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A Criterion for Existence of Bivariate Vector Valued Rational Interpolants

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Abstract In this paper, a necessary and sufficient condition for the existence of a kind of bivariate vector valued rational interpolants over rectangular grids is given. This criterion is an algebraic method, i.e., by solving a system of equations based on the given data, we can directly test whether the relevant interpolant exists or not. By coming up with our method, the problem of how to deal with scalar equations and vector equations in the same system of equations is solved. After testing existence, an expression of the corresponding bivariate vector-valued rational interpolant can be constructed consequently. In addition, the way to get the expression is different from the one by making use of Thiele-type bivariate branched vector-valued continued fractions and Samelson inverse which are commonly used to construct the bivariate vector-valued rational interpolants. Compared with the Thiele-type method, the one given in this paper is more direct. Finally, some numerical examples are given to illustrate the result.

Keywords bivariate Newton interpolation formula; bivariate vector-valued rational interpolants; existence; necessary and sufficient conditions.

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1. Introduction

In [1–5], the univariate vector valued rational interpolation has been well studied, and its many results have been generalized to the bivariate vector valued rational interpolation[6−13]. However, very few results can be found so far about the criterion of the existence for the bivariate vector valued rational interpolation. In this paper, a necessary and sufficient condition for the existence of a kind of bivariate vector valued rational interpolation over the rectangular grids is given by means of Newton interpolation formula. We present a method with some inheritance to construct a bivariate vector valued rational interpolating function which is different from Thiele-type vector valued fractions[8,9]. Finally, some numerical examples are given to illustrate the result.

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2. Preliminaries

A set on $\mathbb{R}^2$ is called rectangular grids if it has following form

$$\prod^{m,n} = \{(x_i, y_j)|(x_i, y_j) \in \mathbb{R}^2, \quad i = 0, 1, \ldots, m; \ j = 0, 1, \ldots, n\}. \tag{1}$$

The bivariate vector valued rational interpolation is to seek a vector-valued rational function $\boldsymbol{R}(x, y) = \boldsymbol{P}(x, y)/Q(x, y)$ which satisfies interpolating conditions:

$$\boldsymbol{R}(x_i, y_j) = \mathbf{v}_{ij}, \quad i = 0, 1, \ldots, m; \ j = 0, 1, \ldots, n, \tag{2}$$

where $\boldsymbol{P}(x, y)$ is a d dimension complex bivariate vector-valued polynomial, $Q(x, y)$ is a real bivariate polynomial and $\boldsymbol{v}_{ij} \in \mathbb{C}^d$ for $i = 0, 1, \ldots, m$; $j = 0, 1, \ldots, n$.

We denote that

$$\begin{cases} \omega_0(z) \equiv 1, \\ \omega_j(z) = (z - z_0)(z - z_1) \cdots (z - z_{j-1}), \quad j = 1, 2, \ldots, \end{cases} \tag{3}$$

and matrix $\boldsymbol{F} = \begin{pmatrix} 1 & & & \\ 1 & \omega_1(z_1) & & \\ \vdots & \vdots & \ddots & \\ 1 & \omega_1(z_k) & \cdots & \omega_k(z_k) \end{pmatrix}$. Let $z_k = x_k, k = 1, 2, \ldots, m$ and $z_k = y_k, k = 1, 2, \ldots, n$. We obtain two matrices $\boldsymbol{V}$ and $\boldsymbol{U}$ whose inverse matrices can be showed respectively as follows:

$$\boldsymbol{V}^{-1} = \begin{pmatrix} v_0^{(0)} & & & & \\ v_1^{(0)} & v_1^{(1)} & & & \\ v_2^{(0)} & v_2^{(1)} & v_2^{(2)} & & \\ \vdots & \vdots & \vdots & \ddots & \\ v_m^{(0)} & v_m^{(1)} & v_m^{(2)} & \cdots & v_m^{(m)} \end{pmatrix}, \quad \boldsymbol{U}^{-1} = \begin{pmatrix} u_0^{(0)} & & & & \\ u_1^{(0)} & u_1^{(1)} & & & \\ u_2^{(0)} & u_2^{(1)} & u_2^{(2)} & & \\ \vdots & \vdots & \vdots & \ddots & \\ u_n^{(0)} & u_n^{(1)} & u_n^{(2)} & \cdots & u_n^{(n)} \end{pmatrix}, \tag{4}$$

