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A Note on Approximating Fixed Points of Pseudocontractive Mappings

YAO Yong Hong
(Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China)
(E-mail: yuyanrong@tjpu.edu.cn; yaoyonghong@yahoo.cn)

Abstract Let K be a nonempty bounded closed convex subset of a real reflexive Banach space E with a uniformly Gateaux differentiable norm. Let $T : K \to K$ be a uniformly continuous pseudocontractive mapping. Suppose every closed convex and bounded subset of K has the fixed point property for nonexpansive mappings. Let $\{\lambda_n\} \subset (0, \frac{1}{2}]$ be a sequence satisfying the conditions: (i) $\lim_{n\to\infty} \lambda_n = 0$; (ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$. Let the sequence $\{x_n\}$ be generated from arbitrary $x_1 \in K$ by $x_{n+1} = (1-\lambda_n)x_n + \lambda_n Tx_n - \lambda_n(x_n - x_1)$, $n \geq 1$. Suppose $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Then $\{x_n\}$ converges strongly to a fixed point of T.

Keywords pseudocontractive mapping; fixed point; uniformly Gateaux differentiable norm; strong convergence.

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1. Introduction and preliminaries

Let E be a real Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

It is well known that if E is smooth, then J is single-valued. In the sequel, we shall denote the single-valued normalized duality mapping by j. We use $D(T)$ and $R(T)$ to denote the domain and range of T, respectively. An operator $T : D(T) \to R(T)$ is called pseudocontractive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x-y\|^2,$$

for all $x, y \in D(T)$.

Within the past 30 years or so, many authors have been devoted to the iterative construction of fixed points of pseudocontractive mappings[1−6]. In 1974, Ishikawa[7] introduced an iterative scheme to approximate the fixed points of Lipschitzian pseudocontractive mappings. He proved the following result.

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Theorem IS[7] *If K is a compact convex subset of a Hilbert space H, $T : K \to K$ is a Lipschitzian pseudocontractive mapping. For $x_0 \in K$, define the sequence $\{x_n\}$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n; \quad y_n = (1-\beta_n)x_n + \beta_n T x_n, \ n \ge 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying the conditions:

(i) $0 \le \alpha_n \le \beta_n < 1$; (ii) $\lim_{n\to\infty} \beta_n = 0$; (iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

We note that $\beta_n \neq 0$ for all $n \ge 0$ in Theorem IS. In connection with the iterative approximation of fixed points of pseudocontractions, the following question is still open: Does the Mann iterative algorithm always converge for continuous pseudocontractions or even Lipschitz pseudocontractions? However, in 2001, Chidume and Mutangadura[8] provided an example of a Lipschitz pseudocontractive mapping with a unique fixed point for which the Mann iterative algorithm failed to converge. In 2003, Chidume and Zegeye[9] introduced a new iterative scheme for approximating the fixed points of pseudocontractive mappings.

Theorem CZ[9] *Let E be a real reflexive Banach space with a uniformly Gateaux differentiable norm. Let K be a nonempty closed convex subset of E. Let $T : K \to K$ be an L-Lipschitz pseudocontractive mapping such that $F(T) \neq \emptyset$. Suppose that every nonempty closed convex bounded subset of K has the fixed point property for nonexpansive self-mappings. Let $\{\lambda_n\}$ and $\{\theta_n\}$ be two sequences in $(0,1]$ satisfying the following conditions: (i) $\lim_{n\to\infty} \theta_n = 0$; (ii) $\lambda_n(1+\theta_n) \le 1, \sum_{n=0}^{\infty} \lambda_n\theta_n = \infty, \lim_{n\to\infty} \frac{\lambda_n}{\theta_n} = 0$; (iii) $\lim_{n\to\infty} \frac{(\frac{\theta_{n-1}}{\theta_n} - 1)}{\lambda_n\theta_n} = 0$. For arbitrarily given $x_1 \in K$, let the sequence $\{x_n\}$ be defined iteratively by*

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n - \lambda_n\theta_n(x_n - x_1), \ n \ge 1. \tag{1}$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Prototypes for the iteration parameters are, for example, $\lambda_n = \frac{1}{(n+1)^a}, \theta_n = \frac{1}{(n+1)^b}$, $0 < b < a$ and $a + b < 1$. But we observe that the canonical choices of $\lambda_n = \frac{1}{n}, \theta_n = \frac{1}{n}$ are impossible. This brings us a question: Under what conditions, $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$ are sufficient to guarantee the strong convergence of the iterative scheme (1) to a fixed point of T? It is our purpose in this paper that we introduce a new iterative scheme to approximate the fixed points of Lipschitzian pseudocontractive mappings. Our results improve and extend many results in the literature.

We need the following lemmas for proving our main results.

