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A Note on Adjacent-Vertex-Distinguishing Total Chromatic Numbers for $P_m \times P_n, P_m \times C_n$ and $C_m \times C_n$

CHEN Xiang En[1], *ZHANG Zhong Fu*[1,2], *SUN Yi Rong*[1]
(1. College of Mathematics and Information Science, Northwest Normal University, Gansu 730070, China;
2. Institute of Applied Mathematics, Lanzhou Jiaotong University, Gansu 730070, China)
(E-mail: chenxe@nwnu.edu.cn)

Abstract Let G be a simple graph. Let f be a mapping from $V(G) \cup E(G)$ to $\{1, 2, \ldots, k\}$. Let $C_f(v) = \{f(v)\} \cup \{f(vw)|w \in V(G), vw \in E(G)\}$ for every $v \in V(G)$. If f is a k-proper-total-coloring, and for $u, v \in V(G), uv \in E(G)$, we have $C_f(u) \neq C_f(v)$, then f is called a k-adjacent-vertex-distinguishing total coloring (k-$AVDTC$ for short). Let $\chi_{at}(G) = \min\{k|G$ have a k-adjacent-vertex-distinguishing total coloring$\}$. Then $\chi_{at}(G)$ is called the adjacent-vertex-distinguishing total chromatic number ($AVDTC$ number for short). The $AVDTC$ numbers for $P_m \times P_n, P_m \times C_n$ and $C_m \times C_n$ are obtained in this paper.

Keywords total coloring; adjacent-vertex-distinguishing total coloring; adjacent-vertex-distinguishing total chromatic number.

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1. Introduction

The graphs considered in this paper are connected, finite, undirected and simple graphs. In [1, 2, 3, 5] the vertex-distinguishing proper edge coloring (i.e. strong coloring), proper edge coloring of a graph in which no two of its vertices is incident to edges colored with the same set of colors, was introduced and investigated. In [7] the adjacent strong edge coloring (i.e. adjacent-vertex-distinguishing proper edge coloring), proper edge coloring of a graph G in which no two adjacent vertices of G is incident to edges colored with the same set of colors, was introduced and studied by ZHANG Zhongfu et al. These concepts can be generalized. The adjacent-vertex-distinguishing total coloring was introduced in [8]. A k-proper-total-coloring f of a graph G is a mapping from $V(G) \cup E(G)$ to $\{1, 2, \ldots, k\}$ such that the following 3 conditions are valid:

1) $\forall u, v \in V(G)$, if $uv \in E(G)$, then $f(u) \neq f(v)$;
2) $\forall e_1, e_2 \in E(G), e_1 \neq e_2$, if e_1, e_2 have a common end vertex, then $f(e_1) \neq f(e_2)$;
3) $\forall u \in V(G), e \in E(G)$, if u is an end vertex of e, then $f(u) \neq f(e)$.

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Suppose f is a k-proper-total-coloring of a graph G. Let $C_f(u) = \{f(u)\} \cup \{f(uw)|w \in V(G), uw \in E(G)\}$ and $f(u) = \{1, 2, \ldots, k\} \backslash C_f(u)$ for every $u \in V(G)$. If $\forall u, v \in V(G), uv \in E(G)$, we have $C_f(u) \neq C_f(v)$, i.e., $\overline{C}_f(u) \neq \overline{C}_f(v)$, then f is called a k-adjacent-vertex-distinguishing total coloring (k-$AVDTC$ for short). The number $\min\{k|G$ has a k-adjacent-vertex-distinguishing total-coloring$\}$ is called the adjacent-vertex-distinguishing total chromatic number ($AVDTC$ number for short) of G and is denoted by $\chi_{at}(G)$. The adjacent-vertex-distinguishing total chromatic numbers of cycles, complete graphs, complete bipartite graphs, fans, wheels and trees are obtained[8]. From these results, the authors in [8] proposed the following conjecture.

Conjecture 1[8] *For every graph G with order at least 2, we have $\chi_{at}(G) \leq \Delta(G) + 3$.*

Note that for complete graph G with order odd and at least 3, we have $\chi_{at}(G) = \Delta(G) + 3$.

Let G and H be graphs. Suppose that $V(G) = \{u_1, u_2, \ldots, u_m\}$, $V(H) = \{v_1, v_2, \ldots, v_n\}$. The Cartesian product of G and H , denoted by $G \times H$, is defined as follows: $V(G \times H) = \{w_{ij}|i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$, $E(G \times H) = \{w_{ij}w_{rs}|i = r, v_jv_s \in E(H) \text{or} j = s, v_iv_r \in E(G)\}$. Let P_n be a path with n vertices and C_n be a cycle with n vertices. The adjacent-vertex-distinguishing total coloring on $P_m \times P_n, P_m \times C_n$ and $C_m \times C_n$ are studied and the corresponding chromatic numbers are obtained by constructing 4, 5, 6-$AVDTC$ in this paper. Theorems 1, 2 and 3 in this paper will indicate that Conjecture 1 is valid for $P_m \times P_n, P_m \times C_n$ and $C_m \times C_n$. For the graph-theoretic terminology the reader is referred to [4, 6]. The following lemma is obvious.

Lemma 1 *If arbitrary two distinct vertices of maximum degree in G are not adjacent, then $\chi_{at}(G) \geq \Delta(G) + 1$; If G has two distinct vertices of maximum degree which are adjacent, then $\chi_{at}(G) \geq \Delta(G) + 2$.*

2. The AVDTC number for $P_m \times P_n$

Theorem 1 *Let $2 \leq m \leq n$. Then*
$$\chi_{at}(P_m \times P_n) = \begin{cases} 4, & m = n = 2; \\ 5, & m = 2, n \geq 3 \text{ or } m = n = 3; \\ 6, & m = 3, n \geq 4 \text{ or } n \geq m \geq 4. \end{cases}$$

Proof Assume that $P_m = u_1u_2 \cdots u_m$, $P_n = v_1v_2 \cdots v_n$, and $V(P_m \times P_n) = \{w_{ij}|i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$, $E(P_m \times P_n) = \{w_{ij}w_{rs}|i = r, v_jv_s \in E(P_n) \text{ or } j = s, v_iv_r \in E(P_m)\}$. There are 4 cases to be considered.

Case 1 $m = n = 2$.

