Journal of Mathematical Research & Exposition
Nov., 2008, Vol. 28, No. 4, pp. 823–828
DOI:10.3770/j.issn:1000-341X.2008.04.010
Http://jmre.dlut.edu.cn

Equitable Total Coloring of Some Join Graphs

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Abstract The total chromatic number $\chi_t(G)$ of a graph $G(V,E)$ is the minimum number of total independent partition sets of $V\bigcup E$, satisfying that any two sets have no common element. If the difference of the numbers of any two total independent partition sets of $V\bigcup E$ is no more than one, then the minimum number of total independent partition sets of $V\bigcup E$ is called the equitable total chromatic number of G, denoted by $\chi_{et}(G)$. In this paper, we have obtained the equitable total chromatic number of $W_m\bigvee K_n$, $F_m\bigvee K_n$ and $S_m\bigvee K_n$ while $m\geq n\geq 3$.

Keywords equitable total coloring; equitable total chromatic number; join graph; equitable edge coloring.

Document code A
MR(2000) Subject Classification 05C15
Chinese Library Classification O157.5

1. Introduction

In this paper we only consider finite simple graphs and we use the standard notation of graph theory. Definitions not given here may be found in [2]. Let $G(V,E)$ be a graph with the set of vertices V and the edge set E. Total coloring was introduced by Vizing[4] and Behzad[1]. They both conjectured that for any graph G the following inequality holds:

$$\Delta(G)+1\leq\chi_t(G)\leq\Delta(G)+2.$$

It is obvious that $\Delta(G)+1$ is the best possible lower bound. The conjecture is proved so far for some specific classes of graphs. And the concept of equitable total coloring was presented in [3] and [5]. In general the equitable total coloring problem is more difficult than the total coloring problem. In [6], the equitable total chromatic numbers of some join graphs were given. In this paper, the equitable total coloring of $W_m\bigvee K_n$, $F_m\bigvee K_n$ and $S_m\bigvee K_n$ $(m\geq n\geq 3)$ have been studied.

Definition 1 *For a k-proper edge coloring f of graph G, if $||E_i(G)|-|E_j(G)||\leq 1$, $i,j=$*

Received date: 2006-11-13; **Accepted date**: 2007-03-23
Foundation item: the National Natural Science Foundation of China (No. 10771091).

$0,1,\ldots,k-1$, where $E_i(G)$ is the set of edges of color i in G, then f is called a k-equitable edge coloring of graph G, and

$$\chi'_e(G) = \min\{k | \text{there is a } k\text{-equitable edge coloring of graph } G\}$$

is called the equitable edge chromatic number of G.

Definition 2[6] *For a simple graph $G(V,E)$, the edges and vertices are called the elements of $G(V,E)$, and elements are called independent if any two of them are neither incident nor adjacent. If k is a natural number and $V\bigcup E = \bigcup_{i=0}^{k-1}(V_i\bigcup E_i)$ satisfies:*

(1) *The elements of $V_i\bigcup E_i$ are independent, $i=0,1,2,\ldots,k-1$;*

(2) $(V_i\bigcup E_i)\bigcap(V_j\bigcup E_j)=\emptyset$, $i,j=0,1,2,\ldots,k-1$, and $i\neq j$;

(3) $||V_i\bigcup E_i|-|V_j\bigcup E_j||\leq 1$, $i,j=0,1,2,\ldots,k-1$,

then the partition $\{V_i\bigcup E_i \mid 0\leq i\leq k-1\}$ is called a k-equitable total coloring of G, and

$$\chi_{et}(G) = \min\{k | \text{there is a } k\text{-equitable total coloring of graph } G\}$$

is called the equitable total chromatic number of G.

