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The Necessary and Sufficient Condition for Strong Irreducibility of Cowen-Douglas Operators

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Abstract In this note, we show that a Cowen-Douglas operator is strongly irreducible if and only if its commutant algebra mod its Jocobson radical is isomorphic to a closed subalgebra of $H^\infty(D)$, where D is the open unit disk, and $H^\infty(D)$ denotes the collection of bounded holomorphic functions on D.

Keywords strongly irreducible; Cowen-Douglas operator; commutant algebra; Jacobson radical.

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1. Introduction and preliminary results

Let $\mathcal{H}$ be a complex separable Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the collectio of all bounded linear operators on $\mathcal{H}$. For an operator $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}'(T)$ denote the commutant of T, and rad$\mathcal{A}'(T)$ denote the Jacobson radical of $\mathcal{A}'(T)$. In 1978, Cowen and Douglas[1] introduced a class of operators related to complex geometry, which are now referred to as Cowen-Douglas operators.

Definition 1.1[1] *For a connected open subset Ω of C and a positive integer n, let $\mathcal{B}_n(\Omega)$ denote the operators T in $\mathcal{L}(\mathcal{H})$ which satisfy*

(i) $\Omega \subset \sigma(T) = \{z \in C; T - z$ *not invertible* $\}$;

(ii) $\mathrm{ran}(T - z) = \mathcal{H}$ *for z in Ω;*

(iii) $\bigvee_{z\in\Omega} \ker(T - z) = \mathcal{H}$*; and*

(iv) $\dim\ker(T - z) = n$ *for z in Ω.*

We call an operator in $\mathcal{B}_n(\Omega)$ a Cowen-Douglas operator with index n. Cowen-Douglas operators have nice properties.

Proposition 1.1[1] *Let Ω_1 and Ω_2 be connected open subsets of C. If $\Omega_1 \subset \Omega_2$, then $\mathcal{B}_n(\Omega_2) \subset \mathcal{B}_n(\Omega_1)$.*

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Proposition 1.2[1] *Every operator in $\mathcal{B}_1(\Omega)$ is strongly irreducible. If $n \geq 2$, there exist many strongly operators in $\mathcal{B}_n(\Omega)$.*

An operator T in $\mathcal{L}(\mathcal{H})$ is said to be strongly irreducible, if there are no non-trivial idempotent operators in $\mathcal{A}'(T)$ (in short $T \in (SI)$).

It is important in the study of an operator T to characterize its commutant $\mathcal{A}'(T)$ explicitly. Jiang[2] studied the commutant of Cowen-Douglas operators and obtained the following result:

Theorem 1.1[2] *If $T \in \mathcal{B}_n(\Omega) \cap (SI)$, then $\mathcal{A}'(T)/\mathrm{rad}\mathcal{A}'(T)$ is commutative.*

If $\mathcal{A}/\mathrm{rad}\mathcal{A}$ is commutative, then $\mathcal{A}$ is essential commutative. By Theorem 1.1, the commutant of every strongly irreducible Cowen-Douglas operator is essential commutative.

Cowen and Douglas[1] obtained the following result:

Theorem 1.2[1] *The commutant of a Cowen-Douglas operator with index 1 is isomorphic to a closed subalgebra of $H^\infty(D)$.*

In 2003, Jiang and He got the following result:

Theorem 1.3[3] *Let A be in $\mathcal{B}_n(\Omega)$. Then the following statements are equivalent.*

(i) $A \in (SI)$;

(ii) There exists a connected open subset Ω_1 of C, such that $A \in \mathcal{B}_m(\Omega_1)$ and $\sigma((\Gamma_A X)(z))$ is connected for each z in Ω_1 and each X in $\mathcal{A}'(A)$, where $m \leq n$ and m is the minimal index of A;

(iii) There exists a connected open subset Ω_1 of C, such that $A \in \mathcal{B}_m(\Omega_1)$ and $\sigma((\Gamma_A X)(z_0))$ is connected for each X in $\mathcal{A}'(A)$ and some z_0 in Ω_1 where $m \leq n$ and m is the minimal index of A.

Inspired by the work of Herreo[4], Jiang raised the following question:

Let $T \in \mathcal{B}_n(\Omega)$. Is it true or not that $T \in (SI)$ if and only if its commutant mod its Jocobson radical is isomorphic to a closed subalgebra of $H^\infty(D)$?