In this case, $P_2 \times P_2 = C_4$. Obviously, we have that $\chi_{at}(P_2 \times P_2) = \chi_{at}(C_4) = 4$.

Case 2 $m = 2, n \geq 3$.

In this case, there exist two adjacent vertices of degree 3. So $\chi_{at}(P_2 \times P_n) \geq 5$. In order to prove $\chi_{at}(P_2 \times P_n) = 5$, we only prove that $P_2 \times P_n$ has a 5-$AVDTC$. we construct a mapping f from $V(P_2 \times P_n) \cup E(P_2 \times P_n)$ to $\{1, 2, 3, 4, 5\}$ as follows:

$f(w_{ij}) \in \{1, 2, 3, 4\}$, and $f(w_{ij}) \equiv i + j - 1 \pmod 4, i = 1, 2, j = 1, 2, \ldots, n;$

$f(w_{ij}w_{i,j+1}) \in \{1,2,3,4\}$, and $f(w_{ij}w_{i,j+1}) \equiv i+j+1 \pmod 4, i=1,2, j=1,2,\dots,n-1$;

$f(w_{1j}w_{2j}) = 5, j=1,2,\dots,n$.

Obviously, f is a 5-proper-total-coloring. For $j=2,3,\dots,n-1$, we have

$\overline{C}_f(w_{1j}) = \{1\}, j\equiv 2 \pmod 4$; $\overline{C}_f(w_{1j}) = \{2\}, j\equiv 3 \pmod 4$; $\overline{C}_f(w_{1j}) = \{3\}, j\equiv 0 \pmod 4$; $\overline{C}_f(w_{1j}) = \{4\}, j\equiv 1 \pmod 4$. $\overline{C}_f(w_{2j}) = \{2\}, j\equiv 2 \pmod 4$; $\overline{C}_f(w_{2j}) = \{3\}, j\equiv 3 \pmod 4$; $\overline{C}_f(w_{2j}) = \{4\}, j\equiv 0 \pmod 4$; $\overline{C}_f(w_{2j}) = \{1\}, j\equiv 1 \pmod 4$.

And $\overline{C}_f(w_{11}) \neq \overline{C}_f(w_{21})$, $\overline{C}_f(w_{1n}) \neq \overline{C}_f(w_{2n})$. So f is a 5-$AVDTC$.

Case 3 $m=n=3$.

In this case, there exists only one vertex of maximum degree (=4). So $\chi_{at}(P_3\times P_3)\geq 5$. To prove $\chi_{at}(P_3\times P_3)=5$, we only prove that $P_3\times P_3$ has a 5-$AVDTC$. we construct a mapping f from $V(P_3\times P_3)\cup E(P_3\times P_3)$ to $\{1,2,3,4,5\}$ as follows:

$f(w_{ij}) \in \{1,2,3\}$, and $f(w_{ij}) \equiv i+j-1 \pmod 3, i=1,2,3; j=1,2,3$;

$f(w_{ij}w_{i,j+1}) \in \{1,2,3\}$, and $f(w_{ij}w_{i,j+1}) \equiv i+j+1 \pmod 3, i=1,2,3; j=1,2$;

$f(w_{1j}w_{2j}) = 4, f(w_{2j}w_{3j}) = 5, j=1,2,3$.

Obviously, f is a 5-proper-total-coloring. For every $xy\in E(P_3\times P_3)$, we have $d(x)\neq d(y)$. So f is a 5-$AVDTC$.

Case 4 $m=3, n\geq 4$ or $4\leq m\leq n$.

In this case, there exist two adjacent vertices of maximum degree (=4). So $\chi_{at}(P_m\times P_n)\geq 6$. To prove $\chi_{at}(P_m\times P_n)=6$, we only prove that $P_m\times P_n$ has a 6-$AVDTC$. we construct a mapping f from $V(P_m\times P_n)\cup E(P_m\times P_n)$ to $\{1,2,3,4,5,6\}$ as follows:

$f(w_{ij}) \in \{1,2,3,4\}$, and $f(w_{ij}) \equiv i+j-1 \pmod 4, i=1,2,\dots,m, j=1,2,\dots,n$;

$f(w_{ij}w_{i,j+1}) \in \{1,2,3,4\}$, and $f(w_{ij}w_{i,j+1}) \equiv i+j+1 \pmod 4, i=1,2,\dots,m, j=1,2,\dots,n-1$;

$f(w_{ij}w_{i+1,j}) = 5, j=1,2,\dots,n, i=1,2,\dots,m-1, i$ is odd;

$f(w_{ij}w_{i+1,j}) = 6, j=1,2,\dots,n, i=1,2,\dots,m-1, i$ is even.

Obviously, f is a 6-proper-total-coloring of $P_m\times P_n$.

And for $j=2,3,\dots,n-1$, we have

$C_f(w_{1j}) = \{2,3,4,5\}, j\equiv 2 \pmod 4$; $C_f(w_{1j}) = \{3,4,1,5\}, j\equiv 3 \pmod 4$;

$C_f(w_{1j}) = \{4,1,2,5\}, j\equiv 0 \pmod 4$; $C_f(w_{1j}) = \{1,2,3,5\}, j\equiv 1 \pmod 4$.

For $j=2,3,\dots,n-1$, we have

$C_f(w_{mj}) = \{a(m+j-1), a(m+j), a(m+j+1), 5\}, m$ is even;

$C_f(w_{mj}) = \{a(m+j-1), a(m+j), a(m+j+1), 6\}, m$ is even,

where $a(z)\in\{1,2,3,4\}, a(z)\equiv z \pmod 4$ for integer z. For $i=2,3,\dots,m-1$, we have

$C_f(w_{i1}) = \{2,4,5,6\}, i$ is even; $C_f(w_{i1}) = \{3,1,5,6\}, i$ is odd; $C_f(w_{in}) = \{a(i+n-1), a(i+n), 5, 6\}$.