Definition 3[2] *The join graph $G\bigvee H$ of disjoint graphs G and H is defined as follows:*

$$V(G\bigvee H) = V(G)\bigcup V(H),\ E(G\bigvee H) = E(G)\bigcup E(H)\bigcup\{uv|u\in V(G), v\in V(H)\}.$$

Definition 4 *Let $m\geq 2$, $n\geq 3$, We define star S_m, fan F_m and wheel W_n as follows:*

$$\begin{aligned}
&V(S_m) = \{u_i \mid i=0,1,2,\ldots,m\},\\
&E(S_m) = \{u_0u_i \mid i=1,2,\ldots,m\};\\
&V(F_m) = \{u_i \mid i=0,1,2,\ldots,m\},\\
&E(F_m) = \{u_0u_i \mid i=1,2,\ldots,m\}\bigcup\{u_iu_{i+1} \mid i=1,2,\ldots,m-1\};\\
&V(W_n) = \{v_i \mid i=0,1,2,\ldots,n\},\\
&E(W_n) = \{v_0v_i \mid i=1,2,\ldots,n\}\bigcup\{v_iv_{i+1} \mid i=1,2,\ldots,n-1\}\bigcup\{v_nv_1\}.
\end{aligned}$$

2. Main results

Lemma 1[2] *For a complete graph K_p of order p,*

$$\chi'(K_p) = \begin{cases} p & if\ p \equiv 1 \pmod 2 \\ p-1 & if\ p \equiv 0 \pmod 2 \end{cases}$$

Lemma 2 *For any subgraph H of a graph G, $\chi'(H)\leq\chi'(G)$.*

Lemma 3 *For a finite simple graph G, $\chi'_e(G)=\chi'(G)$.*

Proof Let f_1 be a k-proper edge coloring of G, where $k=\chi'(G)$. If there exist two colors i and j, such that $||E_i(G)|-|E_j(G)||\geq 2$, then notice that the graph G_{ij} is composed of the edges colored with color i and color j, each branch of G_{ij} is either a path or an even cycle. It is obvious that we can recolor the edges of G_{ij} just with color i and j, such that $||E_i(G_{ij})| - |E_j(G_{ij})|| \leq 1$. After recoloring the corresponding edges of G, we get a new edge coloring f_2 of G. Under edge coloring f_2, $||E_i(G)| - |E_j(G)|| \leq 1$. If there also exist two colors i_1 and j_1, such that $||E_{i_1}| - |E_{j_1}|| \geq 2$, repeat the process above. After finite steps, we can get a k-proper edge coloring f of G. Under edge coloring f, for any $i, j = 0, 1, \ldots, k-1$, $||E_i(G)| - |E_j(G)|| \leq 1$.

Lemma 4[7] *For a simple graph G, $\chi_{et}(G) \geq \chi_t(G) \geq \Delta(G) + 1$.*

Lemma 5 *For a simple graph G of order p, if $\Delta(G) = p-1$, and $\chi'(G \bigvee \{w\}) = p$ for $w \notin V(G)$, then $\chi_{et}(G) = p$.*

Proof By Lemma 3, $\chi'_e(G \bigvee \{w\}) = \chi'(G \bigvee \{w\})$. Notice that $G' = G \bigvee \{w\}$ is obtained by adding a new vertex w to G and adding an edge joining w to each vertex in G. Let g be a p-equitable edge coloring of $G \bigvee \{w\}$. Now we turn g into a p-equitable total coloring f of G as follows:

$$\forall\ v \in V(G),\ f_G(v) = g_{G'}(wv); \forall\ e \in E(G),\ f_G(e) = g_{G'}(e).$$

It is obvious that f is a p-equitable total coloring of G. By Lemma 4, $\chi_{et}(G) \geq \chi_t(G) \geq \Delta(G) + 1 = p$, so $\chi_{et}(G) = p$.

Lemma 6 *For an edge coloring of a graph G, $E_i(G)$ denotes the set of edges of color i. Let the color of some edges of graph G be restricted to be j. Under this restriction, if there exists an α-edge coloring of graph G, such that $\mid E_j(G) \mid = \lfloor \varepsilon/\alpha \rfloor$ or $\lceil \varepsilon/\alpha \rceil$, where $\varepsilon = \mid E(G) \mid$, $\alpha = \chi'(G)$, then under this restriction there exists an α-equitable edge coloring of graph G.*

Proof Let g be an α-edge coloring of graph G under the restriction. So the coloring under $G - E_j(G)$ is an $(\alpha - 1)$-edge coloring. By Lemma 3, there exists an $(\alpha - 1)$-equitable edge coloring g'_e of $G - E_j(G)$. Then $E_j(G)$ combining with the edge independent sets of g'_e constitutes an α-edge coloring of graph G.