In this note, we give an affirmative answer to this question by the tools of complex geometry, and show that a Cowen-Douglas operator is strongly irreducible if and only if its commutant algebra mod its Jocobson radical is isomorphic to a closed subalgebra of $H^\infty(D)$, where D is the open unit disk, and $H^\infty(D)$ denotes the collection of bounded holomorphic functions on D.

Now Cowen-Douglas operators is an especially rich class of operators containing the adjoint of many subnormal, hyper-normal and weighted unilateral shift operators. Atiyah set a high value on the work of Cowen-Douglas theory in Mathematic Review (MR501368, 80f:47012).

In order to prove our main result, we need to introduce the notation of Hermitian holomorphic vector bundle. Let Λ be a manifold with a complex structure and n be a positive integer. A rank n holomorphic vector bundle over Λ consists of a manifold with a complex structure and a holomorphic map π from E onto Λ such that each fibre $E_z = \pi^{-1}(z)$ is isomorphic to C^n and such that for each z_0 in Λ there exists a neighborhood Δ of λ_0 and holomorphic functions $e_1(z), \ldots, e_n(z)$ from Δ to E whose values form a basis for E_z at each z in Δ. The functions $e_1, \ldots, e_n$ are said to be a frame for E on Δ. The bundle is said to be trivial if Δ can be taken to be all of Λ.

For an operator T in $\mathcal{B}_n(\Omega)$, the mapping $z \longrightarrow \ker(T-z)$ defines a rank n holomorphic vector bundle. Let (E_T, π) denote the sub-bundle of trivial bundle $\Omega \times \mathcal{H}$ defined by

$$E_T = \{(z,x) \in \Omega \times \mathcal{H};\ x \in \ker(T-z) \ \text{ and } \ \pi(z,x) = z\}.$$

A Hermitian holomorphic vector bundle E over Λ is a holomorphic vector bundle such that each fibre E_z is an inner product space. Obviously, E_T is a Hermitian holomorphic vector bundle over Ω for T in $\mathcal{B}_n(\Omega)$[1, Corollary 1.12].

Let $T \in \mathcal{B}_n(\Omega)$, $z \in \Omega$ and $S \in \mathcal{A}'(T)$. Then $S(T-z) = (T-z)S$ and $S\ker(T-z) \subseteq \ker(T-z)$. So we can define $(\Gamma_T S)(z) = S|_{\ker(T-z)} =: S(z)$. By Proposition 1.21 in [1], Γ_T is a contractive monomorphism from $\mathcal{A}'(T)$ to $H^\infty_{\mathcal{L}(E)}(\Omega)$, where $H^\infty_{\mathcal{L}(E)}(\Omega)$ denotes the collection of bounded bundle endomorphisms on E.

For $S \in \mathcal{A}'(T)$, let $\{e_1(z), e_2(z), \ldots, e_n(z)\}$ be a holomorphic frame of $\ker(T-z)$. It is easy to see that $S(z) \in M_n(H^\infty(\Omega))$.

Definition 1.2[3] *Let $A \in \mathcal{B}_n(\Omega)$ and $B \in \mathcal{A}'(A)$. If $\sigma(B(z))$ is disconnected at $z = z_0 \in \Omega$, then there exists a positive number δ such that $\sigma(B(z))$ is disconnected for $z \in \{z, |z-z_0| < \delta\} \triangleq D(z_0,\delta)$. Hence, we can find a positive number ε such that $\sigma(B(z)) \cap \bar{D}(\lambda(z_0),\varepsilon) = \lambda(z_0)$, $z \in D(z_0,\delta)$, where $\lambda(z_0)$ is an eigenvalue of $B(z_0)$. Let*

$$P(z) = \int_{\partial D(\lambda(z_0),\varepsilon)} (B(z)-\lambda)^{-1} \mathrm{d}\lambda.$$

Then $P(z)$ is said to be a holomorphic idempotent element defined on $D(\lambda(z_0),\varepsilon)$ induced by $\mathcal{A}'(A)$. If each holomorphic idempotent element defined on connected open set Φ induced by $\mathcal{A}'(A)$ satisfies $\mathrm{dimran}(A|_{\bigvee_{z\in\Phi}} - z_0) < n$, *then n is called minimal index of A, or A is said to have minimal index n.*

Lemma 1.1 *Let $A \in \mathcal{B}_n(\Omega)$ and n be the minimal index of A. And let $P(z)$ be a holomorphic idempotent element defined on open set Φ induced by $\mathcal{A}'(A)$ and* $\mathrm{ran}P(z) = k < n$, $z \in \Phi$. *Set $\mathcal{H}_1 = \bigvee_{z\in\Phi} \mathrm{ran}P(z)$. Then $A|_{\mathcal{H}_1} \in \mathcal{B}_k(\Omega)$.*

