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The Regularity of the Weak Solutions for the N-Dimensional Quasilinear Elliptic Equations with Discontinuous Coefficients

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Abstract The n-dimensional quasilinear elliptic equations with discontinuous coefficients are studied. Using estimate and difference approach methods, we prove that the first derivatives of the weak solutions are continuous in the sense of Hölder up to the inner boundary on which the coefficients are discontinuous.

Keywords Quasilinear elliptic equation; discontinuous coefficients; weak solution; regularity.

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1. Introduction

The problems about elliptic equations with discontinuous coefficients are also known as the diffraction problems. The coefficients of the equations are discontinuous on inner boundary. The papers[1−4] studied the existence and the regularity of the solutions for the linear equations with discontinuous coefficients. The regularity results are very important for the linear diffraction problems and some porus medium problems[5,6]. Therefore, the regularity results of the semi-linear and quasilinear equations are also useful and important for us. In 2005, the paper[7] considered the regularity of the solutions of a special semi-linear second order elliptic equations with discontinuous coefficients that is derived from the electric field. In this paper, we are going to study the regularity of the weak solutions for the n-dimensional quasilinear elliptic equations with discontinuous coefficients.

We first introduce some notations:

Notation 1.1 *(a1)* $\mathbf{R}^n$ *is the n-dimensional Euclidean space, $n \geq 2$, with points* $x = (x_1, x_2, \ldots, x_n)$. $p := (p_1, p_2, \ldots, p_n)$, $w_x := (w_{x_1}, w_{x_2}, \ldots, w_{x_n}) \in \mathbf{R}^n$, *where* $w_{x_i} := \frac{\partial w}{\partial x_i}$. *Let Ω be an open bounded domain in $\mathbf{R^n}$. It satisfies $\overline{\Omega} = \overline{D^{(1)}} \cup \overline{D^{(2)}}$, where $D^{(1)}$ and $D^{(2)}$ are two subdomains which are separated by inner boundary $\Gamma := \{x : x_n = 0\}$. K_ρ is an arbitrary open ball in $\mathbf{R^n}$ of radius ρ and $K_{2\rho}$ is concentric with K_ρ.*

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(a2) Function $a_i(x,w,p)=a_i^{(k)}(x,w,p)$ *if* $x\in D^{(k)}$ *and* $f(x,w,p)=f^{(k)}(x,w,p)$ *if* $x\in D^{(k)}$, $k=1,2$.

(a3) Everywhere pairs of equal indices imply a summation from 1 to n. *For example,* $\frac{\mathrm{d}}{\mathrm{d}x_i}a_i(x,w,w_x)=\sum_{i=1}^n\frac{\mathrm{d}}{\mathrm{d}x_i}a_i(x,w,w_x)$. *"*$A:=B$*" is the definition of* A *by means of the expression* B.

This paper considers the regularity of the following n-dimensional quasilinear elliptic equations with discontinuous coefficients:

Problem 1.1 Find function $w=w(x)$, such that

$$-\frac{\mathrm{d}}{\mathrm{d}x_i}a_i(x,w,w_x)+f(x,w,w_x)=0 \quad \text{in} \quad D^{(k)},\ k=1,2, \tag{1.1}$$

$$[w]_\Gamma=0, \quad \left[\frac{\mathrm{d}w}{\mathrm{d}N}\right]_\Gamma=0, \tag{1.2}$$

$$w(x)=g(x) \quad \text{on} \quad \partial\Omega, \tag{1.3}$$

where $\frac{\mathrm{d}w}{\mathrm{d}N}:=a_i(x,w,w_x)\cos(\mathbf{n},x_i)$, $\mathbf{n}$ is the outward (from $D^{(k)}$) unit normal to Γ, and $[\cdot]_\Gamma$ represents the jump across the inner boundary Γ.

Definition 1.1 *By a bounded weak solution* $w(x)$ *of problem 1.1 we will mean a function* $w(x)$ *from* $W_2^1(\Omega)$ *with* $M:=\operatorname{vari}\max_\Omega|w|<\infty$ *that satisfies the integral identity*

$$\int_\Omega\Big\{a_i(x,w,w_x)\eta_{x_i}+f(x,w,w_x)\eta\Big\}\mathrm{d}x=0 \tag{1.4}$$

for any function $\eta(x)\in\overset{\circ}{W}{}_2^1(\Omega)$ *with* $\operatorname{vari}\max_\Omega|\eta|<\infty$.

Suppose that the following hypotheses are satisfied throughout this paper.

Hypotheses 1.1 *(H1) There exists at least one bounded weak solution for Problem 1.1. The solution is represented by* $w(x)$. *The bound of the solution is denoted by* M.

(H2) Although functions $a_i(x,w,p)$, $f(x,w,p)$ *may be discontinuous for* $x\in\Omega$, *they have partial derivatives for all variables if* x *belongs to the same subdomain* $D^{(k)}$ *and* (w,p) *belongs to* $[-M,M]\times\mathbf{R}$, *and they satisfy the conditions*

$$\nu\sum_{i=1}^n\zeta_i^2\le\frac{\partial}{\partial p_j}a_i(x,w,p)\zeta_i\zeta_j\le\mu\sum_{i=1}^n\zeta_i^2, \quad |a_i(x,w,p)|\le\mu(1+|p|),$$

$$\Big|\frac{\partial a_i(x,w,p)}{\partial x_j}\Big|\le\psi_0(x)+\mu|p|, \quad \Big|\frac{\partial a_i(x,w,p)}{\partial w}\Big|\le\psi_1(x)+\mu|p|,$$

$$|f(x,w,p)|+\Big|\frac{\partial f(x,w,p)}{\partial x_j}\Big|\le\varphi_1+\mu|p|^2,$$

$$\Big|\frac{\partial f(x,w,p)}{\partial w}\Big|\le\varphi_2(x)+\mu|p|^2, \quad \Big|\frac{\partial f(x,w,p)}{\partial p_j}\Big|\le\varphi_3(x)+\mu|p|,$$

where $k=1,2$, $\|\psi_0^2,\psi_1^2,\varphi_1^2,\varphi_2,\varphi_3^2\|_{L_{\frac{q}{2}}(\Omega)}+||\psi_1||_{L_4(\Omega)}\le\mu_1$, ν, μ, μ_1, $q>0$ *and* $q>n$.