For $i=2,3,\dots,m-1, j=2,3,\dots,n-1$, we have that $C_f(w_{ij}) = \{a(i+j-1), a(i+j), a(i+j+1), 5, 6\}$. By careful examination, we can get that for every two adjacent vertices x and y of $P_m\times P_n$, $C(x)\neq C(y)$. So f is a 6-$AVDTC$. The proof is completed. □

3. The adjacent-vertex-distinguishing total chromatic number for $P_m \times C_n$

Theorem 2 *Let $m \geq 2, n \geq 3$. Then* $\chi_{at}(P_m \times C_n) = \begin{cases} 5, & m = 2; \\ 6, & m \geq 3. \end{cases}$

Proof Assume that $P_m = u_1u_2\cdots u_m, C_n = v_1v_2\cdots v_nv_1$, and $V(P_m \times C_n) = \{w_{ij}|i = 1,2,\ldots,m, j = 1,2,\ldots,n\}$, $E(P_m \times C_n) = \{w_{ij}w_{rs}|w_{ij}, w_{rs} \in V(P_m \times C_n), i = r, u_ju_s \in E(P_n)$ or $j = s, v_iv_r \in E(C_m)\}$. If $r > m, s > n$, then we assume that $w_{rs} = w_{ij} = w_{rj} = w_{is}$, where $i = 1,2,\ldots,m, j = 1,2,\ldots,n$, and $i \equiv r \pmod m$, $j \equiv s \pmod n$. There are three cases to be considered.

Case 1 $m = 2, n = 3$.

In this case, there are two adjacent vertices of maximum degree (=3). So $\chi_{at}(P_2 \times C_3) \geq 5$. To prove $\chi_{at}(P_2 \times C_3) = 5$, we only prove that $P_2 \times C_3$ has a 5-$AVDTC$. we construct a mapping f from $V(P_2 \times C_3) \cup E(P_2 \times C_3)$ to $\{1,2,3,4,5\}$ as follows:

$f(w_{11}) = 1, f(w_{12}) = 2, f(w_{13}) = 3, f(w_{21}) = 2, f(w_{22}) = 1, f(w_{23}) = 5$; $f(w_{11}w_{12}) = 3, f(w_{12}w_{13}) = 4, f(w_{13}w_{11}) = 2$; $f(w_{21}w_{22}) = 4, f(w_{22}w_{23}) = 2, f(w_{23}w_{21}) = 3$; $f(w_{11}w_{21}) = 5, f(w_{12}w_{22}) = 5, f(w_{13}w_{23}) = 1$.

We may easily verify that f is a 5-proper-total-coloring. And $C_f(w_{11}) = C_f(w_{23}) = \{1,2,3,5\}$; $C_f(w_{12}) = C_f(w_{21}) = \{2,3,4,5\}$; $C_f(w_{22}) = \{1,2,4,5\}, C_f(w_{13}) = \{1,2,3,4\}$; Thus for arbitrary $xy \in E(P_2 \times C_3)$, we have $C(x) \neq C(y)$. So f is a 5-$AVDTC$.

Case 2 $m \geq 3, n = 3$.

In this case, there are two adjacent vertices of maximum degree (=4). So $\chi_{at}(P_m \times C_3) \geq 6$ according to Lemma 1. To prove $\chi_{at}(P_m \times C_3) = 6$, we only prove that $P_m \times C_3$ has a 6-$AVDTC$. we construct a mapping f from $V(P_m \times C_3) \cup E(P_m \times C_3)$ to $\{1,2,3,4,5,6\}$ as follows:

$f(w_{i1}) \in \{1,2,3\}$ and $f(w_{i1}) \equiv i \pmod 3, i = 1,2,\ldots,m$; $f(w_{i1}w_{i+1,1}) \in \{1,2,3\}$ and $f(w_{i1}w_{i+1,1}) \equiv i + 2 \pmod 3, i = 1,2,\ldots,m-1$. $f(w_{ij}) \in \{1,2,3\}$ and $f(w_{ij}) \equiv f(w_{i1}) + j - 1 \pmod 3, j = 2,3, i = 1,2,\ldots,m$; $f(w_{ij}w_{i+1,j}) \in \{1,2,3\}$ and $f(w_{ij}w_{i+1,j}) \equiv f(w_{i1}w_{i+1,1}) + j - 1 \pmod 3, j = 2,3, i = 1,2,\ldots,m-1$; $f(w_{11}w_{12}) = 4, f(w_{12}w_{13}) = 5, f(w_{13}w_{11}) = 6$; $f(w_{ij}w_{i,j+1}) \in \{4,5,6\}$ and $f(w_{ij}w_{i,j+1}) \equiv f(w_{1j}w_{1,j+1}) + i - 1 \pmod 3, i = 2,3,\ldots,m; j = 1,2,3$; So f is a 6-proper-total-coloring.

Note that $C(w_{11})$ does not contain 5, but contains 4 and 6; $C(w_{12})$ does not contain 6, but contains 4 and 5; $C(w_{13})$ does not contain 4, but contains 5 and 6. One of $C(w_{m1})$, $C(w_{m2})$ and $C(w_{m3})$ does not contain 4, but contains 5 and 6; Another does not contain 5, but contains 4 and 6; And the third one does not contain 6, but contains 5 and 4. If $C(w_{ij})$ does not contain 4 (5 or 6), then $C(x)$ must contain 4 (5 or 6) for $i = 3,4,\ldots,m-2, j = 1,2,3$, and $w_{ij}x \in E(P_m \times C_3)$. So f is a 6-$AVDTC$.

Case 3 $m \geq 2, n \geq 4$.

In this case, $\chi_{at}(P_m \times C_n) \geq 5$ if $m = 2$ and $\chi_{at}(P_m \times C_n) \geq 6$ if $m \geq 3$ according to Lemma

1. To prove $\chi_{at}(P_m \times C_n) = 5$ when $m = 2$ or $\chi_{at}(P_m \times C_n) = 6$ when $m \geq 3$, we only prove that $P_m \times C_n$ has a 5-$AVDTC$ when $m = 2$ or 6-$AVDTC$ when $m \geq 3$. we construct a mapping f from $V(P_m \times C_n) \cup E(P_m \times C_n)$ to $\{1, 2, 3, 4, 5\}$ when $m = 2$ or $\{1, 2, 3, 4, 5, 6\}$ when $m \geq 3$ as follows.

Firstly, we give a 4-$AVDTC$ for $(P_m \times C_n)[w_{11}, w_{12}, \dots, w_{1n}]$, which is an n-cycle induced by the vertices $w_{11}, w_{12}, \dots, w_{1n}$.

