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A Convergence Theorem for a Kind of Composite Power Series Expansions

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Abstract Presented in this paper is a convergence theorem for a kind of composite power series expansions whose coefficients can be expressed by using Faà di Bruno's formula. A related problem is proposed as a remark, and a few examples are given as applications.

Keywords power series expansion; Faà di Bruno's formula; convergence theorem.

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1. Introduction

Denote by R and C the real and complex number fields respectively, and let $a \in C$. Suppose that $f(a+t)$ has a formal power series expansion in t, and let $\sum_{n=1}^{\infty} \alpha_n z^n$ be a formal series over C. Given a formal power series expansion of the form

$$f\Big(a + \sum_{n=1}^{\infty} \alpha_n z^n\Big) = f(a) + \sum_{n=1}^{\infty} \beta_n z^n. \tag{1}$$

It is known that the coefficients$\beta_n$can be expressed by means of Faà di Bruno's formula[1,2]

$$\beta_n = \sum_{\sigma(n)} f^{(k)}(a) \frac{\alpha_1^{k_1} \cdots \alpha_n^{k_n}}{k_1! \cdots k_n!} \tag{2}$$

where $f^{(k)}(a) = (\mathrm{d}/\mathrm{d}t)^k f(a+t)|_{t=0}$ denotes the kth formal derivative of $f(a+t)$ at $t = 0$, and the summation involved in (2) is taken over the set $\sigma(n)$ of all possible partitions of n, that is, over all nonnegative integral solutions $(k_1, k_2, \ldots, k_n)$ of the equation $1k_1 + 2k_2 + \cdots nk_n = n, k_1 + k_2 + \cdots k_n = k$, $k = 1, 2, \ldots, n$.

In this paper all functions are assumed to be analytic having convergence circular regions in C. We shall prove (in §2) a convergence theorem for the series expansion (1), in which a basic convergence condition is stated in terms of the convergence radii of $f(z)$ and $\sum_{n=1}^{\infty} \alpha_n z^n$, and also involving free parameters $p > 1$ and $q > 1$ with $1/p + 1/q = 1$.

2. A convergence theorem and its corollaries

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Theorem *Let the following conditions be fulfilled:*

(i) $\sum_{n=1}^{\infty} \alpha_n z^n$ *is an analytic function of* z *defined for* $|z| < \rho$*;*

(ii) $f(z)$ *is analytic within the circle* $|z-a| < r$ *of center at* $a \in C$*;*

(iii) s *is a finite number defined by*

$$s := \Big(\sum_{n=1}^{\infty} |\alpha_n|^p \rho_0^{pn}\Big)^{1/p} \tag{3}$$

where $p > 1$ *and* $0 < \rho_0 < \rho$*. Then the power series expansion (1) is absolutely convergent for any* z *such that*

$$|z| < \rho_0 r/(r^q + s^q)^{1/q} \tag{4}$$

where $q = p/(p-1)$*, and the coefficients* β_n*contained in (1) are given by Faà di Bruno's formula (2).*

Proof First, notice that the convergence of the series in the RHS (right-hand side) of (3) follows from condition (i) and Cauchy's root-test. Clearly, condition (inequality) (4) implies $z < \rho_0 < \rho$. Thus it suffices to show that (4) also implies the inequality

$$\sum_{n=1}^{\infty} |\alpha_n||z|^n < r \tag{5}$$

which could ensure the absolute convergence of the series expansion (1). Indeed, condition (ii) leads to an absolutely convergent Taylor expansion of $f(z)$ at $z = a$ within the circle $|z-a| < r$, so that we have

$$|f(a+z)| \le |f(a)| + \sum_{k=1}^{\infty} \frac{|f^{(k)}(a)}{k!} |z|^k < +\infty, \quad |z| < r.$$

Consequently, we see that (5) implies the following

$$\begin{aligned}\Big|f\big(a + \sum_{n=1}^{\infty} |\alpha_n||z^n|\big)\Big| &\le |f(a)| + \sum_{k=1}^{\infty} \frac{|f^{(k)}(a)|}{k!} \Big(\sum_{n=1}^{\infty} |\alpha_n||z^n|\Big)^k \\ &= |f(a)| + \sum_{k=1}^{\infty} \frac{|f^{(k)}(a)|}{k!} \sum_{m=1}^{\infty} S_k(m)|z|^m < +\infty,\end{aligned}$$

where $s_k(m)$ denotes the k-fold convolution of the $|\alpha_n|$ coefficients, viz.

$$s_k(m) = \sum_{n_1+\cdots n_k = m} |\alpha_{n-1}| \cdots |\alpha_{n_k}|.$$

By exchanging the order of the double summation occurring above, we get

$$\Big|f\big(a + \sum_{n=1}^{\infty} |\alpha_n||z^n|\big)\Big| \le |f(a)| + \sum_{m=1}^{\infty} \Big(\sum_{k=1}^{m} \frac{f^{(k)}(a)}{k!} S_k(m)\Big)|z|^m. \tag{6}$$

Certainly this implies that the RHS of (1) is bounded absolutely by the RHS of (6).