Remark 1.1 The hypotheses about the inner boundary ($\{x|x_n=0\}$) and the coefficients are suitable for application, because we can transform the general inner boundary into the form $\{x|\ x_n = 0\}$ by a transformation (see [3, Chapter 3, Section 16], [5, 6]).

The main result of this paper is the following theorem:

Theorem 1.1 *Suppose Hypotheses 1.1 is satisfied. Then any bounded weak solution $w(x)$ of Problem 1.1 belongs to the classes $C^{\alpha}(\overline{\Omega'})$ and $W_2^2(\Omega' \bigcap D^{(k)})$, where Ω' is an arbitrary strictly interior subdomain of the domain Ω, $k = 1, 2$, and $\alpha > 0$ is determined only by ν, μ, μ_1, q and M. Furthermore, the solution $w(x)$ has first derivatives that belong to the class $C^{\beta}(\overline{\Omega' \bigcap D^{(k)}})$ and has the estimates*

$$\int_{\Omega' \cap D^{(k)}} |w_{xx}|^2 \mathrm{d}x \leq C, \tag{1.5}$$

$$\max_{K_\rho} |w_{x_i}| + \rho^{-\beta} osc\{w_{x_i}, K_\rho \cap \overline{D^{(k)}}\} \leq C \tag{1.6}$$

for $1 \leq i \leq n$, $k = 1, 2$, $K_\rho \subset \Omega'$, where C and β are positive constants depending only on ν, μ, μ_1, q, M and the distance from Ω' to $\partial\Omega$. In addition, the solution $w(x)$ satisfies the equation (1.1) for almost all $x \in \Omega'$ and the inner boundary condition (1.2) for almost all $x \in \Gamma \bigcap \Omega'$.

2. The proof of the main theorem

In order to prove Theorem 1.1, we need some lemmas. First, in addition to notation 1.1 we introduce another notation:

Notation 2.1 *(A1) Let Ω' be any given strictly interior subdomain of the domain Ω and Ω_i be a series of subdomains, such that $\Omega_3 = \Omega'$, $\overline{\Omega}_1 \subset \Omega$, $\overline{\Omega}_2 \subset \Omega_1$ and $\overline{\Omega}_3 \subset \Omega_2$. Without loss of generality, assume that for $i = 1, 2$, the distance from Ω_{i+1} to $\partial\Omega_i$ equals to the distance from Ω_1 to $\partial\Omega$ which is denoted by d.*

(A2) $C, C_1, C_2, \ldots, \alpha, \alpha_0, \alpha_1, \ldots$ are constants which depend on parameters ν, μ, μ_1, q and M. It is convenient that we only list the other parameters except ν, μ, μ_1, q, M.

(A3) $G := 1 + \psi_0^2 + \varphi_1^2$ and $G_1 := 1 + \psi_1^2 + \varphi_3^2 + \varphi_2$.

The following two useful inequalities can be easily gotten from the hypotheses:

$$a_i(x, w, p)p_i \geq \frac{\nu}{2}|p|^2 - \mu_2, \tag{2.1}$$

$$\int_{K_\rho} (G + G_1)\mathrm{d}x \leq \|G + G_1\|_{L_{\frac{q}{2}}(K_\rho)} \Big(\int_{K_\rho} \mathrm{d}x \Big)^{1-\frac{2}{q}} \leq C\rho^{n-2+2\alpha_0},\ \alpha_0 = 1 - \frac{n}{q}. \tag{2.2}$$

Lemma 2.1 *There exist constants α, ρ_0, $0 < \alpha \leq 1$, $\rho_0 > 0$ such that*

$$\int_{\Omega_1} |w_x|^2 \mathrm{d}x \leq C(d); \quad \rho^{-\alpha} osc\{w, K_\rho\} \leq C(d) \quad \text{if} \quad K_\rho \subset \Omega_1; \tag{2.3}$$

$$\int_{K_\rho} (1 + |w_x|)^2 \zeta^2(x)\mathrm{d}x \leq C\rho^{\alpha_1} \int_{K_\rho} |\zeta_x|^2 \mathrm{d}x \quad \text{if} \quad K_\rho \subset \Omega_1,\ \rho \leq \rho_0, \tag{2.4}$$

where $\zeta(x)$ is an arbitrary bounded function in $\overset{\circ}{W}{}_2^1(K_\rho)$. Here and below, α_1 equals to $\min(\alpha, 2\alpha_0)$.

Proof First, applying the hypotheses and integral identity (1.4), we can get the two estimates of (2.3) from the similar proof to that of [3, Chapter 4, Lemma 1.1 and Theorem 1.1].