If $n \equiv 0 \pmod 4$, then we let

$f(w_{1i}w_{1,i+1}) \in \{1, 2, 3, 4\}$ and $f(w_{1i}w_{1,i+1}) \equiv i \pmod 4, i = 1, 2, \dots, n;$

$f(w_{1j}) \in \{1, 2, 3, 4\}$ and $f(w_{1j}) \equiv j + 1 \pmod 4; j = 1, 2, \dots, n.$

If $n \equiv 1 \pmod 4$, then we let

$f(w_{1i}w_{1,i+1}) \in \{1, 2, 3, 4\}$ and $f(w_{1i}w_{1,i+1}) \equiv i \pmod 4, i = 1, 2, \dots, n - 5;$

$f(w_{1j}) \in \{1, 2, 3, 4\}$, and $f(w_{1j}) \equiv j + 1 \pmod 4; j = 1, 2, \dots, n - 5;$

$f(w_{1,n-4}w_{1,n-3}) = 1, f(w_{1,n-3}w_{1,n-2}) = 2,$ $f(w_{1,n-2}w_{1,n-1}) = 4,$ $f(w_{1,n-1}w_{1n}) = 3,$$f(w_{1n}w_{11}) =$ 4;

$f(w_{1,n-1}) = 2, f(w_{1,n-3}) = 3,$ $f(w_{1,n-2}) = 1,$ $f(w_{1,n-1}) = 2, f(w_{1n}) = 1.$

If $n \equiv 2 \pmod 4$, then we let

$f(w_{1i}w_{1,i+1}) \in \{1, 2, 3, 4\}$ and $f(w_{1i}w_{1,i+1}) \equiv i \pmod 4, i = 1, 2, \dots, n - 6;$

$f(w_{1j}) \in \{1, 2, 3, 4\}$, and $f(w_{1j}) \equiv j + 1 \pmod 4; j = 1, 2, \dots, n - 6;$

$f(w_{1,n-5}w_{1,n-4}) = 1, f(w_{1,n-4}w_{1,n-3}) = 2,$ $f(w_{1,n-3}w_{1,n-2}) = 3,$

$f(w_{1,n-2}w_{1,n-1}) = 4,$ $f(w_{1,n-1}w_{1n}) = 3,$ $f(w_{1n}w_{11}) = 4;$

$f(w_{1,n-5}) = 2, f(w_{1,n-4}) = 3,$ $f(w_{1,n-3}) = 4, f(w_{1,n-2}) = 1,$ $f(w_{1,n-1}) = 2, f(w_{1,n}) = 1.$

If $n \equiv 3 \pmod 4$, then we let

$f(w_{1i}w_{1,i+1}) \in \{1, 2, 3, 4\}$ and $f(w_{1i}w_{1,i+1}) \equiv i \pmod 4, i = 1, 2, \dots, n - 7;$

$f(w_{1j}) \in \{1, 2, 3, 4\}$ and $f(w_{1j}) \equiv j + 1 \pmod 4; j = 1, 2, \dots, n - 7;$

$f(w_{1,n-6}w_{1,n-5}) = 1, f(w_{1,n-5}w_{1,n-4}) = 2,$ $f(w_{1,n-4}w_{1,n-3}) = 3, f(w_{1,n-3}w_{1,n-2}) = 1,$

$f(w_{1,n-2}w_{1,n-1}) = 4, f(w_{1,n-1}w_{1n}) = 3,$ $f(w_{1n}w_{11}) = 4;$

$f(w_{1,n-6}) = 2, f(w_{1,n-5}) = 3,$ $f(w_{1,n-4}) = 4, f(w_{1,n-3}) = 2,$

$f(w_{1,n-2}) = 3, f(w_{1,n-1}) = 2, f(w_{1n}) = 1.$

In all above 4 situations, f is a 4-$AVDTC$ of $(P_m \times C_n)[w_{11}, w_{12}, \dots, w_{1n}]$.

Secondly, we extend f. For $i = 2, 3, \dots, m$, let

$f(w_{ij}) \in \{1, 2, 3, 4\}$ and $f(w_{ij}) \equiv f(w_{1j}) \pmod 4, j = 1, 2, \dots, n, i$ is odd;

$f(w_{ij}) \in \{1, 2, 3, 4\}$ and $f(w_{ij}) \equiv f(w_{1,j+1}) \pmod 4, j = 1, 2, \dots, n, i$ is even;

$f(w_{ij}w_{i,j+1}) \in \{1, 2, 3, 4\}$ and $f(w_{ij}w_{i,j+1}) \equiv f(w_{1j}w_{1,j+1}) \pmod 4, j = 1, 2, \dots, n, i$ is odd;

$f(w_{ij}w_{i,j+1}) \in \{1, 2, 3, 4\}$ and $f(w_{ij}w_{i,j+1}) \equiv f(w_{1,j+1}w_{1,j+2}) \pmod 4, j = 1, 2, \dots, n, i$ is even.

For all $i = 1, 2, \dots, m - 1, j = 1, 2, \dots, n$, we let

$f(w_{ij}w_{i+1,j}) = 5$ when i is odd; $f(w_{ij}w_{i+1,j}) = 6$ when i is even.

By simple verification, we know that f is a 5-$AVDTC$ when $m = 2$ or 6-$AVDTC$ when $m \geq 3$. The proof is completed. □

4. The AVDTC number for $C_m \times C_n$

Theorem 3 *Let $m \geq 3, n \geq 3$. Then $\chi_{at}(C_m \times C_n) = 6$.*

Proof Assume that $C_m = u_1u_2 \cdots u_m u_1$, $C_n = v_1v_2 \cdots v_n v_1$, and

$V(C_m \times C_n) = \{w_{ij} | i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$,

$E(C_m \times C_n) = \{w_{ij}w_{rs} | w_{ij}, w_{rs} \in V(C_m \times C_n)$, and $i = r, v_jv_s \in E(C_n)$ or $j = s, u_iu_r \in E(C_m)\}$.

If $r > m$, $s > n$, then we assume that $w_{rs} = w_{ij} = w_{rj} = w_{is}$, where $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$, and $i \equiv r \pmod m, j \equiv s \pmod n$.

Obviously, $\chi_{at}(C_m \times C_n) \geq 6$. To prove $\chi_{at}(C_m \times C_n) = 6$, we only prove that $C_m \times C_n$ has a 6-$AVDTC$. There are three cases to be considered.