Case 1 If $\mid E_j(G) \mid = \lfloor \varepsilon/\alpha \rfloor$, we denote $\lfloor \varepsilon/\alpha \rfloor$ by c. So $c \leq \varepsilon/\alpha < c + 1$, $c\alpha \leq \varepsilon < (c+1)\alpha$, $c(\alpha - 1) \leq \varepsilon - c < c(\alpha - 1) + \alpha$, $c \leq (\varepsilon - c)/(\alpha - 1) < c + \alpha/(\alpha - 1)$. We will show that $(\varepsilon - c)/(\alpha - 1) > (c + 1)$ does not hold:

Else, $\varepsilon > (c+1)\alpha - 1$, and $(c+1)\alpha - 1 < \varepsilon < (c+1)\alpha$, a contradiction because ε is an integer. So we have $c \leq (\varepsilon - c)/(\alpha - 1) \leq c + 1$, and $E_j(G)$ combining with the edge independent sets of g'_e constitutes an α-equitable edge coloring of G.

Case 2 If $\mid E_j(G) \mid = \lceil \varepsilon/\alpha \rceil$, we assume that ε/α is not an integer. Otherwise, $\lceil \varepsilon/\alpha \rceil = \lfloor \varepsilon/\alpha \rfloor$, it is the same as Case 1. Now we denote $\lceil \varepsilon/\alpha \rceil$ by c. So $c < \varepsilon/\alpha < c + 1$, $c - 1/(\alpha - 1) < (\varepsilon - c - 1)/(\alpha - 1) < c + 1$, we will show that $(\varepsilon - c - 1)/(\alpha - 1) < c$ does not hold:

Else, $\varepsilon < c\alpha + 1$, and $c\alpha < \varepsilon < c\alpha + 1$, a contradiction because ε is an integer. We have $c \leq (\varepsilon - c - 1)/(\alpha - 1) < c + 1$, so $E_j(G)$ combining with the edge independent sets of g'_e constitutes an α-equitable edge coloring of G.

Lemma 7 *For a graph G, $x, y \in V(G)$, $xy \notin E(G)$, $w, z \notin V(G)$, let $G^* = (G \bigvee \{w\} -$*

$\{wx\}) \bigcup \{zx\}$. If g is an α-equitable edge coloring of graph G^*, and $g(zx) = g(wy)$, then there exists an α-equitable total coloring of graph G.

Proof We define an α-total coloring f of graph G as follows:

(1) For $\forall\ e \in E(G)$, $f_G(e) = g_{G^*}(e)$.

(2) For any vertex $v \in V(G)$, $v \neq x$, $f_G(v) = g_{G^*}(wv)$; and $f_G(x) = f_G(y)$.

It is obvious that f is an α-equitable total coloring of graph G.

Theorem 1 *For $m \geq n \geq 3$, $\chi_{et}(W_m \bigvee K_n) = m + n + 1$.*

Proof For $W_m \bigvee K_n = C_m \bigvee K_{n+1}$, let $G' = G \bigvee \{w\}$, where $G = W_m \bigvee K_n$, $w \notin V(G)$. So we have $G' = C_m \bigvee K_{n+1} \bigvee \{w\} = C_m \bigvee K_{n+2}$. And by Lemma 4, $\chi_{et}(W_m \bigvee K_n) \geq m + n + 1$.

Case 1 If $m + n$ is even, then $m + n + 2$ is even, and for $G' \subseteq K_{m+n+2}$, by Lemmas 1 and 2, $\chi'(G') \leq \chi'(K_{m+n+2}) = m + n + 1$, and for $\chi'(G') \geq \Delta(G') = m + n + 1$, we have $\chi'(G') = m + n + 1$. For $\Delta(G) = m + n$, by Lemma 5, $\chi_{et}(W_m \bigvee K_n) = m + n + 1$.

Case 2 If $m + n$ is odd.