where elements $u_k^{(i)}$ and $v_k^{(i)}$ can be calculated by (5) and (6) as follows:

$$\begin{cases} v_j^{(j)} = \frac{1}{\omega'_{j+1}(x_j)}, \omega'_{k+1}(x_j) = \prod\limits_{\substack{i=0 \\ i \neq j}}^{k} (x_j - x_i), \\ v_k^{(j)} = v_{k-1}^{(j)} \dfrac{1}{x_j - x_k}, j = 0, 1, \ldots, m; k = j+1, \ldots, m, \end{cases} \tag{5}$$

$$\begin{cases} u_j^{(j)} = \frac{1}{\omega'_{j+1}(y_j)}, \omega'_{k+1}(y_j) = \prod\limits_{\substack{i=0 \\ i \neq j}}^{k} (y_j - y_i), \\ u_k^{(j)} = u_{k-1}^{(j)} \dfrac{1}{y_j - y_k}, j = 0, 1, \ldots, n; k = j+1, \ldots, n. \end{cases} \tag{6}$$

One can generalize bivariate difference coefficient with scale to that with vector value. It follows that formula

$$\sum_{i=0}^{h} \sum_{j=0}^{k} v_h^{(i)} u_k^{(j)} \boldsymbol{g}(x_i, y_j), \quad h = 0, 1, \ldots, m; \ k = 0, 1, \ldots, n \tag{7}$$

is a bivariate difference coefficient of vector-valued function $\boldsymbol{g}(x, y)$ over rectangular grids (1).

Lemma 2.1 *Bivariate vector-valued Newton interpolating polynomial which satisfies interpolation conditions* $\boldsymbol{g}(x_i, y_j) = \boldsymbol{g}_{ij}$ *for* $i = 0, 1, \ldots, m$; $j = 0, 1, \ldots, n$ *is*

$$\boldsymbol{g}(x, y) = \sum_{h=0}^{m} \sum_{k=0}^{n} \boldsymbol{d}_{hk} \omega_h(x) \omega_k(y), \tag{8}$$

where

$$\boldsymbol{d}_{hk} = \sum_{i=0}^{h} \sum_{j=0}^{k} v_h^{(i)} u_k^{(j)} \boldsymbol{g}_{ij}, \quad h = 0, 1, \ldots, m;\ k = 0, 1, \ldots, n. \tag{9}$$

To carry out discussion conveniently, we arrange points over rectangular grids (1) end to end on diagonals bordering upon each other and obtain a sequence:

$$(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), \ldots, (m, n-1), (m-1, n), (m, n). \tag{10}$$

Let $L = mn + m + n$ and let E denote the following bivariate index set

$$E = \{(\eta_0, \xi_0), (\eta_1, \xi_1), \ldots, (\eta_L, \xi_L)\}. \tag{11}$$

We rearrange indices practically in E and yield a new bivariate index set I:

$$I = \{(i_0, j_0), (i_1, j_1), \ldots, (i_L, j_L)\}, \tag{12}$$

and let

$$N = \{(i_0, j_0), (i_1, j_1), \ldots, (i_s, j_s)\} \subset I, \tag{13}$$

$$D = \{(i_0, j_0), (i_1, j_1), \ldots, (i_t, j_t)\} \subset I, \tag{14}$$

where $s + t = L$.

Definition 2.1 $\boldsymbol{R}(x, y) = \boldsymbol{P}(x, y)/Q(x, y)$ *is called bivariate vector-valued rational function with* $[N/D]_I$ *type if*

$$\begin{aligned} \boldsymbol{P}(x, y) &= \sum_{(i_k, j_k) \in N} \boldsymbol{d}_{i_k, j_k} \omega_{i_k}(x) \omega_{j_k}(y), \\ Q(x, y) &= \sum_{(i_k, j_k) \in D} d_{i_k, j_k} \omega_{i_k}(x) \omega_{j_k}(y). \end{aligned} \tag{15}$$