Proof Let P be an orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_1$ and $A_1 = A|_{\mathcal{H}_1}, A_2 = (A^*|_{\mathcal{H}_1^\perp})^*$. Then

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_1^\perp \end{matrix}.$$

By Lemma 1.2 in [4], $\sigma_P(A^*) = \emptyset$, where $\sigma_P(A^*)$ denotes the point spectrum of A^*. So $\dim\mathcal{H}_1^\perp = +\infty$. Since $A-z$ is right invertible for z in Ω, we can find an operator

$$B = \begin{pmatrix} B_1 & B_{12} \\ 0 & B_2 \end{pmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_1^\perp \end{matrix}$$

such that

$$\begin{aligned}(A-z)B &= \begin{pmatrix} A_1 - z & A_{12} \\ 0 & A_2 - z \end{pmatrix}\begin{pmatrix} B_1 & B_{12} \\ 0 & B_2 \end{pmatrix} \\ &= \begin{pmatrix} I_{\mathcal{H}_1} & 0 \\ 0 & I_{\mathcal{H}_2} \end{pmatrix}.\end{aligned}$$

Hence $(A_2 - z)B_2 = I_{\mathcal{H}_1^\perp}$. This shows that $A_2 - z$ is right invertible for z in Ω. It is easily seen that $A_2 \in \mathcal{B}_m(\Omega)$, where $m < n$. Let π be the canonical map from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, where $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ denotes the Calkin algebra. Then

$$\pi(B)\pi(A-z) = \begin{pmatrix} \pi(I_{\mathcal{H}_1}) & 0 \\ 0 & \pi(I_{\mathcal{H}_1^\perp}) \end{pmatrix}.$$

This shows that $\operatorname{ran}(A_1 - z)$ is closed for z in Ω. Let $(e_1(z), \ldots, e_k(z))$ be a holomorphic frame of $\operatorname{ran}P(z)$. Then $(A_1 - z)e_j(y) = (y - z)e_j(y)$, for $1 \le j \le k$. Note that $\mathcal{H}_1 = \bigvee_{z\in\Phi} \operatorname{ran}P(z)$. Thus $\operatorname{ran}(A_1 - z) = \mathcal{H}_1$, for $z \in \Phi$.

Lemma 1.2[5] *If $T \in \mathcal{B}_n(\Omega) \cap (SI)$, then for any $z \in \Omega$ and $S \in \mathcal{A}'(T)$, $\sigma(S(z))$ is singleton, where $S(z) = S|_{\ker(T-z)}$.*

By Lemmas 1.1 and 1.2, we get the following corollary.

Corollary 1.1 *Let $T \in \mathcal{B}_n(\Omega) \cap (SI)$. For any $z \in \Omega$ and $S \in \mathcal{A}'(T)$, let $\mathcal{A}_T^z = \{S(z); S \in \mathcal{A}'(T)\}$. Then $\mathcal{A}_T^z/\operatorname{rad}\mathcal{A}_T^z \cong C$.*

Lemma 1.3[5] *Let $T \in \mathcal{B}_n(\Omega)$. Then the following statements are equivalent.*

(i) $\mathcal{A}'(T)$ is essential commutative;

(ii) For any $z \in \Omega$, $\mathcal{A}_T^z$ is essential commutative;

(iii) For any $z \in D(z_0, \epsilon) \subseteq \Omega$, $\mathcal{A}_T^z$ is essential commutative, where, $D(z_0, \epsilon) = \{z \in C; |z - z_0| < \epsilon\}$ and $\epsilon > 0$.

Lemma 1.4 *Let $T \in \mathcal{L}(H)$. Then the following statements are equivalent.*

(i) $T \in (SI)$;

(ii) For each operator $B \in \mathcal{A}'(T)$, $\sigma(B)$ is connected.

Proof (1)$\Rightarrow$(2). If there exists an operator $B \in \mathcal{A}'(T)$ such that $\sigma(B)$ is not connected, then there exist two closed subsets σ_1 and σ_2 in complex plane C such that $\sigma(B) = \sigma_1 \cup \sigma_2$ and $\sigma_1 \cap \sigma_2 = \emptyset$. Hence there is a Cauchy domain Ω such that $\bar{\Omega} \cap \sigma_2 = \emptyset$ and $\sigma_1 \subset \Omega$. Let

$$P = \int_{\partial\Omega} (\lambda - B)^{-1} \mathrm{d}\lambda.$$

By Riesz decomposition theorem, P is a nontrivial idempotent in $\mathcal{A}'(T)$. This contradicts $T \in (SI)$.