Lemma 1.1[10] *Let E be a Banach space. Suppose that K is a nonempty closed convex subset of E and $T : K \to E$ is a continuous pseudocontractive mapping satisfying the weakly inward condition: $T(x) \in \overline{I_K(x)}$ ($\overline{I_K(x)}$ is the closure of $I_K(x)$) for each $x \in K$, where $I_K(x) = x + \{c(u-x) : u \in E \text{ and } c \ge 1\}$. Then for each $z \in K$, there exists a unique continuous path $t \mapsto y_t \in K, t \in [0,1)$, satisfying the following equation*

$$y_t = tTy_t + (1-t)z.$$

Furthermore, if E is a reflexive Banach space with a uniformly Gateaux differentiable norm and every nonempty closed convex bounded subset of K has the fixed point property for nonexpansive self-mappings. Then as $t \to 1$, y_t converges strongly to a fixed point of T.

Lemma 1.2[11] *(i) If E is smooth Banach space, then the duality mapping J is single valued and strong-weak* continuous.*

(ii) If E is a Banach space with a uniformly Gâteaux differentiable norm, then the duality mapping $J : E \to E^*$ is single valued and norm to weak star uniformly continuous on bounded sets of E.

Lemma 1.3[12] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\beta_n, n \geq 0$, where $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\}$ are two sequences of real numbers such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \beta_n \leq 0$. Then $\{a_n\}$ converges to zero as $n \to \infty$.*

2. Main results

Theorem 2.1 *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space E with a uniformly Gateaux differentiable norm. Let $T : K \to K$ be a uniformly continuous pseudocontractive mapping. Suppose every closed convex and bounded subset of K has the fixed point property for nonexpansive mappings. Let $\{\lambda_n\} \subset (0, \frac{1}{2}]$ be a sequence satisfying the conditions: (i) $\lim_{n\to\infty} \lambda_n = 0$; (ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$. Let the sequence $\{x_n\}$ be generated from arbitrary $x_1 \in K$ by*

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n - \lambda_n(x_n - x_1), \quad n \geq 1. \tag{2}$$

Suppose $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof First, we note that (2) can be written as

$$x_{n+1} = \lambda_n x_1 + (1-2\lambda_n)x_n + \lambda_n T x_n, \quad n \geq 1.$$

Therefore, as $\lambda_n \in (0, \frac{1}{2}]$, we have $\{x_n\} \subset K, \forall n \geq 1$. Take $p \in F(T)$. Let $S = I - T$. Then we have

$$\langle Sx - Sy, j(x-y)\rangle \geq 0. \tag{3}$$

From (2), we obtain

$$\begin{aligned} x_n =& x_{n+1} + \lambda_n x_n - \lambda_n T x_n + \lambda_n x_n - \lambda_n x_1 \\ =& x_{n+1} + \lambda_n x_n + \lambda_n S x_n - \lambda_n x_1 \\ =& x_{n+1} + \lambda_n[x_{n+1} + \lambda_n x_n + \lambda_n S x_n - \lambda_n x_1] + \lambda_n S x_n - \lambda_n x_1 \\ =& (1+\lambda_n)x_{n+1} + \lambda_n^2(x_n + Sx_n) - \lambda_n^2 x_1 + \lambda_n S x_n - \lambda_n x_1 \\ =& (1+\lambda_n)x_{n+1} + \lambda_n S x_{n+1} + \lambda_n^2(x_n + Sx_n) - \lambda_n^2 x_1 + \\ & \lambda_n(Sx_n - Sx_{n+1}) - \lambda_n x_1. \end{aligned} \tag{4}$$

From (4), we have

$$\begin{aligned} x_n - p =&(1+\lambda_n)(x_{n+1}-p) + \lambda_n(Sx_{n+1} - Sp) + \\ &\lambda_n^2(x_n + Sx_n) - \lambda_n^2 x_1 + \lambda_n(Sx_n - Sx_{n+1}) + \lambda_n(p - x_1). \end{aligned} \tag{5}$$

Combining (3) and (5), we have

$$\begin{aligned} &\langle x_n - p - \lambda_n^2(x_n + Sx_n) + \lambda_n^2 x_1 - \lambda_n(Sx_n - Sx_{n+1}) + \lambda_n(x_1 - p), j(x_{n+1} - p)\rangle \\ &= (1+\lambda_n)\|x_{n+1} - p\|^2 + \lambda_n\langle Sx_{n+1} - Sp, j(x_{n+1} - p)\rangle \\ &\geq (1+\lambda_n)\|x_{n+1} - p\|^2. \end{aligned} \tag{6}$$

