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Study of Modules over 3×3 Formal Triangular Matrix Rings

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Abstract In this paper we carry out a study of modules over a 3×3 formal triangular matrix ring

$$\Gamma = \begin{pmatrix} T & 0 & 0 \\ M & U & 0 \\ N\otimes_U M & N & V \end{pmatrix},$$

where T, U, V are rings, M, N are U-T, V-U bimodules, respectively. Using the alternative description of left Γ-module as quintuple $(A,B,C;f,g)$ with $A \in \mathrm{mod}T$, $B \in \mathrm{mod}U$ and $C \in \mathrm{mod}V$, $f: M\otimes_T A \to B \in \mathrm{mod}U$, $g: N\otimes_U B \to C \in \mathrm{mod}V$, we shall characterize uniform, hollow and finitely embedded modules over Γ, respectively. Also the radical as well as the socle of ${}_\Gamma(A\oplus B\oplus C)$ is determined.

Keywords triangular matrix ring; uniform module; hollow module; radical; socle.

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1. Introduction

All the rings are associative with the identity and all modules are unital modules. Unless otherwise mentioned, all modules are left modules. For any ring R, we denote by $\mathrm{mod}R$ the category of finitely generated left R-module.

Given a 3×3 formal triangular matrix ring

$$\Gamma = \begin{pmatrix} T & 0 & 0 \\ M & U & 0 \\ N\otimes_U M & N & V \end{pmatrix}.$$

Let Ω be the category whose objects are the quintuple $(A,B,C;f,g)$ with $A\in\mathrm{mod}T$, $B\in\mathrm{mod}U$, $C\in\mathrm{mod}V$ and $f: M\otimes_T A\to B$ a U-morphism, $g: N\otimes_U B\to C$ a V-morphism. The morphism between two objects $(A,B,C;f,g)$ and $(A',B',C';f',g')$ are triples of morphisms (α,β,γ) where $\alpha: A\to A'$ is a T-morphism, $\beta: B\to B'$ is a U-morphism, and $\gamma: C\to C'$ is a V-morphism such that the following two diagrams commute.

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$$\begin{tikzcd} M\otimes_T A \arrow[r, "1_M\otimes\alpha"] \arrow[d, "f"] & M\otimes_T A' \arrow[d, "f'"] \\ B \arrow[r, "\beta"] & B', \end{tikzcd} \qquad \begin{tikzcd} N\otimes_U B \arrow[r, "1_N\otimes\beta"] \arrow[d, "g"] & N\otimes_U B' \arrow[d, "g'"] \\ C \arrow[r, "\gamma"] & C'. \end{tikzcd}$$

The composition between two morphisms (α,β,γ) and (α',β',γ') is defined by $(\alpha',\beta',\gamma')(\alpha,\beta,\gamma) = (\alpha'\alpha,\beta'\beta,\gamma'\gamma)$. By [1], The category Ω is equivalent to the category modΓ. The equivalent functor $F:\Omega\to \mathrm{mod}\Gamma$ is defined as follows. For $(A,B,C;f,g)\in \mathrm{Obj}\Omega$ we define $F(A,B,C;f,g):=A\oplus B\oplus C$ as an abelian group and the Γ-module structure on it is defined as

$$\begin{pmatrix} t & 0 & 0\\ m & u & 0\\ \overline{m}\otimes\overline{n} & n & v\end{pmatrix}(a,b,c)=(ta, f(m\otimes a)+ub, g(\overline{n}\otimes f(\overline{m}\otimes a))+g(n\otimes b)+vc)$$

for $t\in T$, $u\in U$, $v\in V$, $m,\overline{m}\in M$, $n,\overline{n}\in N$. If $(\alpha,\beta,\gamma):(A,B,C;f,g)\to(A',B',C',f',g')$ in Ω, then $F(\alpha,\beta,\gamma):=\alpha\oplus\beta\oplus\gamma : A\oplus B\oplus C\to(A'\oplus B'\oplus C')$. Denote by ${}_\Gamma(A\oplus B\oplus C)$ the left Γ-module associated to $(A,B,C;f,g)$. In Section 2, corresponding to submodules and quotient modules of ${}_\Gamma(A\oplus B\oplus C)$, we describe subobjects and quotient objects of Ω. We give some equivalent conditions for a submodule ${}_\Gamma(A'\oplus B'\oplus C')$ of ${}_\Gamma(A\oplus B\oplus C)$ to be essential (resp. small) in ${}_\Gamma(A\oplus B\oplus C)$. We then use these conditions to characterize uniform and hollow modules over Γ respectively. In Section 3, the radical as well as the socle of ${}_\Gamma(A\oplus B\oplus C)$ is determined. Using the description of the socle, we find necessary and sufficient conditions for Γ-modules to be finitely embedded.

2. Uniform and hollow over Γ respectively

We explain the notations adopted in this paper firstly. Let $(A,B,C;f,g)\in \mathrm{Obj}\Omega$ and ${}_\Gamma(A\oplus B\oplus C)$ be the associated left Γ-module. We will describe subobjects (resp. quotient objects) of $(A,B,C;f,g)$ which correspond to submodules (resp. quotient modules) of ${}_\Gamma(A\oplus B\oplus C)$. Assume ${}_\Gamma(A'\oplus B'\oplus C')\leq_\Gamma (A\oplus B\oplus C)$, where ${}_TA'\leq_T A$, ${}_UB'\leq_U B$ and ${}_VC'\leq_V C$ satisfying $f(M\otimes_T A')\leq B'$ and $g(N\otimes_U B')\leq C'$. Denoting the inclusions $A'\hookrightarrow A$, $B'\hookrightarrow B$, $C'\hookrightarrow C$ by i,j,k, respectively, and writing $f'=f\circ(1_M\otimes i)$, $g'=g\circ(1_N\otimes j)$, we have the following two commutative diagrams

$$\begin{tikzcd} M \otimes_T A' \arrow[r, "1_M \otimes i"] \arrow[d, "f'"] & M \otimes_T A \arrow[d, "f"] & N \otimes_U B' \arrow[r, "1_N \otimes j"] \arrow[d, "g'"] & N \otimes_U B \arrow[d, "g"] \\ B' \arrow[r, "j"] & B, & C' \arrow[r, "k"] & C. \end{tikzcd}$$