Next, we take $\eta = [w(x) - w(x_0)]\zeta^2(x)$ in identity (1.4), where x_0 is any point in K_ρ. Then, by [2, Chapter 2, Lemma 5.2] and inequalities (2.1), (2.2) we get from Cauchy's inequality that

$$\begin{aligned}\int_{K_\rho} |w_x|^2\zeta^2(x)\mathrm{d}x \le C\int_{K_\rho}\Big\{|w(x)-w(x_0)|\Big[|w_x|^2\zeta^2+|\zeta_x|^2\Big]+\Big[1+\varphi_1)\Big]\zeta^2\Big\}\mathrm{d}x\\ \le C_1\rho^\alpha\int_{K_\rho}\Big\{|w_x|^2\zeta^2(x)+|\zeta_x|^2\Big\}\mathrm{d}x + C\rho^{2\alpha_0}\int_{K_\rho}|\zeta_x|^2\mathrm{d}x.\end{aligned}$$

If ρ_0 satisfies the condition

$$\rho_0^\alpha C_1 = \frac{1}{2},$$

it follows that estimate (2.4) holds. The lemma is proved. □

Lemma 2.2 *The bounded weak solution w belongs to the class $W_1^2(\Omega_2\bigcap D^{(k)})$. Moreover, weak derivatives $w_{x_jx_s}$ belong to the class $L_2(\Omega_2\bigcap D^{(k)})$ for $1\le i,j\le n$, $1\le s\le n-1$ and have the estimate*

$$\int_{\Omega_2\bigcap D^{(k)}}\Big\{\sum_{j=1}^{n}\sum_{s=1}^{n-1} w_{x_jx_s}^2 + |w_{x_nx_n}|\Big\}\mathrm{d}x \le C(d). \tag{2.5}$$

Proof For an arbitrary sphere K_ρ, $K_{2\rho}\subset\Omega_1$, let $\xi=\xi(x)$ be a smooth function of compact support on K_ρ taking values in $[0,1]$, such that $|\xi_x|\le\frac{C}{\rho}$ in K_ρ and $\xi=1$ in $K_{\frac{\rho}{2}}$. Let

$$x+\Delta x_s := (x_1,...,x_{s-1},x_s+\Delta x_s,...,x_n),\quad w_{(s)} := \frac{\Delta w}{\Delta x_s} = \frac{w(x+\Delta x_s)-w(x)}{\Delta x_s},$$

where $s\le n-1$. Then taking

$$\eta = \frac{\Delta}{\Delta x_s}\Big(\frac{\Delta w(x-\Delta x_s)}{\Delta x_s}\xi^2(x-\Delta x_s)\Big) = \Big(w_{(s)}(x-\Delta x_s)\xi^2(x-\Delta x_s)\Big)_{(s)}$$

in the integral identity (1.4) and applying [2, Chapter 2, formula (4.9)] gives

$$\int_{\Omega_1}\Big\{-\Big(a_i(x,w,w_x)\Big)_{(s)}(w_{(s)}\xi^2)_{x_i} + (f)_{(s)}w_{(s)}\xi^2\Big\}\mathrm{d}x = 0.$$

Note that

$$\begin{aligned}\Big(a_i(x,w,w_x)\Big)_{(s)} =& w_{(s)x_j}\int_0^1\frac{\partial a_i(x^t,w^t,w_x^t)}{\partial w_{x_j}^t}\mathrm{d}t+\\ & w_{(s)}\int_0^1\frac{\partial a_i(x^t,w^t,w_x^t)}{\partial w^t}\mathrm{d}t+\int_0^1\frac{\partial a_i(x^t,w^t,w_x^t)}{\partial x_s^t}\mathrm{d}t,\end{aligned}$$

where $x^t := (1-t)x+t(x+\Delta x_s)$, $w^t(x) := (1-t)w(x)+tw(x+\Delta x_s)$. And $(f)_{(s)}$ has similar equality. Therefore, by Hypotheses 1.1 and Cauchy's inequality we deduce

$$\begin{aligned}\frac{\nu}{2}\int_{\Omega_1}|w_{(s)x}|^2\xi^2\mathrm{d}x \le& \varepsilon\int_{\Omega_1}|w_{(s)x}|^2\xi^2\mathrm{d}x + C(\varepsilon)\int_{\Omega_1}\Big\{\Big[1+|w_{(s)}(x)|^2\Big]\xi^2\int_0^1|w_x^t|^2\mathrm{d}t+\\ & w_{(s)}^2\int_0^1\Big[G_1(x^t)\xi^2(x)+|\xi_x(x)|^2\Big]\mathrm{d}t+\int_0^1\Big[G(x^t)\xi^2(x)+|\xi_x(x)|^2\Big]\mathrm{d}t\Big\}\mathrm{d}x.\end{aligned}$$

Choose $\varepsilon = \nu/4$. From definition of $\xi(x)$ it follows that

$$\begin{aligned}\frac{\nu}{4}\int_{K_\rho}|w_{(s)x}|^2\xi^2\mathrm{d}x \le C\int_{K_\rho}\Big\{&\Big[\Big(|w_x(x)|^2+|w_x(x+\Delta x_s)|^2\Big)\Big(1+w_{(s)}^2\Big)\xi^2\Big]+\\ &\Big[\Big(G_1(x)+G_1(x+\Delta x_s)\Big)w_{(s)}^2\xi^2\Big]+\\ &\Big[\Big(G(x)+G(x+\Delta x_s)\Big)\xi^2\Big]+(w_{(s)}^2+1)|\xi_x|^2\Big\}\mathrm{d}x. \qquad (2.6)\end{aligned}$$

It is easy to know that the function $w(x+\Delta x_s)$ has the similar property to inequality (2.4) of the function $w(x)$. For the right of inequality (2.6), if applying (2.4) with $\zeta(x) = (1+w_{(s)}^2)^{\frac{1}{2}}\xi$ to the first item $[\cdots]$ and applying [2, Chapter 2, Lemma 5.2] to the second item $[\cdots]$ and the third item $[\cdots]$, we obtain

$$\int_{K_\rho}|w_{(s)x}|^2\xi^2\mathrm{d}x \le C_3\int_{K_\rho}\Big\{\rho^{\alpha_1}\Big[|w_{(s)x}|^2\xi^2+(1+w_{(s)}^2)|\xi_x|^2\Big]+(w_{(s)}^2+1)|\xi_x|^2\Big\}\mathrm{d}x.$$