Case 2.1 When $m = n + 1 \geq 4$. For $G = W_m \bigvee K_n = C_m \bigvee K_{n+1} = C_{n+1} \bigvee K_{n+1}$, let

$$V(K_{n+1}) = \{v_0, v_1, \ldots, v_n\},\ V(C_{n+1}) = \{v_{n+1}, v_{n+2}, \ldots, v_{2n+1}\},$$
$$E(C_{n+1}) = \{v_{n+1}v_{n+2}, v_{n+2}v_{n+3}, \ldots, v_{2n}v_{2n+1}, v_{2n+1}v_{n+1}\}.$$

Case 2.1.1 When $m = n + 1 = 4$. We define a total coloring f of $C_4 \bigvee K_4$ as follows:

$$f(v_0) = f(v_1v_6) = f(v_2v_4) = f(v_3v_7) = 0,\ f(v_3) = f(v_0v_5) = f(v_1v_7) = f(v_2v_6) = 3,$$
$$f(v_1) = f(v_0v_7) = f(v_2v_5) = f(v_3v_4) = 1,\ f(v_4) = f(v_6) = f(v_0v_3) = f(v_1v_5) = f(v_2v_7) = 4,$$
$$f(v_2) = f(v_0v_6) = f(v_1v_4) = f(v_3v_5) = 2,\ f(v_5) = f(v_7) = f(v_0v_4) = f(v_1v_2) = f(v_3v_6) = 5,$$
$$f(v_0v_1) = f(v_2v_3) = f(v_4v_5) = f(v_6v_7) = 6,\ f(v_0v_2) = f(v_1v_3) = f(v_4v_7) = f(v_5v_6) = 7.$$

Thus $\chi_{et}(W_4 \bigvee K_3) = 8$.

Case 2.1.2 When $m = n + 1 = 5$. We define a total coloring f of $C_5 \bigvee K_5$ as follows:

$$f(v_0) = 9,\ f(v_7) = 5,\ f(v_8) = 7,\ f(v_9) = 6,\ f(v_i) = i,\ i = 1, 2, 3, 4, 5, 6;$$
$$f(v_0v_i) = 2i,\ i = 1, 2, 3, 4;\ f(v_0v_5) = 0,\ f(v_0v_i) \equiv 2i - 2 \pmod 9,\ i = 6, 7, 8, 9;$$
$$f(v_jv_k) \equiv j + k \pmod{10},\ \text{for } 1 \leq j \leq 4,\ 1 \leq k \leq 9;$$
$$f(v_5v_6) = 3,\ f(v_5v_9) = f(v_7v_8) = 4,\ f(v_6v_7) = 2,\ f(v_8v_9) = 8.$$

For every color of $\{0, 1, 2, \ldots, 9\}$, 5 elements are colored exactly. So f is an equitable total coloring of $W_5 \bigvee K_4$, and $\chi_{et}(W_5 \bigvee K_4) = 10$.

Case 2.1.3 When $m = n + 1 \geq 6$, and n is odd. Let $G^* = (G \bigvee \{w\} - \{wv_{2n}\}) \bigcup \{zv_{2n}\}$, where $w, z \notin V(G)$. We define an edge coloring g of G^* as follows:

$$g(v_0v_j) = 2j,\ 1 \leq j \leq n;\ g(v_0v_{n+1}) = 0;\ g(v_0v_j) \equiv 2j - 2 \pmod{2n+1},\ n + 2 \leq j \leq 2n + 1;$$

$g(v_jv_k) \equiv j+k \pmod{2n+2},\ 1 \le j \le n,\ 1 \le k \le 2n+1;$

$g(wv_0) = 2n+1,\ g(wv_{2n+1}) = 2n,\ g(wv_j) = j,\ 1 \le j \le 2n-1;$

$g(v_{2n}v_{2n+1}) = n+2,\ g(v_{2n+1}v_{n+1}) = n,\ g(v_{n+1}v_{n+2}) = n-1,\ g(zv_{2n}) = g(wv_{n+1}) = n+1.$

Case 2.1.3.1 When $n \equiv 1 \pmod 4$. Let $n = 4k+1$.