Thus, we see that $(A', B', C'; f', g') \in \mathrm{Obj}\Omega$ and $(i, j, k) : (A', B', C'; f', g') \to (A, B, C; f, g)$ is a map in Ω realizing ${}_\Gamma(A' \oplus B' \oplus C')$ as a Γ-submodule of ${}_\Gamma(A \oplus B \oplus C)$. Also it is clear that every Γ-submodule of ${}_\Gamma(A \oplus B \oplus C)$ is obtained in this way. Let ${}_TA''$, ${}_UB''$ and ${}_VC''$ be quotient modules of ${}_TA$, ${}_UB$ and ${}_VC$ respectively with $\eta_1 : A \to A''$, $\eta_2 : B \to B''$ and $\eta_3 : C \to C''$ the canonical quotient maps. Let $A' = \ker\eta_1$, $B' = \ker\eta_2$ and $C' = \ker\eta_3$. Suppose $f(M \otimes_T A') \leq B'$ and $g(N \otimes_U B') \leq C'$. Denote the inclusions $A' \to A$, $B' \to B$ and $C' \to C$ by j_1, j_2 and j_3, respectively. We get two maps $f'' : M \otimes_T A'' \to B''$ and $g'' : N \otimes_U B'' \to C''$ rendering the following two diagrams commutative

$$\begin{tikzcd} M \otimes_T A' \arrow[r, "1_M \otimes j_1"] \arrow[d, "f'"] & M \otimes_T A \arrow[r, "1_M \otimes \eta_1"] \arrow[d, "f"] & M \otimes A'' \arrow[r] \arrow[d, "f''"] & 0 \\ B' \arrow[r, "j_2"] & B \arrow[r, "\eta_2"] & B'' \arrow[r] & 0, \end{tikzcd}$$

$$\begin{tikzcd} N \otimes_U B' \arrow[r, "1_N \otimes \mathrm{j}_2"] \arrow[d, "g'"] & N \otimes_U B \arrow[r, "1_N \otimes \eta_2"] \arrow[d, "g"] & N \otimes_U B'' \arrow[r] \arrow[d, "g''"] & 0 \\ C' \arrow[r, "j_3"] & C \arrow[r, "\eta_3"] & C'' \arrow[r] & 0. \end{tikzcd}$$

In the two diagrams $f' = f \circ (1_M \otimes j_1)$, $g' = g \circ (1_N \otimes j_2)$ and the rows are exact. It is clear that $(\eta_1, \eta_2, \eta_3) : (A, B, C; f, g) \to (A'', B'', C''; f'', g'')$ is a map in Ω realizing ${}_\Gamma(A'' \oplus B'' \oplus C'')$ as a quotient of ${}_\Gamma(A \oplus B \oplus C)$. When we talk of a submodule ${}_\Gamma(A' \oplus B' \oplus C')$ of ${}_\Gamma(A \oplus B \oplus C)$, we have ${}_TA' \leq_T A$, ${}_UB' \leq_U B$, ${}_VC' \leq_V C$, $f \circ (1_M \otimes j_1)(M \otimes_T A') \leq B'$, $g \circ (1_N \otimes j_2)(N \otimes_U B') \leq C'$, and the maps $f' : M \otimes_T A' \to B'$ and $g' : N \otimes_U B' \to C'$ are completely determined, respectively. They have to be $f \circ (1_M \otimes j_1)$ and $g \circ (1_N \otimes j_2)$, respectively. Similarly, when we deal with a quotient ${}_\Gamma(A'' \oplus B'' \oplus C'')$ of ${}_\Gamma(A \oplus B \oplus C)$, the two maps $f'' : M \otimes_T A'' \to B''$ and $g'' : N \otimes_U B'' \to C''$ are completely determined. Because of these facts we will not specifically mention the maps f', g', f'' and g'' in these situations.

Let $({}_TA, {}_UB, {}_VC; f, g) \in \mathrm{Obj}\Omega$, $L = \{a \in A | f(m \otimes a) = 0$, for all $m \in M\}$ and $Q = \{b \in B | g(n \otimes b) = 0$, for all $n \in N\}$. Then clearly ${}_TL \leq_T A$, ${}_UQ \leq_U B$ and ${}_\Gamma(L \oplus Q \oplus 0) \leq_\Gamma (A \oplus B \oplus C)$.

Theorem 2.1 *${}_\Gamma(A' \oplus B' \oplus C')$ is essential in ${}_\Gamma(A \oplus B \oplus C)$ if and only if ${}_T(A' \cap L)$, ${}_U(B' \cap Q)$ and ${}_VC'$ are essential in ${}_TL$, ${}_UQ$ and ${}_VC$, respectively.*

Proof Assume that ${}_\Gamma(A' \oplus B' \oplus C')$ is essential in ${}_\Gamma(A \oplus B \oplus C)$.