Case 1 One of m, n is 3.

Without loss of generality, we assume $m = 3$. There are three subcases to be considered in the following.

Case 1.1 $n \equiv 0 \pmod 3$.

we construct a mapping f from $V(C_3 \times C_n) \cup E(C_3 \times C_n)$ to $\{1, 2, 3, 4, 5, 6\}$ as follows:

$f(w_{1j}) \in \{1, 2, 3\}$ and $f(w_{1j}) \equiv j \pmod 3, j = 1, 2, \ldots, n$;

$f(w_{1j}w_{1,j+1}) \in \{1, 2, 3\}$ and $f(w_{1j}w_{1,j+1}) \equiv j + 2 \pmod 3, j = 1, 2, \ldots, n$.

$f(w_{ij}) \in \{1, 2, 3\}$ and $f(w_{ij}) \equiv f(w_{1j}) + j - 1 \pmod 3, i = 2, 3, j = 1, 2, \ldots, n$;

$f(w_{ij}w_{i,j+1}) \in \{1, 2, 3\}$ and $f(w_{ij}w_{i,j+1}) \in f(w_{1j}w_{1,j+1}) + j - 1 \pmod 3, i = 2, 3, j = 1, 2, \ldots, n$;

$f(w_{11}w_{21}) = 4, f(w_{21}w_{31}) = 5, f(w_{31}w_{11}) = 6$;

$f(w_{ij}w_{i+1,j}) \in \{4, 5, 6\}$ and $f(w_{ij}w_{i+1,j}) \equiv f(w_{i1}w_{i+1,1}) + j - 1 \pmod 3, i = 1, 2, 3; j = 2, 3, \ldots, n$;

Obviously, f is a 6-proper-total-coloring. And

$\overline{C}_f(w_{1j}) = \{5\}, j \equiv 1 \pmod 3; f(w_{1j}) = \{6\}, j \equiv 2 \pmod 3; f(w_{1j}) = \{4\}, j \equiv 0 \pmod 3$;

$f(w_{2j}) = \{6\}, j \equiv 1 \pmod 3; f(w_{2j}) = \{4\}, j \equiv 2 \pmod 3; f(w_{2j}) = \{5\}, j \equiv 0 \pmod 3$;

$f(w_{3j}) = \{4\}, j \equiv 1 \pmod 3; f(w_{3j}) = \{5\}, j \equiv 2 \pmod 3; f(w_{3j}) = \{6\}, j \equiv 0 \pmod 3$. So f is a 6-$AVDTC$.

Case 1.2 $n \equiv 1 \pmod 3$.

we construct a mapping f from $V(C_3 \times C_n) \cup E(C_3 \times C_n)$ to $\{1, 2, 3, 4, 5, 6\}$ as follows:

$f(w_{1j}) \in \{1, 2, 3\}$ and $f(w_{1j}) \equiv j \pmod 3, j = 1, 2, \ldots, n - 3$;

$f(w_{1j}w_{1,j+1}) \in \{1, 2, 3\}$ and $f(w_{1j}w_{1,j+1}) \equiv j + 2 \pmod 3, j = 1, 2, \ldots, n - 3$.

$f(w_{ij}) \in \{1, 2, 3\}$ and $f(w_{ij}) \equiv f(w_{1j}) + j - 1 \pmod 3, i = 2, 3, j = 1, 2, \ldots, n - 3$;

$f(w_{ij}w_{i,j+1}) \in \{1, 2, 3\}$ and $f(w_{ij}w_{i,j+1}) \equiv f(w_{1j}w_{1,j+1}) + j - 1 \pmod 3, i = 2, 3, j = 1, 2, \ldots, n - 3$;

$f(w_{11}w_{21}) = 4, f(w_{21}w_{31}) = 5, f(w_{31}w_{11}) = 6$;

$f(w_{ij}w_{i+1,j}) \in \{4,5,6\}$ and $f(w_{ij}w_{i+1,j}) \equiv f(w_{i1}w_{i+1,1}) + j - 1 \pmod 3, i = 1,2,3; j = 2,3,\ldots,n-3$;

$f(w_{1,n-2}) = 2, f(w_{1,n-1}) = 4, f(w_{1n}) = 3, f(w_{2,n-2}) = 3, f(w_{2,n-1}) = 5$;

$f(w_{2n}) = 1, f(w_{3,n-2}) = 4, f(w_{3,n-1}) = 6, f(w_{3n}) = 2$;

$f(w_{1,n-2}w_{1,n-1}) = 5, f(w_{1,n-1}w_{1,n}) = 1, f(w_{1n}w_{11}) = 2$;

$f(w_{2,n-2}w_{2,n-1}) = 6, f(w_{2,n-1}w_{2,n}) = 2, f(w_{2n}w_{21}) = 3$;

$f(w_{3,n-2}w_{3,n-1}) = 1, f(w_{3,n-1}w_{3,n}) = 3, f(w_{3n}w_{31}) = 1$;

$f(w_{1,n-2}w_{2,n-2}) = 4, f(w_{2,n-2}w_{3,n-2}) = 5, f(w_{3,n-2}w_{1,n-2}) = 6$;

$f(w_{1,n-1}w_{2,n-1}) = 3, f(w_{2,n-1}w_{3,n-1}) = 4, f(w_{3,n-1}w_{1,n-1}) = 2$;

$f(w_{1,n}w_{2,n}) = 6, f(w_{2,n}w_{3,n}) = 4, f(w_{3,n}w_{1,n}) = 5$.

We may verify that f is a 6-proper-total-coloring. And

$\overline{C}_f(w_{1j}) = \{5\}, j \equiv 1 \pmod 3$; $\overline{C}_f(w_{1j}) = \{6\}, j \equiv 2 \pmod 3$; $\overline{C}_f(w_{1j}) = \{4\}, j \equiv 0 \pmod 3$;

$\overline{C}_f(w_{2j}) = \{6\}, j \equiv 1 \pmod 3$; $\overline{C}_f(w_{2j}) = \{4\}, j \equiv 2 \pmod 3$; $\overline{C}_f(w_{2j}) = \{5\}, j \equiv 0 \pmod 3$;

$\overline{C}_f(w_{3j}) = \{4\}, j \equiv 1 \pmod 3$; $\overline{C}_f(w_{3j}) = \{5\}, j \equiv 2 \pmod 3$; $\overline{C}_f(w_{3j}) = \{6\}, j \equiv 0 \pmod 3$.