Next we prove that $\limsup_{n\to\infty}\langle x_1 - p, j(x_n - p)\rangle \leq 0$. Indeed, taking $z = x_1$ in Lemma 1.1, we have $z_t - x_n = (1-t)(Tz_t - x_n) + t(x_1 - x_n)$. Hence

$$\begin{aligned} \|z_t - x_n\|^2 =&(1-t)\langle Tz_t - x_n, j(z_t - x_n)\rangle + t\langle x_1 - x_n, j(z_t - x_n)\rangle \\ =&(1-t)\langle Tz_t - Tx_n, j(z_t - x_n)\rangle + (1-t)\langle Tx_n - x_n, j(z_t - x_n)\rangle + \\ &t\langle x_1 - z_t, j(z_t - x_n)\rangle + t\|z_t - x_n\|^2 \\ \leq&\|z_t - x_n\|^2 + (1-t)\|Tx_n - x_n\|\|z_t - x_n\| + t\langle x_1 - z_t, j(z_t - x_n)\rangle. \end{aligned}$$

Therefore,

$$\langle z_t - x_1, j(z_t - x_n)\rangle \leq \frac{1-t}{t}\|Tx_n - x_n\|\|z_t - x_n\| \leq M\frac{1-t}{t}\|Tx_n - x_n\|,$$

where $M > 0$ is some constant such that $\|z_t - x_n\| \leq M$ for $t \in (0,1], n \geq 1$. Letting $n \to \infty$, we have

$$\limsup_{n\to\infty}\langle z_t - x_1, j(z_t - x_n)\rangle \leq 0.$$

From Lemma 1.1, we know $z_t \to p$ (as $t \to 0$). Since the duality mapping $J : E \to E^*$ is norm to weak star uniformly continuous, we have from Lemma 1.2

$$\limsup_{n\to\infty}\langle x_1 - p, j(x_n - p)\rangle \leq 0. \tag{7}$$

From (6), we get

$$\begin{aligned} (1+\lambda_n)\|x_{n+1} - p\|^2 \leq&\langle x_n - p - \lambda_n^2(x_n + Sx_n) + \\ &\lambda_n^2 x_1 - \lambda_n(Sx_n - Sx_{n+1}) + \lambda_n(x_1 - p), j(x_{n+1} - p)\rangle \\ \leq&\|x_n - p\|\|x_{n+1} - p\| + M\lambda_n^2 + M\lambda_n\|Sx_{n+1} - Sx_n\| + \\ &\lambda_n\langle x_1 - p, j(x_{n+1} - p)\rangle \\ \leq&\frac{1}{2}(\|x_n - p\|^2 + \|x_{n+1} - p\|^2) + M\lambda_n^2 + M\lambda_n\|Sx_{n+1} - Sx_n\| + \\ &\lambda_n\langle x_1 - p, j(x_{n+1} - p)\rangle, \end{aligned}$$

that is,

$$\begin{aligned} \|x_{n+1} - p\|^2 \leq&[1 - \frac{2}{3}\lambda_n]\|x_n - p\|^2 + 2M\lambda_n^2 + \\ &2M\lambda_n\|Sx_{n+1} - Sx_n\| + 2\lambda_n\langle x_1 - p, j(x_{n+1} - p)\rangle. \end{aligned} \tag{8}$$

Note that

$$\|x_{n+1} - x_n\| \leq \lambda_n \|x_n\| + \lambda_n \|Tx_n\| + \lambda_n \|x_n - x_1\| \to 0 (n \to \infty).$$

By the uniform continuity of T, we have

$$\|Sx_{n+1} - Sx_n\| \to 0 (n \to \infty).$$

Finally applying Lemma 1.3 to (8), we can conclude that $x_n \to p$. This completes the proof. □

Theorem 2.2 *Let E be a uniformly smooth Banach space and K a nonempty bounded closed convex subset of E. Let $T : K \to K$ be a uniformly continuous pseudocontractive mapping. Let $\{\lambda_n\} \subset (0, \frac{1}{2}]$ be a sequence satisfying the conditions: (i) $\lim_{n\to\infty} \lambda_n = 0$; (ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$. Let the sequence $\{x_n\}$ be generated from arbitrary $x_1 \in K$ by*

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n Tx_n - \lambda_n(x_n - x_1), \quad n \geq 1.$$

Suppose $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof Since every uniformly smooth Banach space is reflexive and whose norm is uniformly Gateaux differentiable. At the same time, every closed convex and bounded subset of K has the fixed point property for nonexpansive mappings. Hence, from Theorem 2.1, we can obtain desired result. This completes the proof. □

Remark It is clear that our iterative scheme (2) with mild restrictions on parameters is simpler than the iterative scheme (1). We can choose $\lambda_n = \frac{1}{n}$ for all $n \geq 1$. It is worthy of mentioning that our methods are different from those in the literature.

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