Let $0 \neq c \in C$. Then $(0,0,c)$ is non-zero in ${}_{\Gamma}(A \oplus B \oplus C)$. We can find an element

$$\begin{pmatrix} t & 0 & 0 \\ m & u & 0 \\ \overline{n} \otimes \overline{m} & n & v \end{pmatrix} \in \Gamma$$

with

$$(0,0,0) \neq \begin{pmatrix} t & 0 & 0 \\ m & u & 0 \\ \overline{n} \otimes \overline{m} & n & v \end{pmatrix} (0,0,c) \in_{\Gamma} (A' \oplus B' \oplus C').$$

But

$$\begin{pmatrix} t & 0 & 0 \\ m & u & 0 \\ \overline{n} \otimes \overline{m} & n & v \end{pmatrix} (0,0,c) = (0,0,vc).$$

Thus $0 \neq vc \in C'$. This proves that ${}_{V}C'$ is essential in ${}_{V}C$.

Let $0 \neq a \in_{T} L$. Then $(a,0,0) \neq (0,0,0) \in_{\Gamma} (A \oplus B \oplus C)$. We can find an element

$$\begin{pmatrix} t & 0 & 0 \\ m & u & 0 \\ \overline{n} \otimes \overline{m} & n & v \end{pmatrix} \in \Gamma$$

with

$$(0,0,0) \neq \begin{pmatrix} t & 0 & 0 \\ m & u & 0 \\ \overline{n} \otimes \overline{m} & n & v \end{pmatrix} (a,0,0) \in_{\Gamma} (A' \oplus B' \oplus C').$$

But

$$\begin{pmatrix} t & 0 & 0 \\ m & u & 0 \\ \overline{n} \otimes \overline{m} & n & v \end{pmatrix} (a,0,0) = (ta,0,0) \ \ (\text{since} \ \ f(m \otimes a) = 0).$$

Thus $0 \neq ta \in A'$. Since $ta \in_{T} L$, we see $0 \neq ta \in A' \cap L$, showing that ${}_{T}(A' \cap L)$ is essential in ${}_{T}L$.

Similarly, let $0 \neq b \in_{U} Q$. Then $(0,0,0) \neq (0,b,0) \in_{\Gamma} (A \oplus B \oplus C)$. We can find an element

$$\begin{pmatrix} t & 0 & 0 \\ m & u & 0 \\ \overline{n} \otimes \overline{m} & n & v \end{pmatrix} \in \Gamma$$

with

$$(0,0,0) \neq \begin{pmatrix} t & 0 & 0 \\ m & u & 0 \\ \overline{n} \otimes \overline{m} & n & v \end{pmatrix} (0,b,0) \in_{\Gamma} (A' \oplus B' \oplus C').$$

But

$$\begin{pmatrix} t & 0 & 0 \\ m & u & 0 \\ \overline{n} \otimes \overline{m} & n & v \end{pmatrix} (0,b,0) = (0,ub,0) \ \ (\text{since} \ \ g(n \otimes b) = 0).$$

Thus, $0 \neq ub \in B'$. Since $ub \in_U Q$, we see that $0 \neq ub \in B' \cap Q$, showing that ${}_U(B' \cap Q)$ is essential in ${}_UQ$.

Conversely, assume that ${}_T(A' \cap L)$, ${}_U(B' \cap Q)$ and ${}_VC'$ are essential in ${}_TL$, ${}_UQ$, and ${}_VC$, respectively. We need to prove that ${}_\Gamma(A' \oplus B' \oplus C')$ is essential in ${}_\Gamma(A \oplus B \oplus C)$. In order to do so, let $(0,0,0) \neq (a,b,c) \in_\Gamma (A \oplus B \oplus C)$. We divide the proof into the following cases.

(1) $c \neq 0$;

(2) $c = 0,\ b \neq 0$ and $b \notin Q$;

(3) $c = 0,\ b \neq 0$ and $b \in Q$;

(4) $c = 0,\ b = 0,\ a \neq 0$ and $a \notin L$;

(5) $c = 0,\ b = 0,\ a \neq 0$ and $a \in L$.

In Case (1), we can find $0 \neq vc \in C'$. It follows that

$$0 \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & v \end{pmatrix} (0,0,c) = (0,0,vc) \in_\Gamma (A' \oplus B' \oplus C').$$

In Case (2), we can find an $n \in N$ with $0 \neq g(n \otimes b) \in C$. Hence there exists a $v \in V$ with $0 \neq vg(n \otimes b) \in C'$. Thus

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & v \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & n & 0 \end{pmatrix} (0,b,0) = (0,0,vg(n \otimes b))$$

is a non-zero element of ${}_\Gamma(A' \oplus B' \oplus C')$.

In Case (3), we have $g(n \otimes b) = 0$ for all $n \in N$. Since ${}_U(B' \cap Q)$ is essential in ${}_UQ$, we can find an element $u \in U$ with $0 \neq ub \in_U (B' \cap Q)$. Thus

$$0 \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 0 \end{pmatrix} (0,b,0) = (0,ub,0) \in_{\Gamma_3} (A' \oplus B' \oplus C').$$

In Case (4), we can find an $m \in M$ with $0 \neq f(m \otimes a) \in B$. On the one hand, if $f(m \otimes a) \overline{\in} Q$, we can find an $n \in N$ with $0 \neq g(n \otimes f(m \otimes a)) \in C$. Hence there exists $v \in V$ with $0 \neq vg(n \otimes f(m \otimes a)) \in C'$. Thus

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & v \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ n \otimes m & 0 & 0 \end{pmatrix} (a,0,0) = (0,0,vg(n \otimes f(m \otimes a)))$$

is a non-zero element of ${}_\Gamma(A' \oplus B' \oplus C')$. On the other hand, if $f(m \otimes a) \in Q$, we have $g(n \otimes f(m \otimes a)) = 0$ for all $n \in N$. Since ${}_U(B' \cap Q)$ is essential in ${}_UQ$, we can find an element $u \in U$ with $0 \neq uf(m \otimes a) \in_U (B' \cap Q)$. In this case

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (a,0,0) = (0,uf(m \otimes a),0)$$

and $(0,0,0) \neq (0, uf(m \otimes a), 0) \in_\Gamma (A' \oplus B' \oplus C')$.