Let

$$a_{ij}^{(i'j')} = \sum_{k=s+1}^{L} v_{i_k}^{(i)} u_{j_k}^{(j)} v_{i_k}^{(i')} u_{j_k}^{(j')}, (i', j'), (i, j) \in I, \tag{16}$$

$$b_{ij}^{(i'j')} = \sum_{k=t+1}^{L} v_{i_k}^{(i)} u_{j_k}^{(j)} v_{i_k}^{(i')} u_{j_k}^{(j')} \boldsymbol{v}_{ij} \cdot \boldsymbol{v}_{i'j'}^{*}, (i', j'), (i, j) \in I, \tag{17}$$

where the dot means cross product and $\boldsymbol{v}_{i'j'}^{*}$ *is conjugate vector of* $\boldsymbol{v}_{i'j'}$,

$$c_{ij}^{(i'j')} = a_{ij}^{(i'j')} + b_{ij}^{(i'j')}, (i', j'), (i, j) \in I. \tag{18}$$

3. Important result

Theorem 3.1 *The bivariate vector-valued rational interpolation function with $[N/D]_I$ type exists if and only if system of equations*

$$\begin{pmatrix} c_{i_0j_0}^{(i_0j_0)} & c_{i_1j_1}^{(i_0j_0)} & \cdots & c_{i_0j_L}^{(i_0j_0)} \\ c_{i_0j_0}^{(i_1j_1)} & c_{i_1j_1}^{(i_1j_1)} & \cdots & c_{i_0j_L}^{(i_1j_1)} \\ \vdots & \vdots & & \vdots \\ c_{i_0j_0}^{(i_Lj_L)} & c_{i_1j_1}^{(i_Lj_L)} & \cdots & c_{i_0j_L}^{(i_Lj_L)} \end{pmatrix} \begin{pmatrix} q_{i_0j_0} \\ q_{i_1j_1} \\ \vdots \\ q_{i_Lj_L} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{19}$$

has one solution with every element being non-zero, i.e., $q_{ij} \neq 0$ for $i = 0, 1, \ldots, m$; $j = 0, 1, \ldots, n$ and the corresponding bivariate vector valued rational function is $\boldsymbol{R}(x, y) = \boldsymbol{P}(x, y)/Q(x, y)$, where

$$\boldsymbol{P}(x, y) = \sum_{k=0}^{s} \Big(\sum_{i=i_0}^{i_k} \sum_{j=j_0}^{j_k} v_{i_k}^{(i)} u_{j_k}^{(j)} q_{ij} \boldsymbol{v}_{ij} \Big) \omega_{i_k}(x) \omega_{j_k}(y), \tag{20}$$

$$Q(x, y) = \sum_{k=0}^{t} \Big(\sum_{i=i_0}^{i_k} \sum_{j=j_0}^{j_k} v_{i_k}^{(i)} u_{j_k}^{(j)} q_{ij} \Big) \omega_{i_k}(x) \omega_{j_k}(y). \tag{21}$$

Proof Firstly, we prove the necessity. If vector-valued rational function with $[N/D]_I$ type exists, there must be $\boldsymbol{R}(x, y) = \boldsymbol{P}(x, y)/Q(x, y)$ which satisfies (2) and $Q(x_i, y_j) = q_{ij} \neq 0$ for $i = 0, 1, \ldots, m;\ j = 0, 1, \ldots, n$. Thus $\boldsymbol{P}(x_i, y_j) = q_{ij} \boldsymbol{v}_{ij}$ for $i = 0, 1, \ldots, m;\ j = 0, 1, \ldots, n$. By Lemma 2.1, it follows that

$$\boldsymbol{P}(x, y) = \sum_{k=0}^{L} \boldsymbol{d}_{i_k, j_k} \omega_{i_k}(x) \omega_{j_k}(y),$$

$$Q(x, y) = \sum_{k=0}^{L} d_{i_k, j_k} \omega_{i_k}(x) \omega_{j_k}(y), \tag{22}$$

where

$$\boldsymbol{d}_{i_k, j_k} = \sum_{i=i_0}^{i_k} \sum_{j=j_0}^{j_k} v_{i_k}^{(i)} u_{j_k}^{(j)} q_{ij} \boldsymbol{v}_{ij}, \quad k = 0, 1, \ldots, L;$$

$$d_{i_k, j_k} = \sum_{i=i_0}^{i_k} \sum_{j=j_0}^{j_k} v_{i_k}^{(i)} u_{j_k}^{(j)} q_{ij}, \quad k = 0, 1, \ldots, L. \tag{23}$$