Hence, choosing $\rho_1 := (4C_3)^{-\frac{1}{\alpha_1}}$, from Lemma 2.1 we deduce that if $\rho \le \rho_1$,

$$\int_{K_\rho}|w_{(s)x}|^2\xi^2\mathrm{d}x \le C(d,\rho). \qquad (2.7)$$

This estimate and [3, Chapter 2, Lemma 4.6] give the existence of weak derivatives $w_{x_ix_s}$ in $L_2(\Omega_2\bigcap D^{(k)})$ for $k=1,2$, $1\le i\le n$ and $s\le n-1$. In addition, the integral identity (1.4) and the equation 1.1 imply that $w_{x_nx_n}$ exists in $L_1(\Omega_2\bigcap D^{(k)})$ and has the estimate

$$|w_{x_nx_n}| \le C\Big\{\sum_{i=1}^{n}\sum_{s=1}^{n-1}|w_{x_ix_s}|+(1+\psi_0+\varphi_1+\psi_1|w_x|+|w_x|^2)\Big\}. \qquad (2.8)$$

Consequently, estimate (2.5) is concluded from inequalities (2.7) and (2.8). Function w belongs to $W_1^2(\Omega_2\bigcap D^{(k)})$, $k=1,2$. The lemma is proved. □

Here and below, $\int_{K_\rho}|w_{xx}|\mathrm{d}x$ represents $\int_{K_\rho\bigcap D^{(1)}}|w_{xx}|\mathrm{d}x+\int_{K_\rho\bigcap D^{(2)}}|w_{xx}|\mathrm{d}x$.

Lemma 2.3 *There exist positive constants δ_0, ρ_2, such that w_x belongs to the class $L_4(K_{\frac{\rho}{2}})$ if $0<\delta\le\delta_0$, $\rho\le\rho_2$ and $K_\rho\subset\Omega_2$. In addition, w belongs to the class $W_2^2(\Omega_3\bigcap D^{(k)})$, $k=1,2$, and*

$$\|w_x\|_{L_4(K_{\frac{\rho}{2}})} \le C(d,\rho)/\delta, \quad ||w_{xx}||_{L_2(\Omega_3\bigcap D^{(k)})} \le C(d)/\delta. \qquad (2.9)$$

If $\xi(x)$ is a smooth function of compact support on $K_{\frac{\rho}{2}}$ taking values between 0 and 1, then the following inequality is satisfied for any $l\ge 0$

$$\begin{aligned}&\int_{K_{\frac{\rho}{2}}}\Big\{|w_x|^2h^{l+1}\xi^2+\psi_1^2h^{l+1}\xi^2+\delta\psi_1^2|w_x|^2h^l\xi^2+Gh^l\xi^2\Big\}\mathrm{d}x\\ &\le C(d)\cdot\rho^{\alpha_1}\int_{K_{\frac{\rho}{2}}}\Big\{(|w_x|^4+\sum_{j=1}^{n}\sum_{s=1}^{n-1}w_{x_jx_s}^2)h^l\xi^2+\\ &\quad l^2|h_x|^2h^{l-1}\xi^2+(1+|w_x|^2)h^l|\xi_x|^2\Big\}\mathrm{d}x, \qquad (2.10)\end{aligned}$$

where function $h:=\min\{\sigma,N\}$, $\sigma := 1+\delta a_n^2+\sum_{s=1}^{n-1}w_{x_s}^2$, in which N, δ are constants and N is large enough.

Proof Since the functions h and σ coincide when $\sigma \le N$, the function h and σ satisfy inequalities

$$h,\ \delta a_n^2,\ \sigma \le C(w_x^2+1); \quad |h_x|(\delta a_n^2 + w_{x_s}^2) \le |h_x| h,\ \ s \le n-1. \tag{2.11}$$

Cauchy's inequality shows

$$\delta a_n^2 = \delta\Big(w_{x_j}\int_0^1 \frac{\partial}{\partial p_j} a_n(x,w,p)\Big|_{p=tw_x}\mathrm{d}t + a_n(x,w,0)\Big)^2 \ge \frac{\delta}{2}\nu^2 w_{x_n}^2 - \delta[C_4\sum_{s=1}^{n-1} w_{x_s}^2 + 1].$$

Therefore, if $0<\delta\le\delta_0 := \min\{1, (C_4+\frac{\nu^2}{4})^{-1}\}$, then

$$\sigma \ge \max\{\frac{\delta\nu^2}{4}|w_x|^2, 1\}; \quad |h_x||w_x| \le \frac{2}{\nu\delta^{\frac12}}|h_x|h^{\frac12}; \quad (h+\delta|w_x|^2)\sigma^{-1} \le C. \tag{2.12}$$