In Case (5), we have $f(m \otimes a) = 0$ for all $m \in M$. Since $A' \cap L$ is essential in ${}_TL$, we can find an element $t \in T$ with $0 \neq ta \in A' \cap L$. In this case

$$\begin{pmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (a, 0, 0) = (ta, 0, 0)$$

and $(0,0,0) \neq (ta, 0, 0) \in {}_\Gamma(A' \oplus B' \oplus C')$. This proves that ${}_\Gamma(A' \oplus B' \oplus C')$ is essential in ${}_\Gamma(A \oplus B \oplus C)$. $\square$

Definition 2.1[2] *A module is said to be uniform if and only if every nonzero submodule is essential.*

Theorem 2.1 enables us to obtain the following.

Corollary 2.1 *${}_{\Gamma_3}(A \oplus B \oplus C)$ is uniform if and only if one and only one of the following three conditions holds.*

(a) ${}_VC = 0$ and ${}_TL = {}_T A$, ${}_UQ = {}_U B$ are uniform.

(b) ${}_TL = 0$ and ${}_VC$, ${}_UQ = {}_U B$ are uniform.

(c) ${}_UQ = 0$ and ${}_VC$, ${}_TL = {}_T A$ are uniform.

If $V \in \mathrm{mod}R$ and $W \leq_R V$ we write $W \ll V$ to indicate that W is a small (or superfluous) submodule of V. The following proposition gives a necessary and sufficient condition for ${}_{\Gamma_3}(A' \oplus B' \oplus C')$ to be small in ${}_{\Gamma_3}(A \oplus B \oplus C)$.

Theorem 2.2 *${}_{\Gamma_3}(A' \oplus B' \oplus C')$ is small in ${}_{\Gamma_3}(A \oplus B \oplus C)$ if and only if ${}_TA'$, $\beta(B')$ and $\delta(C')$ are small in ${}_TA$, $B/f(M \otimes_T A)$ and $C/g(N \otimes_U B)$, respectively, where $\beta : B \to B/f(M \otimes_T A)$ and $\delta : C \to C/g(N \otimes_U B)$ are the canonical quotient maps.*

Proof Assume that ${}_\Gamma(A' \oplus B' \oplus C') \ll {}_\Gamma(A \oplus B \oplus C)$.

Let ${}_TH \leq {}_TA$ which satisfies $A' + H = A$. If $\mu : H \hookrightarrow A$ denotes the inclusion map, then with $f' = f \circ (1_M \otimes \mu)$ we have $(H, B, C; f', g)$ giving rise to a Γ-submodule ${}_\Gamma(H \oplus B \oplus C)$ of ${}_\Gamma(A \oplus B \oplus C)$ which satisfies ${}_\Gamma(A' \oplus B' \oplus C') +_\Gamma (H \oplus B \oplus C) = {}_\Gamma(A \oplus B \oplus C)$. The assumption ${}_\Gamma(A' \oplus B' \oplus C') \ll {}_\Gamma(A \oplus B \oplus C)$ yields ${}_\Gamma(H \oplus B \oplus C) = {}_\Gamma(A \oplus B \oplus C)$, hence $H = A$. Thus $A' + H = A$ implies $H = A$. This proves that $A' \ll A$.

Let ${}_UE \leq_U (B/f(M \otimes_T A))$ satisfy $\beta(B') + E = \beta(B) = B/f(M \otimes_T A)$ and $D = \beta^{-1}(E)$. Then $f(M \otimes_T A) \subseteq D$. Let $v : D \hookrightarrow B$ be the inclusion map. Then with $g' = g \circ (1_N \otimes v)$ we have $(A, D, C; f, g')$ giving rise to a $\Gamma-$submodule ${}_\Gamma(A \oplus D \oplus C)$ of ${}_\Gamma(A \oplus B \oplus C)$. From $\beta(B') + E = \beta(B)$, we get $B' + D = B$. Hence ${}_\Gamma(A' \oplus B' \oplus C') + {}_\Gamma(A \oplus D \oplus C) = {}_\Gamma(A \oplus B \oplus C)$. The assumption ${}_\Gamma(A' \oplus B' \oplus C') \ll {}_\Gamma(A \oplus B \oplus C)$ yields $D = B$. Since $\beta^{-1}(E) = B$, we get $E = \beta(B) = B/f(M \otimes_T A)$. Thus $\beta(B') + E = \beta(B)$ implies $E = \beta(B)$. This gives $\beta(B') \ll \beta(B)$.

Let ${}_VF \leq_V (C/g(N \otimes_U B))$ satisfy $\delta(C') + F = \delta(C) = C/g(N \otimes_U B)$ and $K = \delta^{-1}(F)$. Then $g(N \otimes_U B) \subseteq K$. Hence $(A, B, K; f, g)$ gives rise to a Γ-submodule ${}_\Gamma(A \oplus B \oplus K)$ of ${}_\Gamma(A \oplus B \oplus C)$. From $\delta(C') + F = \delta(C)$ we get $C' + K = C$. Hence ${}_\Gamma(A' \oplus B' \oplus C') + {}_\Gamma(A \oplus B \oplus K) = {}_\Gamma(A \oplus B \oplus C)$.

The assumption ${}_\Gamma(A' \oplus B' \oplus C') \ll {}_\Gamma(A \oplus B \oplus C)$ yields $K = C$. Since $\delta^{-1}(F) = C$ we get $F = \delta(C) = C/g(N \otimes_U B)$. Thus $\delta(C') + F = \delta(C)$ implies $F = \delta(C)$. This proves $\delta(C') \ll \delta(C)$.