$\overline{C}_f(w_{1,n-2}) = \{1\}, \overline{C}_f(w_{1,n-1}) = \{6\}, \overline{C}_f(w_{1n}) = \{4\}, \overline{C}_f(w_{2,n-2}) = \{2\}, \overline{C}_f(w_{2,n-1}) = \{1\}$,

$\overline{C}_f(w_{2n}) = \{5\}, \overline{C}_f(w_{3,n-2}) = \{3\}, \overline{C}_f(w_{3,n-1}) = \{5\}, \overline{C}_f(w_{3n}) = \{6\}$.

So f is a 6-$AVDTC$.

Case 1.3 $n \equiv 1 \pmod 3$.

we construct a mapping f from $V(C_3 \times C_n) \cup E(C_3 \times C_n)$ to $\{1,2,3,4,5,6\}$ as follows:

$f(w_{1j}) \in \{1,2,3\}$ and $f(w_{1j}) \equiv j \pmod 3, j = 1,2,\ldots,n-4$;

$f(w_{1j}w_{1,j+1}) \in \{1,2,3\}$ and $f(w_{1j}w_{1,j+1}) \equiv j+2 \pmod 3, j = 1,2,\ldots,n-4$.

$f(w_{ij}) \in \{1,2,3\}$ and $f(w_{ij}) \equiv f(w_{1j}) + j - 1 \pmod 3, i = 2,3, j = 1,2,\ldots,n-4$;

$f(w_{ij}w_{i,j+1}) \in \{1,2,3\}$ and $f(w_{ij}w_{i,j+1}) \equiv f(w_{1j}w_{1,j+1}) + j - 1 \pmod 3, i = 2,3, j = 1,2,\ldots,n-4$;

$f(w_{11}w_{21}) = 4, f(w_{21}w_{31}) = 5, f(w_{31}w_{11}) = 6$;

$f(w_{ij}w_{i+1,j}) \in \{4,5,6\}$ and $f(w_{ij}w_{i+1,j}) \equiv f(w_{i1}w_{i+1,1}) + j - 1 \pmod 3, i = 1,2,3; j = 2,3,\ldots,n-4$;

$f(w_{1,n-3}) = 2, f(w_{1,n-2}) = 6, f(w_{1,n-1}) = 4, f(w_{1n}) = 3, f(w_{2,n-3}) = 3, f(w_{2,n-2}) = 4$,

$f(w_{2,n-1}) = 5, f(w_{2n}) = 1, f(w_{3,n-3}) = 4, f(w_{3,n-2}) = 5, f(w_{3,n-1}) = 6, f(w_{3n}) = 2$,

$f(w_{1,n-3}w_{1,n-2}) = 5, f(w_{1,n-2}w_{1,n-1}) = 2, f(w_{1,n-1}w_{1,n}) = 1, f(w_{1n}w_{11}) = 2$;

$f(w_{2,n-3}w_{2,n-2}) = 6, f(w_{2,n-2}w_{2,n-1}) = 3, f(w_{2,n-1}w_{2,n}) = 2, f(w_{2n}w_{21}) = 3$;

$f(w_{3,n-3}w_{3,n-2}) = 1, f(w_{3,n-2}w_{3,n-1}) = 4, f(w_{3,n-1}w_{3,n}) = 3, f(w_{3n}w_{31}) = 1$;

$f(w_{1,n-3}w_{2,n-3}) = 4, f(w_{2,n-3}w_{3,n-3}) = 5, f(w_{3,n-3}w_{1,n-3}) = 6, f(w_{1,n-2}w_{2,n-2}) = 1$;

$f(w_{2,n-2}w_{3,n-2}) = 2, f(w_{3,n-2}w_{1,n-2}) = 3, f(w_{1,n-1}w_{2,n-1}) = 6, f(w_{2,n-1}w_{3,n-1}) = 1$;

$f(w_{3,n-1}w_{1,n-1}) = 5, f(w_{1,n}w_{2,n}) = 6, f(w_{2,n}w_{3,n}) = 4, f(w_{3,n}w_{1,n}) = 5$.

We may verify that f is a 6-proper-total-coloring. And $\overline{C}_f(w_{1j}) = \{5\}, j \equiv 1 \pmod 3; \overline{C}_f(w_{1j}) = \{6\}, j \equiv 2 \pmod 3; \overline{C}_f(w_{1j}) = \{4\}, j \equiv 0 \pmod 3$;

$\overline{C}_f(w_{2j}) = \{6\}, j \equiv 1 \pmod 3; \overline{C}_f(w_{2j}) = \{4\}, j \equiv 2 \pmod 3; \overline{C}_f(w_{2j}) = \{5\}, j \equiv 0 \pmod 3$;

$\overline{C}_f(w_{3j}) = \{4\}, j \equiv 1 \pmod 3; \overline{C}_f(w_{3j}) = \{5\}, j \equiv 2 \pmod 3; \overline{C}_f(w_{3j}) = \{6\}, j \equiv 0 \pmod 3$.

$\overline{C}_f(w_{1,n-3}) = \{1\}, \overline{C}_f(w_{1,n-2}) = \{4\}, \overline{C}_f(w_{1,n-1}) = \{3\}, \overline{C}_f(w_{1n}) = \{4\}$;
$\overline{C}_f(w_{2,n-3}) = \{2\}, \overline{C}_f(w_{2,n-2}) = \{5\}, \overline{C}_f(w_{2,n-1}) = \{4\}, \overline{C}_f(w_{2n}) = \{5\}$;
$\overline{C}_f(w_{3,n-3}) = \{3\}, \overline{C}_f(w_{3,n-2}) = \{6\}, \overline{C}_f(w_{3,n-1}) = \{2\}, \overline{C}_f(w_{3n}) = \{6\}$.
So f is a 6-$AVDTC$.

Case 2 One of m, n is even.

Without loss of generality, we assume that m is even. We construct a mapping f from $V(C_m \times C_n) \cup E(C_m \times C_n)$ to $\{1, 2, 3, 4, 5, 6\}$ as follows.