In order to prove the lemma, first, let $\theta(x)$ be an arbitrary bounded function in $\overset{\circ}{W}{}^1_2(K_\rho)$. In view of (2.8), since h and σ coincide when $\sigma \le N$, we deduce from the second inequality of (2.11) and (2.12) that

$$\begin{aligned}\int_{K_\rho} |(h^{\frac12}\theta)_x|^2\mathrm{d}x \le& C\int_{K_\rho}\Big\{h^{-1}|h_x|^2\theta^2 + h|\theta_x|^2\Big\}\mathrm{d}x\\ =&C\Big\{\int_{K_\rho\bigcap\{x|\sigma\le N\}} h^{-1}|h_x|^2\theta^2\mathrm{d}x + \int_{K_\rho} h|\theta_x|^2\mathrm{d}x\Big\}\\ \le& C\Big\{\int_{K_\rho\bigcap\{x|\sigma\le N\}} h^2\sigma^{-3}\Big[\delta^2 a_n^2\Big(G+\psi_1^2|w_x|^2+|w_x|^4+\\ &\sum_{j=1}^{n}\sum_{s=1}^{n-1} w_{x_jx_s}^2\Big) + \sum_{j=1}^{n}\sum_{s=1}^{n-1}|w_{x_s}^2|w_{x_jx_s}|\Big]\theta^2\mathrm{d}x + \int_{K_\rho} h|\theta_x|^2\mathrm{d}x\Big\}\\ \le& C_{5,1}\int_{K_\rho}\Big\{\Big(G+\sum_{j=1}^{n}\sum_{s=1}^{n-1} w_{x_jx_s}^2 + \psi_1^2 h + |w_x|^2 h\Big)\theta^2 + h|\theta_x|^2\Big\}\mathrm{d}x\\ <&+\infty.\end{aligned} \tag{2.13}$$

Then this inequality and inner boundary condition (1.2) give us $h^{\frac12}\theta \in \overset{\circ}{W}{}^1_2(K_\rho)$. Next, applying [2, Chapter 2, Lemma 5.2] to $\int_{K_\rho}\psi_1^2 h\theta^2\mathrm{d}x$ and $\int_{K_\rho}(1+\psi_0^2+\varphi_1^2)\theta^2\mathrm{d}x$ and applying inequality (2.4) with $\zeta = h^{\frac12}\theta$ to $\int_{K_\rho}|w_x|^2h\theta^2\mathrm{d}x$, from inequality (2.13) we have

$$\begin{aligned}\int_{K_\rho}(\psi_1^2+|w_x|^2)h\theta^2\mathrm{d}x &\le C\Big(\rho^{2\alpha_0}+\rho^{\alpha_1}\Big)\int_{K_\rho}|(h^{\frac12}\theta)_x|^2\mathrm{d}x\\ &\le C_{5,1}\rho^{\alpha_1}\int_{K_\rho}\Big\{\Big(G+\sum_{j=1}^{n}\sum_{s=1}^{n-1} w_{x_jx_s}^2 + \psi_1^2 h + |w_x|^2 h\Big)\theta^2 + h|\theta_x|^2\Big\}\mathrm{d}x,\end{aligned} \tag{2.14}$$

$$\int_{K_\rho} G\theta^2\mathrm{d}x \le C\rho^{2\alpha_0}\int_{K_\rho}|\theta_x|^2\mathrm{d}x. \tag{2.15}$$

Let $\rho_{2,1} := \min\{\rho_0,\ \rho_1,\ [2C_{5,1}]^{-\frac{1}{2\alpha_1}}\}$. It follows that for $\rho \le \rho_{2,1}$,

$$\int_{K_\rho}(\psi_1^2+|w_x|^2)h\theta^2\mathrm{d}x \le C\rho^{\alpha_1}\int_{K_\rho}\Big\{\sum_{j=1}^{n}\sum_{s=1}^{n-1} w_{x_jx_s}^2\theta^2 + (1+h)|\theta_x|^2\Big\}\mathrm{d}x. \tag{2.16}$$

Next, let θ be smooth function, such that $|\theta_x| \le \frac{C}{\rho}$ for all x and $\theta = 1$ in $K_{\frac{\varrho}{2}}$, and let $N \to \infty$ in inequality (2.16). Lemma 2.2 and the first inequality of (2.11) and (2.12) imply that w_x belongs to the class $L_4(K_{\frac{\varrho}{2}})$ and the first estimate of (2.9) holds. Hence from inequalities (2.5) and (2.8) it follows that w belongs to the class $W_2^2(\Omega_3 \bigcap D^{(k)})$ and the second estimate of (2.9) also holds.

Next let θ be $h^{\frac{l}{2}}\xi$, where $\xi(x)$ is a smooth function of compact support on $K_{\frac{\varrho}{2}}$ taking values between 0 and 1. The similar proof to that of estimate (2.13) shows $\theta(x) \in \overset{\circ}{W}{}^{1}_{2}(K_{\frac{\varrho}{2}})$, which implies that estimate (2.16) is also true.

Since $w_x \in L_4(K_{\frac{\varrho}{2}})$, the similar proof to that of estimate (2.13) also shows

$$\begin{aligned}
&\int_{K_{\frac{\varrho}{2}} \bigcap D^{(k)}} \left(\sigma^{\frac{1}{2}} h^{\frac{l}{2}} \xi\right)_x^2 \mathrm{d}x \\
&\le C \int_{K_{\frac{\varrho}{2}} \bigcap D^{(k)}} \left(\psi_0^2 + \psi_1^4 + |w_x|^4 + |w_{xx}|^2\right) h^l \xi^2 \mathrm{d}x + \\
&\quad \int_{K_{\frac{\varrho}{2}} \bigcap D^{(k)} \bigcap \{x|\sigma \le N\}} \sigma |(h^{\frac{l}{2}})_x|^2 \xi^2 \mathrm{d}x + \int_{K_{\frac{\varrho}{2}} \bigcap D^{(k)}} \sigma h^l |\xi_x|^2 \mathrm{d}x \\
&\le CN^l \int_{K_{\frac{\varrho}{2}} \bigcap D^{(k)}} \left(\psi_0^2 + \psi_1^4 + |w_x|^4 + |w_{xx}|^2\right) \mathrm{d}x + \\
&\quad CN^l \int_{K_{\frac{\varrho}{2}} \bigcap D^{(k)}} l^2 h^{-1} |h_x|^2 \xi^2 \mathrm{d}x + CN^l \int_{K_{\frac{\varrho}{2}} \bigcap D^{(k)}} \left(1 + |w_x|^2\right) |\xi_x|^2 \mathrm{d}x \\
&< +\infty.
\end{aligned}$$