We know that $\boldsymbol{R}(x, y) = \boldsymbol{P}(x, y)/Q(x, y)$ is $[N/D]_I$ type function. We hence have

$$\begin{cases} \boldsymbol{d}_{i_k, j_k} = \boldsymbol{0}, & k = s+1, s+2, \ldots, L, \\ d_{i_k, j_k} = 0, & k = t+1, t+2, \ldots, L. \end{cases} \tag{24}$$

It follows that

$$\begin{cases} \sum\limits_{i=i_0}^{i_L} \sum\limits_{j=j_0}^{j_L} v_{i_k}^{(i)} u_{j_k}^{(j)} q_{ij} \boldsymbol{v}_{ij} = \boldsymbol{0}, & k = s+1, s+2, \ldots, L, \\ \sum\limits_{i=i_0}^{i_L} \sum\limits_{j=j_0}^{j_L} v_{i_k}^{(i)} u_{j_k}^{(j)} q_{ij} = 0, & k = t+1, t+2, \ldots, L. \end{cases} \tag{25}$$

Multiplying vector $\sum_{k=s+1}^{L} v_{i_k}^{(i')} u_{j_k}^{(j')} \boldsymbol{v}_{i'j'}^{*}$ and scale $\sum_{k=t+1}^{L} v_{i_k}^{(i')} u_{j_k}^{(j')}$ respectively in both sides of two equalities in (25), we obtain

$$\begin{cases} \sum\limits_{k=s+1}^{L} \Big(v_{i_k}^{(i')} u_{j_k}^{(j')} \mathbf{v}_{i'j'}^{*} \Big) \cdot \Big(\sum\limits_{i=i_0}^{i_L} \sum\limits_{j=j_0}^{j_L} v_{i_k}^{(i)} u_{j_k}^{(j)} \boldsymbol{v}_{ij} \Big) = 0, \\ \sum\limits_{k=s+1}^{L} \Big(v_{i_k}^{(i')} u_{j_k}^{(j')} \Big) \Big(\sum\limits_{i=i_0}^{i_L} \sum\limits_{j=j_0}^{j_L} v_{i_k}^{(i)} u_{j_k}^{(j)} \Big) = 0, \quad i', j' \in I. \end{cases} \tag{26}$$

It follows that

$$\sum_{i=i_0}^{i_L} \sum_{j=j_0}^{j_L} \Big(\sum_{k=s+1}^{L} v_{i_k}^{(i)} u_{j_k}^{(j)} v_{i_k}^{(i')} u_{j_k}^{(j')} \boldsymbol{v}_{ij} \cdot \mathbf{v}_{i'j'}^{*} + \sum_{k=t+1}^{L} v_{i_k}^{(i)} u_{j_k}^{(j)} v_{i_k}^{(i')} u_{j_k}^{(j')} \Big) q_{ij} = 0,$$
$$i', j' \in I. \tag{27}$$

By formulas (16)–(18), one can yield

$$\sum_{i=i_0}^{i_L} \sum_{j=j_0}^{j_L} \Big(a_{ij}^{(i'j')} + b_{ij}^{(i'j')} \Big) q_{ij} = 0. \tag{28}$$

This implies that

$$\sum_{i=i_0}^{i_L} \sum_{j=j_0}^{j_L} c_{ij}^{(i'j')} q_{ij} = 0,$$

which is the system of equations (19).