$$g(v_{n+2}v_{n+3}) = g(v_{n+4}v_{n+5}) = \cdots = g(v_{n+2k-2}v_{n+2k-1}) = n+1.$$
$$g(v_jv_{j+1}) = j-1,\ \text{for } n+3 \le j \le 2n-1,\ \text{and } j \ne n+2, n+4, \ldots, n+2k-2.$$

So g is a $(2n+2)$-edge coloring of G^*, $\chi'(G^*) = 2n+2$. All the edges of color $n+1$ are listed below:

$$v_0v_{(n+1)/2}, v_1v_n, v_2v_{n-1}, \ldots, v_{(n-1)/2}v_{(n+3)/2},$$

$$wv_{n+1}, zv_{2n}, v_{n+2}v_{n+3}, v_{n+4}v_{n+5}, \ldots, v_{n+2k-2}v_{n+2k-1}.$$

So $\mid E_{n+1}(G^*) \mid = (n+1)/2 + (k+1) = 3k+2$. When $n = 5$, i.e. $k = 1$, $\mid E_6(G^*) \mid = 6 = 3k+3 = \lceil 3n/4 + 2 \rceil$. When $n > 5$, we have $\mid E(G^*) \mid = (n+1)(3n/2+4)$, and $\lfloor \mid E(G^*) \mid / \chi'(G^*) \rfloor = \lfloor 3n/4 + 2 \rfloor = 3k+2$, by Lemmas 6 and 7, there exists a $(2n+2)$-equitable total coloring of graph G, thus $\chi_{et}(W_{n+1} \bigvee K_n) = 2n+2$.

Case 2.1.3.2 When $n \equiv 3 \pmod 4$. Let $n = 4k+3$.

$$g(v_{n+2}v_{n+3}) = g(v_{n+4}v_{n+5}) = \cdots = g(v_{n+2k}v_{n+2k+1}) = n+1.$$
$$g(v_jv_{j+1}) = j-1,\ \text{for } n+3 \le j \le 2n-1,\ \text{and } j \ne n+2, n+4, \ldots, n+2k.$$

So g is a $(2n+2)$-edge coloring of G^*, $\chi'(G^*) = 2n+2$. All the edges of color $n+1$ are listed as follows:

$$v_0v_{(n+1)/2}, v_1v_n, v_2v_{n-1}, \ldots, v_{(n-1)/2}v_{(n+3)/2},$$
$$wv_{n+1}, zv_{2n}, v_{n+2}v_{n+3}, v_{n+4}v_{n+5}, \ldots, v_{n+2k}v_{n+2k+1}.$$

So $\mid E_{n+1}(G^*) \mid = (n+1)/2 + (k+2) = 3k+4$. For $\lfloor \mid E(G^*) \mid / \chi'(G^*) \rfloor = \lfloor 3n/4 + 2 \rfloor = 3k+4$, by Lemmas 6 and 7, there exists a $(2n+2)$-equitable total coloring of graph G. Hence we have $\chi_{et}(W_{n+1} \bigvee K_n) = 2n+2$.

Case 2.1.4 When $m = n+1 \ge 7$, and n is even. Let $G^* = (G \bigvee \{w\} - \{wv_{2n+1}\}) \bigcup \{zv_{2n+1}\}$, where $w, z \notin V(G)$. $g(v_0v_j) = 2j$, for $1 \le j \le n$; $g(v_0v_{n+1}) = 0$; $g(v_0v_j) \equiv 2j-2 \pmod{2n+1}$, for $n+2 \le j \le 2n+1$;
$g(v_jv_k) \equiv j+k \pmod{2n+2}$, $1 \le j \le n, 1 \le k \le 2n+1$; $g(wv_0) = 2n+1$, $g(wv_j) = j$, $1 \le j \le 2n$; $g(v_{2n-1}v_{2n}) = 2n-4$, $g(v_{2n}v_{2n+1}) = 2n-2$, $g(v_{2n+1}v_{n+1}) = n$, $g(v_{n+1}v_{n+2}) = n-1$, $g(zv_{2n+1}) = g(wv_{n+2}) = n+2$.