Then inner boundary condition (1.2) gives that $\sigma^{\frac{1}{2}}\theta$ belongs to the class $\overset{\circ}{W}{}^{1}_{2}(K_{\frac{\varrho}{2}})$. Then by [2, Chapter 2, Lemma 5.2] and the similar proof to inequality (2.14), we have

$$\begin{aligned}
&\int_{K_{\frac{\varrho}{2}}} \delta \psi_1^2 |w_x|^2 \theta^2 \le C \int_{K_{\frac{\varrho}{2}}} \psi_1^2 \sigma \theta^2 \le C \rho^{\alpha_1} \int_{K_{\frac{\varrho}{2}}} |(\sigma^{\frac{1}{2}} \theta)_x|^2 \mathrm{d}x \\
&\le C_{5,2} \rho^{\alpha_1} \int_{K_{\frac{\varrho}{2}}} \Big\{ \delta \psi_1^2 |w_x|^2 \theta^2 + \Big[G + |w_x|^4 + \sum_{j=1}^{n} \sum_{s=1}^{n-1} w_{x_j x_s}^2\Big] \theta^2 + \\
&\qquad (1 + |w_x|^2) |\theta_x|^2 \Big\} \mathrm{d}x. \qquad (2.17)
\end{aligned}$$

Finally, inequalities (2.17) and (2.14)–(2.16) yield that when $\rho \le \rho_2 := \min\{\rho_{2,1}, [2C_{5,2}]^{-\frac{1}{2\alpha_1}}\}$, we have

$$\begin{aligned}
&\int_{K_{\frac{\varrho}{2}}} \Big[(\psi_1^2 + |w_x|^2) h + \delta \psi_1^2 |w_x|^2\Big] \theta^2 \mathrm{d}x \\
&\le C \rho^{\alpha_1} \int_{K_{\frac{\varrho}{2}}} \Big\{ \Big(|w_x|^4 + \sum_{j=1}^{n} \sum_{s=1}^{n-1} w_{x_j x_s}^2\Big) \theta^2 + (1 + \sigma) |\theta_x|^2 \Big\} \mathrm{d}x \qquad (2.18)
\end{aligned}$$

with $\theta = h^{\frac{l}{2}}\xi$, which gives inequality (2.10). The lemma is proved. □

On the basis of Lemma 2.3, we are going to estimate $|w_x|_{L_p}$.

Lemma 2.4 *There exist positive constants ρ_3, δ, such that if $0<\rho\le\rho_3$, $K_\rho\subset\Omega_2$, then*

$$\int_{K_{\frac{\rho}{4}}}(|w_{xx}|^2\sigma^r+|w_x|^{2r+4})\mathrm{d}x\le C(r,\delta,1/\rho), \tag{2.19}$$

where $r:=[\frac{q^2+qn}{2(q-n)}-2]+1$, $\sigma=1+\delta a_n^2+\sum_{s=1}^{n-1}w_{x_s}^2$.

Proof Just as above, let h also be $\min\{\sigma,N\}$ and $\xi(x)$ be a smooth function of compact support on $K_{\frac{\rho}{2}}$ taking values between 0 and 1. We shall show that there exists $\rho_{3,1}>0$, such that for $\rho\le\rho_{3,1}$,

$$\begin{aligned}&\int_{K_{\frac{\rho}{2}}}\Big\{\sum_{s=1}^{n-1}\sum_{j=1}^{n}w_{x_jx_s}^2h^l\xi^2+l|h_x|^2h^{l-1}\xi^2\Big\}\mathrm{d}x\\&\le\frac{C(d,r)}{\delta^2}\int_{\frac{\rho}{2}}\Big\{|w_x|^4h^l\xi^2+(|w_x|^2+1)h^l|\xi_x|^2\Big\}\mathrm{d}x.\end{aligned} \tag{2.20}$$

Suppose $0<\delta\le\delta_0$, $s\le n-1$ and $\eta_s(x)$ is a smooth function of compact support on $K_{\frac{\rho}{2}}$ taking values in $[0,1]$. Then integral identity (1.4) and integral by parts yield the equality

$$\begin{aligned}&\sum_{s=1}^{n-1}\int_{K_{\frac{\rho}{2}}}\Big\{\Big[-\frac{\partial}{\partial p_j}a_i(x,w,p)w_{x_jx_s}-\frac{\partial}{\partial x_s}a_i(x,w,p)-\\&\frac{\partial}{\partial w}a_i(x,w,p)w_{x_s}\Big]\Big|_{p=w_x}\eta_{sx_i}+f\eta_{sx_s}\Big\}\mathrm{d}x=0.\end{aligned} \tag{2.21}$$