Supposing that system of equations (19) has a non-zero solution with $q_{ij} \neq 0$ for $i = 0, 1, \ldots, m$; $j = 0, 1, \ldots, n$, by (19) coupled with formula (27), we obtain following formula

$$\sum_{i'=i_0}^{i_L} \sum_{j'=j_0}^{j_L} q_{i'j'} \Big\{ \sum_{i=i_0}^{i_L} \sum_{j=j_0}^{j_L} \Big(\sum_{k=s+1}^{L} v_{i_k}^{(i)} u_{j_k}^{(j)} v_{i_k}^{(i')} u_{j_k}^{(j')} \boldsymbol{v}_{ij} \cdot \boldsymbol{v}_{i'j'}^{*} +$$
$$\sum_{k=t+1}^{L} v_{i_k}^{(i)} u_{j_k}^{(j)} v_{i_k}^{(i')} u_{j_k}^{(j')} \Big) q_{ij} \Big\} = 0. \tag{29}$$

It follows that

$$\sum_{k=s+1}^{L} \Big(\sum_{i'=i_0}^{i_L} \sum_{j'=j_0}^{j_L} v_{i_k}^{(i')} u_{j_k}^{(j')} q_{i'j'} \boldsymbol{v}_{i'j'}^{*} \Big) \cdot \Big(\sum_{i=i_0}^{i_L} \sum_{j=j_0}^{j_L} v_{i_k}^{(i)} u_{j_k}^{(j)} q_{ij} \boldsymbol{v}_{ij} \Big) +$$
$$\sum_{k=t+1}^{L} \Big(\sum_{i'=i_0}^{i_L} \sum_{j'=j_0}^{j_L} v_{i_k}^{(i')} u_{j_k}^{(j')} q_{i'j'} \Big) \cdot \Big(\sum_{i=i_0}^{i_L} \sum_{j=j_0}^{j_L} v_{i_k}^{(i)} u_{j_k}^{(j)} q_{ij} \Big), \tag{30}$$

which means that

$$\sum_{k=s+1}^{L} \Big\| \sum_{i=i_0}^{i_L} \sum_{j=j_0}^{j_L} v_{i_k}^{(i)} u_{j_k}^{(j)} q_{ij} \boldsymbol{v}_{ij} \Big\|^2 + \sum_{k=t+1}^{L} \Big| \sum_{i=i_0}^{i_L} \sum_{j=j_0}^{j_L} v_{i_k}^{(i)} u_{j_k}^{(j)} q_{ij} \Big|^2 = 0. \tag{31}$$

This implies that

$$\sum_{i=i_0}^{i_L} \sum_{j=j_0}^{j_L} v_{i_k}^{(i)} u_{j_k}^{(j)} q_{ij} \boldsymbol{v}_{ij} = \mathbf{0}, \quad k = s+1, \ldots, L;$$

$$\sum_{i=i_0}^{i_L}\sum_{j=j_0}^{j_L} v_{i_k}^{(i)} u_{j_k}^{(j)} q_{ij} = 0, \quad k = t+1, \ldots, L. \tag{32}$$

By formula (30) we may set

$$\begin{aligned}\boldsymbol{P}(x,y) &= \sum_{k=0}^{s}\Big(\sum_{i=i_0}^{i_L}\sum_{j=j_0}^{j_L} v_{i_k}^{(i)} u_{j_k}^{(j)} q_{ij}\boldsymbol{v}_{ij}\Big)\omega_{i_k}(x)\omega_{j_k}(y)\\ &= \sum_{k=0}^{s}\Big(\sum_{i=i_0}^{i_k}\sum_{j=j_0}^{j_k} v_{i_k}^{(i)} u_{j_k}^{(j)} q_{ij}\boldsymbol{v}_{ij}\Big)\omega_{i_k}(x)\omega_{j_k}(y),\end{aligned} \tag{33}$$

$$\begin{aligned}Q(x,y) &= \sum_{k=0}^{t}\Big(\sum_{i=i_0}^{i_L}\sum_{j=j_0}^{j_L} v_{i_k}^{(i)} u_{j_k}^{(j)} q_{ij}\Big)\omega_{i_k}(x)\omega_{j_k}(y)\\ &= \sum_{k=0}^{t}\Big(\sum_{i=i_0}^{i_k}\sum_{j=j_0}^{j_k} v_{i_k}^{(i)} u_{j_k}^{(j)} q_{ij}\Big)\omega_{i_k}(x)\omega_{j_k}(y).\end{aligned} \tag{34}$$

Therefore, the equality $\boldsymbol{R}(x,y) = \boldsymbol{P}(x,y)/Q(x,y)$ is a bivariate vector-valued rational interpolation function with $[N/D]_I$ type which satisfies interpolating conditions (2). Now Theorem is proved completely. $\square$

4. Numerical Examples

In this section, three examples are given. The first one and last one illustrate validity of Theorem 3.1. In fact, many data used practically are floating point numbers. The second one is a corresponding application.