Conversely, assume that $A' \ll A$, $\beta(B') \ll \beta(B)$ and $\delta(C') \ll \delta(C)$.

Let ${}_TX \leq_T A$, ${}_UY \leq_U B$ and ${}_VZ \leq_V C$ satisfy the conditions $f \circ (1_M \otimes \mu)(M \otimes_T X) \subseteq Y$ and $g \circ (1_N \otimes \upsilon)(N \otimes_U Y) \subseteq Z$ where $\mu : X \hookrightarrow A$ and $\upsilon : Y \hookrightarrow B$ denote the inclusions. Write $f' = f \circ (1_M \otimes \mu)$ and $g' = g \circ (1_N \otimes \upsilon)$. Suppose $(X, Y, Z; f', g')$ satisfies ${}_\Gamma(A' \oplus B' \oplus C') + {}_\Gamma(X \oplus Y \oplus Z) = {}_\Gamma(A \oplus B \oplus C)$. Then $A' + X = A$, $B' + Y = B$ and $C' + Z = C$. From $A' \ll A$ we get $X = A$. Hence $\mu = 1_A, f((M \otimes_T A)) \subseteq Y$. From $B' + Y = B$ we get $\beta(B') + \beta(Y) = \beta(B),$ *and* the hypothesis $\beta(B') \ll \beta(B)$ yields $\beta(Y) = \beta(B)$. Since $f(M \otimes_T A) \subseteq Y$, we get $Y = B$. Thus $\upsilon = 1_B$ and $g(N \otimes_U B) \subseteq Z$. From $C' + Z = C$ we get $\delta(C') + \delta(Z) = \delta(C)$, the hypothesis $\delta(C') \ll \delta(C)$ yields $\delta(Z) = \delta(C)$. Since $g(N \otimes_U B) \subseteq Z$, we get $Z = C$. Thus ${}_\Gamma(A' \oplus B' \oplus C') + {}_\Gamma(X \oplus Y \oplus Z) = {}_\Gamma(A \oplus B \oplus C)$ implies $X = A, Y = B$ and $Z = C$. This proves that ${}_\Gamma(A' \oplus B' \oplus C') \ll {}_\Gamma(A \oplus B \oplus C)$. □

Definition 2.2[2] *A module is said to be hollow if and only if it is nonzero and in it every proper submodule is small.*

From Theorem 2.2 and Definition 2.2 we get the following.

Corollary 2.2 *The left Γ-module ${}_\Gamma(A \oplus B \oplus C)$ determined by $(A, B, C;, f, g)$ is hollow if and only if exact one of the following three conditions holds.*

(a) ${}_TA$ *is hollow and* $B = f(M \otimes_T A)$, $C = g(N \otimes_U B)$.

(b) $A = 0$, B *is hollow and* $C = g(N \otimes_U B)$.

(c) $A = 0, B = 0$ *and* ${}_VC$ *is hollow.*

Proof Assume ${}_\Gamma(A \oplus B \oplus C)$ is hollow.

Suppose $A \neq 0$. Then there exists ${}_TA' \leq {}_TA$ with ${}_TA' \neq {}_TA$. If $i : A' \hookrightarrow A$ denotes the inclusion and $f' = f \circ (1_M \otimes i)$, the submodule ${}_\Gamma(A' \oplus B \oplus C)$ corresponding to $(A', B, C; f', g)$ is a proper submodule of ${}_\Gamma(A \oplus B \oplus C)$. Thus ${}_\Gamma(A' \oplus B \oplus C) \ll {}_\Gamma(A \oplus B \oplus C)$. By Theorem 2.2, ${}_TA' \ll {}_TA$, $\beta(B) \ll \beta(B)$ and $\delta(C) \ll \delta(C)$. But both $\beta(B)$ and $\delta(C)$ are zero, i.e., $B = f(M \otimes_T A)$ and $C = g(N \otimes_U B)$. Thus if $A \neq 0$, ${}_TA$ is hollow, $B = f(M \otimes_T A)$ and $C = g(N \otimes_U B)$. Therefore, (a) is true when $A \neq 0$.

In Case $A = 0$ and $B \neq 0$, ${}_\Gamma(A \oplus B \oplus C)$ is hollow by assumption. For any ${}_UB' \leq {}_UB$ with ${}_UB' \neq {}_UB$. ${}_\Gamma(0 \oplus B' \oplus C)$ is a proper submodule of ${}_\Gamma(0 \oplus B \oplus C)$. From Theorem 2.2, we get $\beta(B') \ll \beta(B)$ and $\delta(C') \ll \delta(C)$. Since $A = 0$ with $f : M \otimes_T A \to B$, $f = 0$. Since $\delta : B \to B/f(M \otimes_T A)$, we get $\delta = 1_B$. Thus $B' \ll B$ and ${}_UB$ is hollow. Also $\delta(C) \ll \delta(C)$ means $\delta(C) = 0$, i.e., $C = g(N \otimes_U B)$. Thus if $A = 0$ and $B \neq 0$, we get ${}_UB$ is hollow and $C = g(N \otimes_U B)$. (b) is true in this case.

The last case is $A = 0$ and $B = 0$. ${}_\Gamma(A \oplus B \oplus C)$ is hollow by assumption. For any ${}_VC' \leq_V C$ with ${}_VC' \neq {}_VC$, ${}_\Gamma(0 \oplus 0 \oplus C')$ is a proper submodule of ${}_\Gamma(0 \oplus 0 \oplus C)$. From Theorem 2.2, we get ${}_VC' \ll {}_VC$. This means (c) is true.