Firstly, similar to Case 3 of the proof of Theorem 3, we can give a 4-$AVDTC$ f for $(C_m \times C_n)[w_{11}, w_{12}, \ldots, w_{1n}]$, which is a cycle induced by the vertices $w_{11}, w_{12}, \ldots, w_{1n}$.

Secondly, we extend f. Let

$f(w_{ij}) \in \{1, 2, 3, 4\}$ and $f(w_{ij}) = f(w_{1j})$, $i = 2, 3, \ldots, m, j = 1, 2, \ldots, n, i$ is odd;
$f(w_{ij}) \in \{1, 2, 3, 4\}$ and $f(w_{ij}) = f(w_{1,j+1})$, $i = 2, 3, \ldots, m, j = 1, 2, \ldots, n, i$ is even.
$f(w_{ij}w_{i+1,j}) = 5, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, i$ is odd; $f(w_{ij}w_{i+1,j}) = 6$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, i$ is even.

We may easily verify that f is a 6-$AVDTC$ of $C_m \times C_n$.

Case 3 m, n are all odd and $m \geq 5, n \geq 5$. There are two subcases to be considered.

Case 3.1 $n \equiv 1 \pmod 4$.

We construct a mapping f from $V(C_m \times C_n) \cup E(C_m \times C_n)$ to $\{1, 2, 3, 4, 5, 6\}$ as follows. Let
$f(w_{1j}) \in \{1, 2, 3, 4\}$ and $f(w_{1j}) \equiv j + 1 \pmod 4, j = 1, 2, \ldots, n - 4$;
$f(w_{1j}w_{1,j+1}) \in \{1, 2, 3, 4\}$ and $f(w_{1j}w_{1,j+1}) \equiv j \pmod 4, j = 1, 2, \ldots, n - 4$.
$f(w_{1,n-3}) = 3, f(w_{1,n-2}) = 1, f(w_{1,n-1}) = 2, f(w_{1,n}) = 1$;
$f(w_{1,n-3}w_{1,n-2}) = 2, f(w_{1,n-2}w_{1,n-1}) = 4, f(w_{1,n-1}w_{1,n}) = 3, f(w_{1,n}w_{11}) = 4$.
$f(w_{2j}) \in \{1, 2, 3, 4\}$ and $f(w_{2j}) \equiv j + 2 \pmod 4, j = 1, 2, \ldots, n - 4$;
$f(w_{2j}w_{2,j+1}) \in \{1, 2, 3, 4\}$ and $f(w_{2j}w_{2,j+1}) \equiv j + 1 \pmod 4, j = 1, 2, \ldots, n - 4$.
$f(w_{2,n-3}) = 4, f(w_{2,n-2}) = 2, f(w_{2,n-1}) = 3, f(w_{2,n}) = 2$;
$f(w_{2,n-3}w_{2,n-2}) = 3, f(w_{2,n-2}w_{2,n-1}) = 1, f(w_{2,n-1}w_{1,n}) = 4, f(w_{2,n}w_{21}) = 1$.
$f(w_{mj}) \in \{1, 2, 3, 4\}$ and $f(w_{mj}) \equiv j + 3 \pmod 4, j = 1, 2, \ldots, n - 4$;
$f(w_{mj}w_{m,j+1}) \in \{1, 2, 3, 4\}$ and $f(w_{mj}w_{m,j+1}) \equiv j + 1 \pmod 4, j = 1, 2, \ldots, n - 4$.
$f(w_{m,n-3}) = 6, f(w_{m,n-2}) = 3, f(w_{m,n-1}) = 5, f(w_{m,n}) = 3$;
$f(w_{m,n-3}w_{m,n-2}) = 1, f(w_{m,n-2}w_{1,n-1}) = 2, f(w_{m,n-1}w_{m,n}) = 4, f(w_{m,n}w_{m1}) = 1$.
For $i = 3, 4, \ldots, m - 1, j = 1, 2, \ldots, n$, let
$f(w_{ij}) = f(w_{1,j})$, if i is odd; $f(w_{ij}) = f(w_{2,j})$, if i is even;
$f(w_{ij}w_{i,j+1}) = f(w_{1j}w_{1,j+1})$, if i is odd; $f(w_{ij}w_{i,j+1}) = f(w_{2j}w_{2,j+1})$, if i is even.

For $i = 1, 2, \ldots, m - 2$, $j = 1, 2, \ldots, n - 2$, let $f(w_{ij}w_{i+1,j}) \in \{5, 6\}$, $f(w_{ij}w_{i+1,j}) \equiv i + j - 1 \pmod 2$. For $i = 1, 2, \ldots, m - 2$, $j = n - 1, n$, let $f(w_{ij}w_{i+1,j}) \in \{5, 6\}, f(w_{ij}w_{i+1,j}) \equiv i + j \pmod 2$. And let
$f(w_{m-1,j}w_{m,j}) \in \{5, 6\}, f(w_{m-1,j}w_{m,j}) \equiv j + 1 \pmod 2, j = 1, 2, \ldots, n - 3$;
$f(w_{m-1,n-2}w_{m,n-2}) = 4, f(w_{m-1,n-1}w_{m,n-1}) = 6, f(w_{m-1,n}w_{m,n}) = 5$.

$f(w_{m,j}w_{1,j}) \in \{1,2,3,4\}, f(w_{m,j}w_{1,j}) \equiv j+2 \pmod 4, j=1,2,\ldots,n-3;$

$f(w_{m,n-2}w_{1,n-2})=6, f(w_{m,n-1}w_{1,n-1})=1, f(w_{m,n}w_{1n})=2.$

We may verify that f is a 6-proper-total-coloring. Let $B_i=(\overline{C}_f(w_{i1}),\overline{C}_f(w_{i2}),\ldots,\overline{C}_f(w_{in})), i=1,2,\ldots,m$. We have

$B_1=(6,5,6,5,6,5,6,5,\ldots,6,5,6,5,6,5,3,6,5);$

$B_i=(4,1,2,3,4,1,2,3,\ldots,4,1,2,3,4,1,4,2,3), i=2,3,\ldots,m-2, i$ is even;

$B_i=(3,4,1,2,3,4,1,2,\ldots,3,4,1,2,3,4,3,1,2), i=2,3,\ldots,m-2, i$ is odd;

$B_{m-1}=(4,1,2,3,4,1,2,3,\ldots,4,1,2,3,4,1,6,2,3),$

$B_m=(5,6,5,6,5,6,5,6,\ldots,5,6,5,6,5,3,5,3,6).$

So f is a 6-$AVDTC$ of $C_m \times C_n$.