Example 4.1 Let the interpolating conditions be given in the following table:

$\boldsymbol{v}_{ij}$	$y_0 = 0$	$y_1 = 1$	$y_2 = 2$
$x_0 = 0$	$(\frac{3}{5}, \frac{1}{5})$	$(\frac{3}{4}, 0)$	$(1, -\frac{1}{3})$
$x_1 = 1$	$(\frac{1}{4}, \frac{1}{4})$	$(\frac{1}{3}, 0)$	$(\frac{1}{2}, -\frac{1}{2})$

Table 1 interpolating conditions in Example 4.1

By (12), (13) and (14), we set

$$\begin{aligned}&I = \{(0,0),(1,0),(0,1),(1,1),(0,2),(1,2)\},\\ &N = \{(0,0),(1,0),(0,1)\} \subset I,\\ &D = \{(0,0),(1,0),(0,1),(1,1)\} \subset I.\end{aligned} \tag{35}$$

The question is whether there exists a bivariate vector-valued rational interpolation function with $[N/D]_I$ type which satisfies (2).

Solution By formulas (16), (17) and (18), $a_{ij}^{(i'j')}, b_{ij}^{(i'j')}$ and $c_{ij}^{(i'j')}$ for (i,j) and (i',j') which belongs to I can be calculated respectively. Therefore, we obtain a system of equations:

$$\begin{pmatrix} \frac{17}{10} & -\frac{13}{10} & -\frac{49}{20} & \frac{8}{5} & \frac{23}{30} & -\frac{3}{10} \\ -\frac{13}{10} & \frac{41}{32} & \frac{51}{32} & -\frac{37}{24} & -\frac{7}{24} & -\frac{1}{4} \\ -\frac{49}{20} & \frac{51}{32} & \frac{33}{8} & -\frac{9}{4} & -\frac{7}{4} & \frac{11}{16} \\ \frac{8}{5} & -\frac{37}{24} & -\frac{9}{4} & \frac{19}{9} & \frac{2}{3} & -\frac{7}{12} \\ \frac{23}{30} & -\frac{7}{24} & -\frac{7}{4} & \frac{2}{3} & \frac{19}{18} & -\frac{5}{12} \\ -\frac{3}{10} & \frac{1}{4} & \frac{11}{16} & -\frac{7}{12} & -\frac{5}{12} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} q_{00} \\ q_{10} \\ q_{01} \\ q_{11} \\ q_{02} \\ q_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving it, we obtain solution $\boldsymbol{q} = k[5/2, 2, 2, 3/2, 3/2, 1]^{\mathrm{T}}$, where k is an arbitrary constant. Thus there exists a totally non-zero solution$[5, 4, 4, 3, 3, 2]^{\mathrm{T}}$. By Theorem 3.1, a bivariate vector-valued rational interpolation function with $[N/D]_I$ type $\boldsymbol{R}(x, y) = \boldsymbol{P}(x, y)/Q(x, y)$ which satisfies (2) can be given:

$$\boldsymbol{R}(x, y) = \frac{(-3 + 2x, -1 + y)}{-5 + x + y}.$$

Example 4.2 Let the interpolation conditions be given in the following table:

$\boldsymbol{v}_{ij}$	$y_0 = 0$	$y_1 = 1$	$y_2 = 2$
$x_0 = 0$	$(0.600000, 0.200000)$	$(0.750000, 0)$	$(1.00000, -0, 333333)$
$x_1 = 1$	$(0.250000, 0.250000)$	$(0.333333, 0)$	$(0.500000, -0.500000)$

Table 2 interpolating condition in Example 4.2

By (12), (13) and (14), we set (35). Study whether there exists a bivariate vector-valued rational interpolation function with $[N/D]_I$ type which satisfies (2) with error interval $[-10^{-6}, 10^{-6}]$.