Conversely assume (a) or (b) or (c) is valid.

In Case (a), if ${}_{\Gamma}(A' \oplus B' \oplus C')$ is a proper submodule of ${}_{\Gamma}(A \oplus B \oplus C)$ we should necessarily have $A' \subset A$. Otherwise, assume $A' = A$. Then $B' \supseteq f(M \otimes_T A') = f(M \otimes_T A) = B$ which implies $B = B'$ and $C' \supseteq g(N \otimes_U B') = g(N \otimes_U B) = C$ which implies $C = C'$. Hence ${}_{\Gamma}(A' \oplus B' \oplus C') = {}_{\Gamma}(A \oplus B \oplus C)$ which is a contradiction. Now we need only to prove that ${}_{\Gamma}(A' \oplus B' \oplus C') \ll {}_{\Gamma}(A \oplus B \oplus C)$. But it follows from Theorem 2.2 immediately since $A' \ll A$, $\beta(B) = 0$ and $\delta(C) = 0$.

In Case (b), for any proper submodule ${}_{\Gamma}(0 \oplus B' \oplus C')$ of ${}_{\Gamma}(0 \oplus B \oplus C)$ with $B' \subset B$. Otherwise, assume $B' = B$. Then $C' \supseteq g(N \otimes_U B') = g(N \otimes_U B) = C$ implies $C = C'$. Which is contradict to ${}_{\Gamma}(0 \oplus B' \oplus C') \neq {}_{\Gamma}(0 \oplus B \oplus C)$. Thus we need only to prove that ${}_{\Gamma}(0 \oplus B' \oplus C') \ll {}_{\Gamma}(0 \oplus B \oplus C)$. This follows from Theorem 2.2 immediately since $B' \ll B$ and $C = g(N \otimes_U B)$.

In Case (c), any proper submodule of ${}_{\Gamma}(0 \oplus 0 \oplus C)$ is of the form ${}_{\Gamma}(0 \oplus 0 \oplus C')$ with $C' \subset C$. From (c), $C' \ll C$. By Theorem 2.2 ${}_{\Gamma}(0 \oplus 0 \oplus C') \ll {}_{\Gamma}(0 \oplus 0 \oplus C)$ is gotten immediately. $\square$

We now construct an example to show that ${}_{\Gamma}(A \oplus B \oplus C)$ can be hollow even when ${}_U B$ and ${}_V C$ have infinite dual Goldie dimensions in the sense of [3].

Example 2.1 Let K be a field. Suppose $T = K, U = K, V = K$ and M, N are infinite dimensional vector spaces over K regarded as $(K - K)$ bimodules in the usual way. Let

$$\begin{pmatrix} T & 0 & 0 \\ M & U & 0 \\ N \otimes_K M & N & V \end{pmatrix} = \begin{pmatrix} K & 0 & 0 \\ M & K & 0 \\ N \otimes_K M & N & K \end{pmatrix}.$$

Let $A = K, B = M$ and $C = N \otimes_K M$. Define $f : M \otimes_T A \to B$ by $f(m \otimes a) = ma$ and $g : N \otimes_U B \to C$ by $g(n \otimes m) = n \otimes m$. Then from Theorem 2.2 we see that ${}_{\Gamma}(A \oplus B \oplus C)$ corresponding to $(A, B, C; f, g)$ is hollow, since ${}_T A$ is hollow and $B = f(M \otimes_T A)$, $C = g(N \otimes_U B)$. However the dual Goldie dimensions of ${}_U B$ and ${}_V C$ are infinite.

3. Determination of Rad ${}_{\Gamma}(A \oplus B \oplus C)$ and Soc ${}_{\Gamma}(A \oplus B \oplus C)$

In this section we will determine the radical $\mathrm{Rad}_{\Gamma}(A \oplus B \oplus C)$ and $\mathrm{Soc}_{\Gamma}(A \oplus B \oplus C)$ of ${}_{\Gamma}(A \oplus B \oplus C)$. For this purpose we will describe the maximal (resp. simple) submodules of ${}_{\Gamma}(A \oplus B \oplus C)$ firstly. Let ${}_T L = \{a \in A | f(m \otimes a) = 0 \text{ for all } m \in M\}$ and ${}_U Q = \{b \in B | g(n \otimes b) = 0 \text{ for all } n \in N\}$.

Theorem 3.1 *Let $\Im_1 = \{{}_{\Gamma}(A' \oplus B \oplus C) | A'$ a maximal submodule of $A\}$, $\Im_2 = \{{}_{\Gamma}(A \oplus B' \oplus C) | B'$ a maximal submodule of B with $f(M \otimes_T A) \subseteq B'\}$ and $\Im_3 = \{{}_{\Gamma}(A \oplus B \oplus C') | C'$ a maximal submodule of C with $g(N \otimes_U B) \subseteq C'\}$. Let $\Psi_1 = \{(L' \oplus 0 \oplus 0) | L'$ a simple submodule of $L\}$, $\Psi_2 = \{(0 \oplus Q' \oplus 0) | Q'$ a simple submodule of $Q\}$ and $\Psi_3 = \{(0 \oplus 0 \oplus C') | C'$ a simple submodule of $C\}$. Then*

(a) *The family $\Im$ of maximal submodules of ${}_{\Gamma}(A \oplus B \oplus C)$ is precisely $\Im = \Im_1 \cup \Im_2 \cup \Im_3$.*

(b) *The family Ψ of minimal submodules of ${}_{\Gamma}(A \oplus B \oplus C)$ is precisely $\Psi = \Psi_1 \cup \Psi_2 \cup \Psi_3$.*