Case 2.1.4.1 When $n \equiv 0 \pmod 4$. Let $n = 4k$, $k \ge 2$. $g(v_{n+3}v_{n+4}) = g(v_{n+5}v_{n+6}) = \cdots = g(v_{n+2k-1}v_{n+2k}) = n+2$. $g(v_jv_{j+1}) = j-1$, for $n+2 \le j \le 2n-2$, and $j \ne n+3, n+4, \ldots, n+2k-1$. Hence g is a $(2n+2)$-edge coloring of G^*, $\chi'(G^*) = 2n+2$. All the edges of color $n+2$

are listed below:

$$v_0v_{n/2+1}, v_1v_{n+1}, v_2v_n, v_3v_{n-1}, \ldots, v_{n/2}v_{n/2+2},$$
$$wv_{n+2}, zv_{2n+1}, v_{n+3}v_{n+4}, v_{n+5}v_{n+6}, \ldots, v_{n+2k-1}v_{n+2k}.$$

So $| E_{n+1}(G^*) |= (n/2+1)+(k+1) = 3k+2$. For $\lfloor | E(G^*) | /\chi'(G^*)\rfloor = \lfloor 3n/4+2\rfloor = 3k+2$, by Lemmas 6 and 7, there exists a $(2n+2)$-equitable total coloring of graph G. So we have $\chi_{et}(W_{n+1} \bigvee K_n) = 2n+2$.

Case 2.1.4.2 When $n \equiv 2 \pmod 4$. Let $n = 4k+2$. The coloring is the same as Case 2.1.4.1, and $| E_{n+1}(G^*) |= (n/2+1)+(k+1) = 3k+3$. $\lfloor | E(G^*) | /\chi'(G^*)\rfloor = \lfloor 3n/4+2\rfloor = 3k+3$, by Lemmas 6 and 7, there exists a $(2n+2)$-equitable total coloring of graph G. It is obvious that $\chi_{et}(W_{n+1} \bigvee K_n) = 2n+2$.

Case 2.2 If $m \geq n+3 \geq 6$, for $G' = C_m \bigvee K_{n+1} \bigvee \{w\} = C_m \bigvee K_{n+2}$, let

$$V(C_m) = \{u_0, u_1, \ldots, u_{m-1}\},\ E(C_m) = \{u_0u_1, u_1u_2, \ldots, u_{m-2}u_{m-1}, u_{m-1}u_0\},$$
$$V(K_n+2) = \{v_0, v_1, \ldots, v_{n+1}\}.$$

We define an edge coloring g of $C_m \bigvee K_{n+2}$ as follows:

$$g(v_0v_j) = 2j,\ 1 \leq j \leq n+1;\ g(v_0u_i) \equiv 2i+2n+4 \pmod{m+n},\ \text{for } 0 \leq i \leq m-3;$$
$$g(v_0u_{m-2}) = m+n,\ g(v_0u_{m-1}) = 0,\ g(v_jv_k) = j+k,\ 1 \leq j,k \leq n+1;$$
$$g(u_iv_j) \equiv i+n+2+j \pmod{m+n+1},\ \text{for } 0 \leq i \leq m-1,\ 1 \leq j \leq n+1;$$
$$g(u_0u_1) = n+1,\ g(u_iu_{i+1}) = i+n+2,\ \text{for } 1 \leq i \leq m-3,\ \text{and } i \neq m-4;$$
$$g(u_{m-4}u_{m-3}) = n, g(u_{m-2}u_{m-1}) = n+3, g(u_{m-1}u_0) = n+2.$$

It is easy to see that g is a proper edge coloring of $C_m \bigvee K_{n+1} \bigvee \{w\}$, so $\chi'(C_m \bigvee K_{n+1} \bigvee \{w\}) = m+n+1$. For $\Delta(C_m \bigvee K_{n+1}) = m+n$, by Lemma 5, we have $\chi_{et}(C_m \bigvee K_{n+1}) = m+n+1$. From all above, the theorem holds. □

Theorem 2 *$\chi_{et}(F_m \bigvee K_n) = \chi_{et}(S_m \bigvee K_n) = m+n+1$, for $m \geq n \geq 3$.*

The proof is similar to Theorem 1.

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