It is right to choose $\eta_s=w_{x_s}h^l\xi^2(l\ge0)$ (see [3, page 213]). Since

$$\sum_{s=1}^{n-1}\int_{K_{\frac{\rho}{2}}}\frac{\partial a_i}{\partial p_j}\Big|_{p=w_x}lw_{x_jx_s}w_{x_s}h^{l-1}h_{x_i}\xi^2\mathrm{d}x=\frac{l}{2}\int_{K_{\frac{\rho}{2}}}\frac{\partial a_i}{\partial p_j}\Big[h_{x_j}-2\delta a_n\frac{\mathrm{d}a_n}{\mathrm{d}x_j}\Big]h_{x_i}h^{l-1}\xi^2\mathrm{d}x,$$

from inequalities (2.8), (2.11) we can get that

$$\begin{aligned}I:=&\int_{K_{\frac{\rho}{2}}}\Big\{\nu\sum_{s=1}^{n-1}\sum_{j=1}^{n}w_{x_jx_s}^2h^l\xi^2+\frac{l}{2}|h_x|^2h^{l-1}\xi^2\Big\}\mathrm{d}x\\\le&\int_{K_{\frac{\rho}{2}}}\Big\{\frac{\partial}{\partial p_j}a_i\Big|_{p=w_x}\Big[-2\delta a_n\Big(\frac{\partial a_n}{\partial p_k}w_{x_kx_j}+\frac{\partial a_n}{\partial x_j}+\frac{\partial a_n}{\partial w}w_{x_j}\Big)\Big|_{p=w_x}h_{x_i}h^{l-1}\xi^2-\\&2\sum_{s=1}^{n-1}w_{x_s}w_{x_jx_s}h^l\xi\xi_{x_j}\Big]+f\Big(w_{x_sx_s}h^l\xi^2+lw_{x_s}h^{l-1}h_{x_s}\xi^2+2w_{x_s}h^l\xi\xi_{x_s}\Big)\Big\}\mathrm{d}x\\\le&C\int_{K_{\frac{\rho}{2}}}\Big\{\sum_{s=1}^{n-1}\sum_{j=1}^{n}|w_{x_jx_s}|\Big[|w_x|h^l\xi|\xi_x|+l\delta^{\frac12}|h_x|h^{l-\frac12}\xi^2+(\varphi_1+|w_x|^2)h^l\xi^2\Big]+\\&l\delta^{\frac12}\Big[(\psi_0+\varphi_1+|w_x|^2)|h_x|h^{l-\frac12}\xi^2+l\psi_1|h_x|h^l\xi^2\Big]+\\&(\varphi_1+|w_x|^2)(l|w_x||h_x|h^{l-1}\xi^2+|w_x|h^l\xi|\xi_x|)\Big\}\mathrm{d}x.\end{aligned} \tag{2.22}$$

Besides, from the second inequality of (2.12) it follows

$$l(\varphi_1+|w_x|^2)|w_x||h_x|h^{l-1}\xi^2\le\frac{2}{\nu\delta^{\frac12}}l\varphi_1|h_x|h^{l-\frac12}\xi^2+\frac{4l}{\nu^2\delta}|w_x||h_x|h^l\xi^2. \tag{2.23}$$

Then applying Cauchy's inequality and inequalities (2.22), (2.23), (2.8), (2.11) and (2.12) we get

$$\begin{aligned}I \leq &\Big\{\varepsilon_1 \int_{K_{\frac{\rho}{2}}} \sum_{s=1}^{n-1}\sum_{j=1}^{n} w_{x_jx_s}^2 h^l\xi^2 \mathrm{d}x + l\Big[\delta^{\frac{1}{2}} C_6(\varepsilon_1) + \varepsilon_2 + \varepsilon_3\Big] \int_{K_{\frac{\rho}{2}}} |h_x|^2 h^{l-1}\xi^2 \mathrm{d}x + \\ &\frac{C(\varepsilon_1, \varepsilon_2, \varepsilon_3, r)}{\delta^2}\Big\{ \int_{K_{\frac{\rho}{2}}} \Big[G + \psi_1^2 h + |w_x|^4\Big] h^l\xi^2 + \int_{K_{\frac{\rho}{2}}} |w_x|^2 h^l |\xi_x|^2 \mathrm{d}x\Big\},\end{aligned} \tag{2.24}$$

where $\varepsilon_1 = \frac{\nu}{8}$, $\varepsilon_2 = \varepsilon_3 = \frac{1}{8}$, $\delta = \min\{\delta_0, \, [8C_6(\frac{\nu}{8})]^{-2}\}$. Together with inequality (2.10), it gives

$$I \leq \frac{C_7(d,r)\cdot\rho^{\alpha_1}}{\delta^2} I + \frac{C(d,r)}{\delta^2}\int_{K_{\frac{\rho}{2}}} \Big\{|w_x|^4 h^l \xi^2 + (1+|w_x|^2)|h^l|\xi_x|^2\Big\}\mathrm{d}x.$$

Consequently, setting $\rho_{3,1} := \min\{\rho_2, [\frac{\delta^2}{2C_7(d,r)}]^{\frac{1}{\alpha_1}}\}$, we see that inequality (2.20) holds for $\rho \leq \rho_{3,1}$.

Next, we are going to estimate $\int_{K_{\frac{\rho}{2}}} |w_x|^4 h^l \xi^2$. If $l \geq 0$ and x_0 is any point in $K_{\frac{\rho}{2}}$, we conclude from inequalities (2.1), (2.12) and Young's inequality that

$$J := \int_{K_{\frac{\rho}{2}}} a_i \cdot (w(x) - w(x_0))_{x_i} \sigma h^l \xi^2 \mathrm{d}x \geq \int_{K_{\frac{\rho}{2}}} \Big\{\frac{\delta\nu^3}{16}|w_x|^4 - \frac{C}{\delta}\Big\} h^l\xi^2 \mathrm{d}x. \tag{2.25}$$