Proof Let ${}_{\Gamma}(A' \oplus B' \oplus C')$ be any maximal submodule of ${}_{\Gamma}(A \oplus B \oplus C)$. If $A' \subset A$, since

$(A' \oplus B' \oplus C') \subseteq (A' \oplus B \oplus C) \subset (A \oplus B \oplus C)$, we conclude $B' = B$ and $C' = C$. Also from $(A' \oplus B \oplus C) \subseteq (A'' \oplus B \oplus C) \subset (A \oplus B \oplus C)$ for any $A' \subseteq A'' \subset A$ we see that $A' = A''$. Hence A' is a maximal submodule of A, which shows that $(A' \oplus B' \oplus C') = (A' \oplus B \oplus C)$ is in $\Im_1$. Now suppose $A' = A$. Then $f(M \otimes_T A) \subseteq B'$. On the one hand, if $B' \neq B$, since $(A' \oplus B' \oplus C') \subseteq (A \oplus B' \oplus C) \subset (A \oplus B \oplus C)$, we have $C = C'$. Also from $(A \oplus B' \oplus C) \subseteq (A \oplus B'' \oplus C) \subset (A \oplus B \oplus C)$ for any $B' \subseteq B'' \subset B$ we see that $B' = B''$ immediately. Hence B' is a maximal submodule of B. Thus $(A' \oplus B' \oplus C') = (A \oplus B' \oplus C)$ is in $\Im_2$. On the other hand, if $B' = B$, then $g(N \otimes_U B) \subseteq C'$. Since ${}_\Gamma(A' \oplus B' \oplus C') = {}_\Gamma (A \oplus B \oplus C')$ is a maximal submodule of ${}_\Gamma(A \oplus B \oplus C)$, we see that C' is a maximal submodule of C immediately. Thus $(A' \oplus B' \oplus C') = (A \oplus B \oplus C')$ is in $\Im_3$.

Conversely, it is straightforward to see that any submodule of ${}_\Gamma(A \oplus B \oplus C)$ belonging to $\Im_1 \cup \Im_2 \cup \Im_3$ is a maximal submodule of ${}_\Gamma(A \oplus B \oplus C)$. This proves (a).

Let ${}_\Gamma(A' \oplus B' \oplus C')$ be any simple submodule of ${}_\Gamma(A \oplus B \oplus C)$. If $C' \neq 0$, for any $0 \neq C'' \subseteq C'$, since $(0 \oplus 0 \oplus C'') \subseteq (A' \oplus B' \oplus C') \subset (A \oplus B \oplus C)$, we see that $A' = 0$, $B' = 0$ and C' is a simple submodule of C. If $C' = 0$, we have $g \circ (1_N \otimes j)(N \otimes_U B) = 0$, hence $j(B') \subseteq_U Q$ where $j : B' \hookrightarrow B$ denotes the inclusion. Thus $B' \leq_U Q$. If $B' \neq 0$, for any $0 \neq B'' \subseteq B'$, since $(0 \oplus B'' \oplus 0) \subseteq (A' \oplus B' \oplus 0) \subset (A \oplus B \oplus C)$, we see that $A' = 0$. Thus B' is a simple submodule of B. If $B' = 0$, we have $f \circ (1_M \otimes i)(M \otimes_T A) = 0$ and thus $i(A') \subseteq_T L$ where $i : A' \hookrightarrow A$ denotes the inclusion. Thus $A' \leq_T L$. If $0 \neq A'' \subseteq A' \subset_T L$, we have $(A'' \oplus 0 \oplus 0) \subset (A' \oplus 0 \oplus 0)$. Hence A' is a simple submodule of ${}_T L$. Therefore, $\Psi \subseteq \Psi_1 \cup \Psi_2 \cup \Psi_3$.

For (c), it is straightforward to see that any submodule of ${}_\Gamma(A \oplus B \oplus C)$ belonging to $\Psi_1 \cup \Psi_2 \cup \Psi_3$ is a simple submodule of ${}_\Gamma(A \oplus B \oplus C)$. This proves (c). □

Corollary 3.1 *Let $\beta : B \to B/f(M \otimes_T A)$ and $\delta : C \to C/g(N \otimes_U B)$ denote the canonical quotient maps. Then*

(a) $\mathrm{Rad}(A \oplus B \oplus C) = \mathrm{Rad}A \oplus \beta^{-1}(\mathrm{Rad}B/f(M \otimes_T A)) \oplus \delta^{-1}(\mathrm{Rad}C/g(N \otimes_U B))$.

(b) $\mathrm{Soc}_\Gamma(A \oplus B \oplus C) = {}_\Gamma (\mathrm{Soc}_T L \oplus \mathrm{Soc}_U Q \oplus \mathrm{Soc}_V C)$.

Proof These are immediate consequences of Theorem 3.1. □

Definition 3.1[4,5] *A module V is said to be finitely embedded (or finitely co-generated) if* $\mathrm{Soc}V$ *is finitely generated and essential in V.*

Theorem 3.1 enables us to obtain the following.