(2)$\Rightarrow$(1). If $T \notin (SI)$, then T can be written as the direct sum of T_1 and T_2, i.e., $T = T_1 \dot{+} T_2$, where $T_1 \in \mathcal{L}(\mathcal{H}_1)$ and $T_2 \in \mathcal{L}(\mathcal{H}_2)$, and $\dot{+}$ means algebra direct sum. Let $B = I_{\mathcal{H}_1} \dot{+} \alpha I_{\mathcal{H}_2}$. Then

$B{\in}\mathcal{A}'(T)$ and $\sigma(B)$ is not connected. This contradicts (ii).

2. The proof of main result

Theorem 2.1 *Let $T \in \mathcal{B}_n(\Omega)$. Then T is strongly irreducible if and only if its commutant algebra mod its Jocobson radical is isomorphic to a closed subalgebra of $H^\infty(D)$, where D is the open unit disk, and $H^\infty(D)$ denotes the collection of bounded holomorphic functions on D.*

Proof "only if part". Let $A \in \mathcal{B}_n(\Omega)$. For $z \in \Omega$ and $S \in \mathcal{A}'(T)$, let $\mathcal{A}_T^z = \{S(z); S \in \mathcal{A}'(T)\}$. Note that $S(z) \in M_n(H^\infty(\Omega))$. Therefore, $\mathcal{A}_T^z \subseteq M_n(C)$ is a finite dimensional algebra. Let ϕ_z be a function from $\mathcal{A}'(T)$ to $\mathcal{A}_T^z$ defined by $\phi_z(S) = S(z)$. Then ϕ_z is an isomorphism.

Claim If J is a maximal left ideal of $\mathcal{A}'(T)$, then $\phi_z(J)$ is a maximal left ideal of $\mathcal{A}_T^z$.

Proof of the Claim If it is not true, let $J' \supset \phi_z(J)$ be a maximal left ideal of $\mathcal{A}_T^z$. Then there exists $S'(z) \in J'$, but $S'(z) \notin \phi_z(J)$. Moreover, $S' \notin J$, and for any $S(z) \in \mathcal{A}_T^z$, $S'(z)S(z) \in J'$, i.e. $\phi_z(S'S) \in J'$. ϕ_z is an isomorphism, so $S'S \in \phi_z^{-1}(J)$, where, $S' \notin J$, for any $S \in \mathcal{A}'(T)$. This contradicts our assumption.

For a unital Banach algebra $\mathcal{A}$, its Jacobson radical is the intersection of all maximal left ideals of $\mathcal{A}$, hence, $\mathrm{rad}\mathcal{A}'(T) \cong \mathrm{rad}\mathcal{A}_T^z$. By Lemma 1.3, if $T{\in}\mathcal{B}_n(\Omega) \cap (SI)$, $\mathcal{A}'(T)$ is essential commutative $\Leftrightarrow$ for any $z \in \Omega$, $\mathcal{A}_T^z$ is essential commutative. By the second isomorphism theorem, $\mathcal{A}'(T)/\mathrm{rad}\mathcal{A}'(T) \cong \mathcal{A}_T^z/\mathrm{rad}\mathcal{A}_T^z$. By Corollary 1.1, $\mathcal{A}_T^z/\mathrm{rad}\mathcal{A}_T^z \subseteq H^\infty(\Omega)$. Without loss of generality, we may assume that $\bar{D} \subseteq \Omega$, then $H^\infty(\Omega) \subseteq H^\infty(D)$. Hence, $\mathcal{A}'(T)/\mathrm{rad}\mathcal{A}'(T)$ is isomorphic to a subalgebra of $H^\infty(D)$.

"if part". For $S \in \mathcal{A}'(T)$ and $[S] \in \mathcal{A}'(T)/\mathrm{rad}\mathcal{A}'(T)$, by Theorem 3.1.5 in [6], $\sigma(S) = \sigma([S])$. By our assumption, $\mathcal{A}'(T)/\mathrm{rad}\mathcal{A}'(T)$ is isomorphic to a subalgbra of $H^\infty(D)$, so there exists a function $f \in H^\infty(D)$, such that $\sigma(S) = \sigma([S]) = \sigma(f) = \mathrm{cl}\hat{f}(D)$. By Theorem 6.17 in [7], $\hat{f}$ is an analytic function. So $\sigma(S)$ is connected. Hence $T \in (SI)$ by Lemma 1.4.

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