On the other hand, using (2.11), (2.12), integral by part and the similar proof to that of (2.24) gives

$$\begin{aligned}J = &-\int_{K_{\frac{\rho}{2}}} (w(x) - w(x_0))\Big\{\frac{\mathrm{d}a_i}{\mathrm{d}x_i}\sigma w h^l\xi^2 + a_i\sigma_{x_i} h^l\xi^2 + a_i\sigma l h_{x_i} h^{l-1}\xi^2 + 2a_i\sigma h^l \xi\xi_{x_i}\Big\}\mathrm{d}x \\ \leq & C(d,r)\rho^\alpha \int_{K_{\frac{\rho}{2}}} \Big\{|w_x|^4 h^l\xi^2 + \Big(G + \psi_1^2|w_x|^2 + \sum_{j=1}^{n}\sum_{s=1}^{n-1} w_{x_jx_s}^2\Big)h^l\xi^2 + \\ & l|h_x|^2 h^{l-1}\xi^2 + (1+|w_x|^2)h^l|\xi_x|^2\Big\}\mathrm{d}x.\end{aligned} \tag{2.26}$$

Then, substituting estimates (2.10), (2.20) into inequality (2.26), together with (2.25), we know that there exists $0 < \rho_3 \leq \rho_{3,2}$, such that for $\rho \leq \rho_3$,

$$\int_{K_{\frac{\rho}{2}}} |w_x|^4 h^l\xi^2 \mathrm{d}x \leq C(\delta)\int_{K_{\frac{\rho}{2}}} h^l\xi^2\mathrm{d}x + C(d,r,\delta)\cdot\rho^\alpha \int_{K_{\frac{\rho}{2}}} (1+|w_x|^2)|h^l|\xi_x|^2\mathrm{d}x. \tag{2.27}$$

Finally, to prove estimate (2.19), let $\rho_l^* := \frac{\rho}{2} - \frac{l}{4(r+1)}\rho$ for $l = 0, 1, 2, \ldots, r$ and $\xi_l(x)$ be smooth function of compact support on $K_{\rho_l^*}$ taking values between 0 and 1, such that $|(\xi_l)_x| \leq \frac{C}{\rho}$ in $K_{\rho_l^*}$ and $\xi_l = 1$ in $K_{\rho_{l+1}^*}$. We replace K_ρ and ξ by $K_{\rho_l^*}$ and ξ_l respectively in inequalities (2.20) and (2.27). By considering in succession the inequality (2.20) and (2.27) for $l = 1, 2, \ldots, r$ and letting $N \to \infty$, we can get the estimate (2.19). $\square$

The Proof of Theorem 1.1 Lemma 2.1 gives us the Hölder estimate of w and Lemma 2.3 gives us that $w(x)$ has weak derivatives $w_{x_ix_j}$ in $L_2((\Omega' \cap D^{(k)})$ for $1 \leq i, j \leq n$, which have the estimate (1.5). Furthermore, we can get that w satisfies the equation (1.1) for almost all $x \in \Omega'$ and the inner boundary condition (1.2) for almost all $x \in \Gamma \bigcap \Omega'$ (see [2, Chapter 3, Section 13]). To complete the proof, we need to estimate the bound and Hölder bound of the derivative w_x.

In the identity (2.21), we set $\eta_s = \xi^2 \max\{w_{x_s}(x) - \tau, 0\}$, where $s \le n-1$, τ is an arbitrary number and $\xi(x)$ is a smooth function of compact support on K_ρ taking values in $[0,1]$. If we substitute the function η_s into (2.21) and make the elementary estimates, as we have done several times before, we obtain

$$\begin{aligned}
\int_{A_{\tau,\rho}} |w_{x_s x}|^2 \xi^2 \mathrm{d}x &\le C \int_{A_{\tau,\rho}} \Big\{ \Big[G + \psi_1^2 |w_x|^2 + |w_x|^4 \Big] \xi^2 + (w_{x_s} - \tau)^2 |\xi_x|^2 \Big\} \mathrm{d}x \\
&\le ||G||_{L_{\frac{q}{2}}(K_\rho)} mes^{1-\frac{2}{q}}(A_{\tau,\rho}) + ||\psi_1^2||_{L_{\frac{q}{2}}(K_\rho)} \Big\| |w_x|^2 \Big\|_{L_{qq'/(2(q-q'))}} mes^{1-\frac{2}{q'}}(A_{\tau,\rho}) + \\
&\quad \Big\| |w_x|^4 \Big\|_{L_{s'}(K_\rho)} mes^{1-\frac{1}{s'}}(A_{\tau,\rho}) + C \int_{A_{\tau,\rho}} (w_{x_s} - \tau)^2 |\xi_x|^2 \mathrm{d}x \\
&\le C \int_{A_{\tau,\rho}} (w_{x_s} - \tau)^2 |\xi_x|^2 + C mes^{1-\frac{2}{q''}}(A_{\tau,\rho}),
\end{aligned} \tag{2.28}$$

where $\rho \le \rho_3/4$, $A_{\tau,\rho} := \{x : x \in K_\rho, w_{x_s} > \tau\}$, $q' := \frac{(2r+4)q}{q+2r+4}$, $s' := \frac{r}{2} + 1$ and $q'' := \min\{q, q', 2s'\}$. Since $r = [\frac{q^2+qn}{2(q-n)} - 2] + 1$, we have $2r + 4 > \frac{q^2+qn}{q-n} = \{\frac{q+n}{2} \cdot q\} \div \{q - \frac{q+n}{2}\}$, $\frac{qq'}{q-q'} = 2r+4$, $q > q' > \frac{q+n}{2} > n$, $\frac{1}{s'} < \frac{2}{n}$ and $q'' > n$. Analogous inequalities hold for the sets $B_{\tau,\rho}$, where $w_{x_s} < \tau$. Consequently, together with above inequality and estimate (2.8), the estimate (1.6) follows from the similar proof to that of [3]. The theorem is proved. □

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