Theorem 3.2 *(1)* $\mathrm{Soc}_\Gamma(A \oplus B \oplus C)$ *is finitely generated if and only if* $\mathrm{Soc}_T L$, $\mathrm{Soc}_U Q$ *and* $\mathrm{Soc}_V C$ *are finitely generated.*

(2) $\mathrm{Soc}_\Gamma(A \oplus B \oplus C)$ *is essential in* ${}_\Gamma(A \oplus B \oplus C)$ *if and only if* $\mathrm{Soc}_T L$, $\mathrm{Soc}_U Q$ *and* $\mathrm{Soc}_V C$ *are essential in* ${}_T L$, ${}_U Q$ *and* ${}_V C$ *respectively.*

(3) ${}_\Gamma(A \oplus B \oplus C)$ *is finitely embedded if and only if* ${}_T L$, ${}_U Q$ *and* ${}_V C$ *are finitely embedded.*

Proof From Corollary 3.1 (b) we have $\mathrm{Soc}_\Gamma(A \oplus B \oplus C) = {}_\Gamma (\mathrm{Soc}_T L \oplus \mathrm{Soc}_U Q \oplus \mathrm{Soc}_V C)$. Note that ${}_\Gamma(\mathrm{Soc}_T L \oplus \oplus \mathrm{Soc}_U Q \oplus \mathrm{Soc}_V C)$ corresponds to the quintuple $(\mathrm{Soc}_T L, \mathrm{Soc}_U Q, \mathrm{Soc}_V C; 0, 0)$. From a well-known result (Exercise $1D(b)$ on p.7 of [6]) we see that $\mathrm{Soc}_\Gamma(A \oplus B \oplus C)$ is finitely generated $\Leftrightarrow$ $\mathrm{Soc}_T L$, $\mathrm{Soc}_U Q/(M \otimes \mathrm{Soc}_T L)$, $\mathrm{Soc}_V C/(N \otimes \mathrm{Soc}_U Q)$ are finitely generated. This proves (1). (2) is an immediate consequence of Theorem 2.1. (3) is immediate from (1) and (2).

Often one is interested in finding conditions implying $\mathrm{Rad}(V) \ll V$. In this connection, we have the following.

Theorem 3.3 $\mathrm{Rad}_\Gamma(A \oplus B \oplus C)$ *is small in* ${}_\Gamma(A \oplus B \oplus C)$ *if and only if* $\mathrm{Rad}_T A$, $\mathrm{Rad}_U(B/f(M \otimes_T A))$ *and* $\mathrm{Rad}_V(C/g(N \otimes_U B))$ *are small in* ${}_T A$, ${}_U(B/f(M \otimes_T A))$ *and* ${}_V(C/g(N \otimes_U B))$, *respectively.*

Proof This is an immediate consequence of Corollary 3.1(a) and Theorem 2.2. □

Definition 3.2[7] *Let R be a ring and $V \in \mathrm{Mod}R$. V is said to be co-hopfian if every injective endomorphism $f : V \to V$ is automatically an isomorphism.*

In [7] we know that any quasi-injective finitely embedded module V is co-hopfian. Using (3) of Theorem 3.2 which characterizes finitely embedded modules over 3×3 formal triangular matrix rings, we will construct an example of a finitely embedded module which is not co-hopfian, thus showing that quasi-injectivity of V cannot be dispensed with for the validity of the above result.

Example 3.1 Let

$$\Gamma = \begin{pmatrix} Z & 0 & 0 \\ Z_{p^\infty} & Z & 0 \\ Z_{p^\infty} & Z_{p^\infty} & Z \end{pmatrix},$$

where p is a prime. Consider the Γ-module ${}_\Gamma(Z \oplus Z_{p^\infty} \oplus Z_{p^\infty})$ associated to the quintuple $(Z, Z_{p^\infty}, Z_{p^\infty}; f, g)$ where $f : Z_{p^\infty} \otimes Z \to Z_{p^\infty}$ by $x \otimes k \to xk$ and $g : Z_{p^\infty} \otimes Z_{p^\infty} \to Z_{p^\infty}$ by $x \otimes y \to xy$ for all $k \in Z$ and $x, y \in Z_{p^\infty}$. In this case Z_{p^∞} is finitely embedded in Mod Z. Also $L = \{k \in Z \mid xk = 0 \text{ for all } x \in Z_{p^\infty}\}$ is finitely embedded in $\mathrm{Mod}Z$, $Q = \{x \in Z_{p^\infty} \mid yx = 0 \text{ for all } y \in Z_{p^\infty}\}$ is finitely embedded in $\mathrm{Mod}Z$. From (3) of Theorem 3.2 we see that ${}_\Gamma(Z \oplus Z_{p^\infty} \oplus Z_{p^\infty})$ is finitely embedded. Let n be an integer ≥ 2 and relatively prime to p. Let $\sigma_1 : Z \to Z$, $\sigma_2 : Z_{p^\infty} \to Z_{p^\infty}$, $\sigma_3 : Z_{p^\infty} \to Z_{p^\infty}$ be given by multiplication by n. Clearly

$$\begin{tikzcd} Z_{p^\infty} \otimes Z \arrow[r, "1_{Z_{p^\infty}} \otimes \sigma_1"] \arrow[d, "f"] & Z_{p^\infty} \otimes Z \arrow[d, "f"] \\ Z_{p^\infty} \arrow[r, "\sigma_2"] & Z_{p^\infty}, \end{tikzcd} \qquad \begin{tikzcd} Z_{p^\infty} \otimes Z_{p^\infty} \arrow[r, "1_{Z_{p^\infty}} \otimes \sigma_2"] \arrow[d, "g"] & Z_{p^\infty} \otimes Z_{p^\infty} \arrow[d, "g"] \\ Z_{p^\infty} \arrow[r, "\sigma_3"] & Z_{p^\infty} \end{tikzcd}$$

are two commutative diagrams. The map $\sigma = (\sigma_1, \sigma_2, \sigma_3) :_\Gamma (Z \oplus Z_{p^\infty} \oplus Z_{p^\infty}) \to_\Gamma (Z \oplus Z_{p^\infty} \oplus Z_{p^\infty})$ is an injective endomorphism. This is because σ_1 is injective, σ_2 and σ_3 are isomorphisms (since $(p, n) = 1$). However σ is not surjective, because σ_1 is not. Thus ${}_\Gamma(Z \oplus Z_{p^\infty} \oplus Z_{p^\infty})$ is not co-hopfian.

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