, we mainly study the geometric properties of hyperbolic Darboux image and rectifying Gaussian surface. By the propositions in last section, we know that the function $(\tau/\kappa)'(s)$ and the rectifying Darboux vector $\tilde{D}(s)=(\tau/\kappa)(s)t(s)-b(s)$ are important subjects to study. When $(\tau/\kappa)'(s)=0$, the curve $\gamma(s)$ is a helix in R_1^3. For the unit speed curve $\gamma: I\to R_1^3$, the unit tangent curve $t: I\to H^2, (t: I\to S_1^2)$ is called *the hyperbolic (pseudo spherical) tangent image* of γ. By calculation, the geodesic curvature of hyperbolic (pseudo spherical) tangent image of γ is the function $-(\delta(\gamma)\tau/\kappa)(s)$. Now we assume that γ is a timelike helix, i.e., the tangent vectors of the curve are all timelike vectors (when γ is a spacelike curve, the situation is completely similar). The geodesic curvature of γ is $-(\tau/\kappa)(s)$. We have the following propositions.

Proposition 4.1 *Let $\gamma : I \to R_1^3$ be a regular timelike curve. Then we have*

(1) γ is a helix if and only if the rectifying Darboux vector $\tilde{D}(s) = (\tau/\kappa)(s)t(s) - b(s)$ is a constant vector.

(2) $t(s)$, the hyperbolic tangent image of γ is a hyperbola if γ is a helix, and its center is determined by the constant vector $\overline{d}(s) = \frac{-(\tau/\kappa)(s)}{\sqrt{(\tau/\kappa)^2(s)-1}}d(s)$, where $(\tau/\kappa)^2(s) - 1 > 0$.

Proof (1) By Frenet-Serret type formula, we have $\tilde{D}'(s) = (\tau/\kappa)'(s)t(s)$, so γ is a helix if and only if the rectifying Darboux vector satisfies the following condition: $\tilde{D}'(s) \equiv 0$, i.e., $\tilde{D}(s)$ is constant.

(2) $d(s)$ is constant if γ is a helix and $\frac{-(\tau/\kappa)(s)}{\sqrt{(\tau/\kappa)^2(s)-1}}$ is constant, so $\overline{d}(s)$ is constant. As $\langle t(s)-\overline{d}(s), \overline{d}(s)\rangle = 0$, so $t(s)-\overline{d}(s)$ is pseudo perpendicular to $\overline{d}(s)$, i.e., for any $s \in I$, $t(s)-\overline{d}(s)$ are on the same plan. Since

$$\langle t(s) - \overline{d}(s), t(s) - \overline{d}(s)\rangle = -1 - \frac{(\tau/\kappa)^2(s)}{(\tau/\kappa)^2(s)+1}$$

is a constant number less than zero, the hyperbolic tangent image $t(s)$ of γ is a hyperbola, and its center is determined by the terminal of the constant vector $\overline{d}(s)$.

Let $\gamma_i : I \to R_1^3$ $(i = 1, 2)$ be a regular curve. If $\gamma_1^p(s_0) = \gamma_2^{(p)}(t_0)$, $0 \le p \le k$, $\gamma_1^{k+1}(s_0) \ne \gamma_2^{(k+1)}(t_0)$, then we call $\gamma_1(s_0)$ and $\gamma_2(t_0)$ have $(k+1)$-point contact. By the definitions of curvature and torsion, we have the following two propositions:

Proposition 4.2 *If $\gamma_1(s_0)$ and $\gamma_2(t_0)$ have k+1-point contact, then $(\tau/\kappa)_1^{(p)}(s_0) = (\tau/\kappa)_2^{(p)}(t_0)(0 \le p \le k-3)$, and $(\tau/\kappa)_1^{(k-2)}(s_0) \ne (\tau/\kappa)_2^{(k-2)}(t_0)$.*

Proposition 4.3 *Let $\gamma : I \to R_1^3$ be a regular nonlightlike curve with $\tau(s_0) \ne 0$ and $\kappa(s_0) \ne 0$. Then there exists an open interval $s_0 \in J \subset I$ and a unique helix $\rho : J \to R_1^3$ such that $\rho(s_0) = \gamma(s_0)$, the curvature of ρ at s_0 is $\kappa(s_0)$, the torsion of ρ at s_0 is $\tau(s_0)$, and γ and ρ have at least 4-point contact at s_0.*

We call the helix ρ in Proposition 4.3 the osculating helix of γ at s_0 and denote it by $\rho_{s_0}(s)$. So the singularities of the hyperbolic Darboux image and the rectifying Gaussian surface describe the contacting extent between the curve and the helix.

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