Solution By formulas (16), (17) and (18), $a_{ij}^{(i'j')}, b_{ij}^{(i'j')}$ and $c_{ij}^{(i'j')}$ for (i, j) and (i', j') which belongs to I can be calculated respectively. Therefore, we obtain a system of equations :

$$\begin{pmatrix} 1.70000 & -1.30000 & -2.45000 & 1.60000 & 0.766667 & -0.300000 \\ -1.30000 & 1.28125 & 1.59375 & -1.54167 & -0.291667 & 0.25000 \\ -2.45000 & 1.59375 & 4.12500 & -2.45000 & -1.75000 & 0.687500 \\ 1.60000 & -1.54167 & -2.45000 & 2.11111 & 0.666667 & -0.583333 \\ 0.766667 & -0.291667 & -1.75000 & 0.666667 & 1.05556 & -0.416667 \\ -0.300000 & 0.250000 & 0.687500 & -0.583333 & -0.416667 & 0.375000 \end{pmatrix} \times$$

$$\begin{pmatrix} q_{00} \\ q_{10} \\ q_{01} \\ q_{11} \\ q_{02} \\ q_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving it, we obtain solutions

$$\boldsymbol{q} = k[0.562544, 0.450064, 0.450064, 0.337526, 0.337527, 0.225017]^{\mathrm{T}},$$

where k is an arbitrary constant. Thus there exists a totally non-zero solution

$$[0.562544, 0.450064, 0.450064, 0.337526, 0.337527, 0.225017]^{\mathrm{T}}.$$

As a matter of fact, $|q_{ij}| > 10^{-6}, i = 0, 1, 2$. By Theorem 3.1, a bivariate vector-valued rational interpolation function with $[N/D]_I$ type $\boldsymbol{R}(x, y) = \boldsymbol{P}(x, y)/Q(x, y)$ which satisfies (2) can be given:

$$\boldsymbol{R}(x, y) = \frac{(0.337527 - 0.225018x, 0.112509 - 0.112509y)}{0.562544 - 0.112510x - 0.112509y}.$$

After assigning $(x_i, y_j), i = 0, 1, j = 0, 1, 2$ to (x, y) in $\boldsymbol{R}(x, y)$, we have a new table as follows:

$\boldsymbol{R}(x_i, y_j)$	$y_0 = 0$	$y_1 = 1$	$y_2 = 2$
$x_0 = 0$	$(0.600000, 0.200000)$	$(0.750000, 0)$	$(1.00000, -0, 333334)$
$x_1 = 1$	$(0.250001, 0.250000)$	$(0.333334, -5.4953 \times 10^{-7})$	$(0.500001, -0.500001)$

Table 3 interpolating result at interpolating points in Example 4.2

and the error is $|\boldsymbol{v}_{ij} - \boldsymbol{R}(x_i, y_j)| \le 10^{-6}, i = 0, 1, j = 0, 1, 2$.

Example 4.3 Let the interpolating conditions be given in the following table:

$\boldsymbol{v}_{ij}$	$y_0 = 0$	$y_1 = 1$	$y_2 = 2$
$x_0 = 0$	$(1, 1)$	$(1, 2)$	$(1, 3)$
$x_1 = 1$	$(2, 1)$	$(1, 2)$	$(1, 3)$

Table 4 interpolating condition in Example 4.3

Study whether there exists a bivariate vector-valued rational interpolation function with $[N/D]_I$ type satisfying (2).

Solution Analogously, we get a system of equations:

$$\frac{1}{4}\begin{pmatrix} 10 & -8 & -20 & 12 & 10 & -5 \\ -8 & 10 & 14 & -14 & -6 & 6 \\ -20 & 14 & 52 & -28 & -32 & 16 \\ 12 & -14 & -28 & 28 & 16 & -16 \\ 10 & -6 & -32 & 16 & 22 & -11 \\ -5 & 6 & 16 & -16 & -11 & 11 \end{pmatrix}\begin{pmatrix} q_{00} \\ q_{10} \\ q_{01} \\ q_{11} \\ q_{02} \\ q_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving it, we obtain solutions $\boldsymbol{q} = k[1, 0, 1, 0, 1, 0]^{\mathrm{T}}$, where k is an arbitrary constant. It is clear that $q_{10} = q_{11} = q_{12} \equiv 0$. Therefore by Theorem 3.1, there does not exist an $[N/D]_I$ type bivariate vector-valued rational interpolation function which satisfies (2).

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