Case 3.2 $n \equiv 3 \pmod 4$.

We construct a mapping f from $V(C_m \times C_n) \cup E(C_m \times C_n)$ to $\{1,2,3,4,5,6\}$ as follows. Let

$f(w_{1j}) \in \{1,2,3,4\}$ and $f(w_{1j}) \equiv j+1 \pmod 4, j=1,2,\ldots,n-6;$

$f(w_{1j}w_{1,j+1}) \in \{1,2,3,4\}$ and $f(w_{1j}w_{1,j+1}) \equiv j \pmod 4, j=1,2,\ldots,n-6.$

$f(w_{1,n-5})=3, f(w_{1,n-4})=4, f(w_{1,n-3})=2, f(w_{1,n-2})=3, f(w_{1,n-1})=2, f(w_{1,n})=1;$

$f(w_{1,n-5}w_{1,n-4})=2, f(w_{1,n-4}w_{1,n-3})=3, f(w_{1,n-3}w_{1,n-2})=1,$

$f(w_{1,n-2}w_{1,n-1})=4, f(w_{1,n-1}w_{1,n})=3, f(w_{1,n}w_{11})=4.$

$f(w_{2j}) \in \{1,2,3,4\}$ and $f(w_{2j}) \equiv j+2 \pmod 4, j=1,2,\ldots,n-6;$

$f(w_{2j}w_{2,j+1}) \in \{1,2,3,4\}$ and $f(w_{2j}w_{2,j+1}) \equiv j+1 \pmod 4, j=1,2,\ldots,n-6.$

$f(w_{2,n-5})=4, f(w_{2,n-4})=1, f(w_{2,n-3})=3, f(w_{2,n-2})=4, f(w_{2,n-1})=3, f(w_{2,n})=2;$

$f(w_{2,n-5}w_{2,n-4})=3, f(w_{2,n-4}w_{2,n-3})=4, f(w_{2,n-3}w_{2,n-2})=2,$

$f(w_{2,n-2}w_{2,n-1})=1, f(w_{2,n-1}w_{1,n})=4, f(w_{2,n}w_{21})=1.$

$f(w_{mj}) \in \{1,2,3,4\}$ and $f(w_{mj}) \equiv j+3 \pmod 4, j=1,2,\ldots,n-6;$

$f(w_{mj}w_{m,j+1}) \in \{1,2,3,4\}$ and $f(w_{mj}w_{m,j+1}) \equiv j+1 \pmod 4, j=1,2,\ldots,n-6.$

$f(w_{m,n-5})=1, f(w_{m,n-4})=5, f(w_{m,n-3})=6, f(w_{m,n-2})=5, f(w_{m,n-1})=1, f(w_{m,n})=3;$

$f(w_{m,n-5}w_{m,n-4})=3, f(w_{m,n-4}w_{1,n-3})=2, f(w_{m,n-3}w_{m,n-2})=1,$

$f(w_{m,n-2}w_{1,n-1})=3, f(w_{m,n-1}w_{m,n})=4, f(w_{m,n}w_{m1})=1.$

For $i=3,4,\ldots,m-1, j=1,2,\ldots,n$, let

$f(w_{ij})=f(w_{1,j})$, if i is odd; $f(w_{ij})=f(w_{2,j})$, if i is even;

$f(w_{ij}w_{i,j+1})=f(w_{1j}w_{1,j+1})$, if i is odd; $f(w_{ij}w_{i,j+1})=f(w_{2j}w_{2,j+1})$, if i is even.

For $i=1,2,\ldots,m-2, j=1,2,\ldots,n-2$, let $f(w_{ij}w_{i+1,j}) \in \{5,6\}$, $f(w_{ij}w_{i+1,j}) \equiv i+j-1 \pmod 2$. For $i=1,2,\ldots,m-2, j=n-1,n$, let $f(w_{ij}w_{i+1,j}) \in \{5,6\}$, $f(w_{ij}w_{i+1,j}) \equiv i+j \pmod 2$. And let

$f(w_{m-1,j}w_{m,j}) \in \{5,6\}, f(w_{m-1,j}w_{m,j}) \equiv j+1 \pmod 2, j=1,2,\ldots,n-2;$

$f(w_{m-1,n-1}w_{m,n-1})=2, f(w_{m-1,n}w_{m,n})=5$

$f(w_{m,j}w_{1,j}) \in \{1,2,3,4\}, f(w_{m,j}w_{1,j}) \equiv j+2 \pmod 4, j=1,2,\ldots,n-4;$

$f(w_{m,n-3}w_{1,n-3})=4, f(w_{m,n-2}w_{1,n-2})=2, f(w_{m,n-1}w_{1,n-1})=6, f(w_{m,n}w_{1n})=2.$

We may verify that f is a 6-proper-total-coloring.

Let $B_i=(\overline{C}_f(w_{i1}),\overline{C}_f(w_{i2}),\ldots,\overline{C}_f(w_{in}))$, $i=1,2,\ldots,m$. We have

$B_1 = (6, 5, 6, 5, 6, 5, 6, 5, \dots, 6, 5, 6, 5, 6, 5, 6, 5, 6, 1, 5)$;

$B_i = (4, 1, 2, 3, 4, 1, 2, 3, \dots, 4, 1, 2, 3, 4, 1, 2, 1, 3, 2, 3), i = 2, 3, \dots, m-2, i$ is even;

$B_i = (3, 4, 1, 2, 3, 4, 1, 2, \dots, 3, 4, 1, 2, 3, 4, 1, 4, 2, 1, 2), i = 2, 3, \dots, m-2, i$ is odd;

$B_{m-1} = (4, 1, 2, 3, 4, 1, 2, 3, \dots, 4, 1, 2, 3, 4, 1, 2, 1, 3, 6, 3)$,

$B_m = (5, 6, 5, 6, 5, 6, 5, 6, \dots, 5, 6, 5, 6, 5, 6, 4, 3, 4, 5, 6)$.

So f is a 6-$AVDTC$ of $C_m \times C_n$.

The proof is completed. □

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