What remains is to deduce (5) from (4). Denote $z = \rho_0 t$ and note that $1/p + 1/q = 1$. Thus, making use of Hölder's inequality, we see that the LHS (left-hand side) of (5) with $|z| < \rho_0$ may be estimated as follows

$$\sum_{n=1}^{\infty}|\alpha_n|\,|z|^n = \sum_{n=1}^{\infty}|\alpha_n|\,|\rho_0^n|\,|t|^n \le \Big(\sum_{n=1}^{\infty}|\alpha_n|^p\,\rho_0^{np}\Big)^{1/p}\Big(\sum_{n=1}^{\infty}|t|^{nq}\Big)^{1/q}$$
$$= s\cdot|t|/(1-|t|^q)^{1/q},\quad |t|<1.$$

Thus the truth of (5) is implied by the inequality

$$s\cdot|t|/(1-|t|^q)^{1/q} < r \tag{7}$$

which is equivalent to

$$s^q|t|^q < r^q(1-|t|^q)|,\ \text{i.e.,}\ |t| < r/(r^q+s^q)^{1/q}.$$

Recalling $z=\rho_0 t$, we can rewrite the above inequality as

$$|z| < \rho_0 r/(r^q+s^q)^{1/q}.$$

Hence (4) really implies (5) and the proof is completed. □

It is easily seen that the theorem and its proof imply the following propositions.

Corollary 1 *Let the condition (i) of Theorem be replaced by*

(i) $\sum_{n=1}^{\infty}\alpha_n z^n$ is absolutely convergent on the disc $|z|<\rho_0$. Also, let the statement $0<\rho_0<\rho$ in (iii) be omitted. Then the conclusion of Theorem does hold under the (i)*, (ii) and (iii).*

Observe that the inequality (6) in the proof holds for every entire function $f(z)$, provided that $\sum_{n=1}^{\infty}\alpha_n z^n$ is convergent for $|z|<\rho$. This leads to the following

Corollary 2 *Let $f(z)$ be an entire function and let $\sum_{n=1}^{\infty}\alpha_n z^n$ be analytic within the circle $|z|<\rho$. Then the expansion formula (1) with $\beta_n$being given by (2) holds for $|z|<\rho$.*

Obviously, the condition $|z|<\rho$ also follows from (4) by letting $r\to\infty$ and then replacing ρ_0 by ρ. Moreover, by taking $p=q=2$ in Theorem we have

Corollary 3 *Given $\sum_{n=1}^{\infty}\alpha_n z^n$ and $f(z)$ as in Theorem. Let s be defined by*

$$s := \Big(\sum_{n=1}^{\infty}|\alpha_n|^2\rho_0^{2n}\Big)^{1/2},\quad 0<\rho_0<\rho. \tag{8}$$

Then the expansion (1) converges absolutely for z such that

$$|z| < \rho_0 r/(r^2+s^2)^{1/2}. \tag{9}$$

Here the condition (9) may be replaced by

$$|z| < \sup\{\rho_0 r/(r^2+s^2)^{1/2}|0<\rho_0<\rho\}. \tag{10}$$

As may be observed, the classical Taylor expansion theorem is a particular case included in Corollary 3 with $\alpha_1=1$, $\alpha_n=0$ $(n\ge 2)$, so that $\rho=+\infty$ and $s=\rho_0$. Indeed, (1) and (2) imply

$$f(a+z) = f(a) + \sum_{n=1}^{\infty}\frac{f^{(n)}(a)}{n!}z^n$$

and the convergence condition (10) may be now written as

$$|z| < \sup\{\rho_0 r/(r^2+\rho_0^2)^{1/2} | 0 < \rho_0 < \infty\} = r.$$

Here we would like to mention a remark involving an unsolved problem.

Remark Notice that the convergence result described in the theorem is proved without reference to Faà di Bruno's formula (2). We guess that an appropriate estimate for $|\beta_n|$ via (2) may possibly lead to some more sharp conditions than (4) or (9) for the absolute convergence of (1). Of course, the exact radius of convergence for the series expansion of (1) is given by the Cauchy-Hadamard formula $\bar{r} = 1/\overline{\lim_n} \, ||\beta_n|^{1/n}$. However this expression is not utilizable in practice.

3. Examples

Here we give a few examples to illustrate certain applications of the convergence result of §2.

Example 1 Consider the power series expansion (Corollary 1 of [1])

$$\exp\Big(\sum_{n=1}^{\infty} \alpha_n z^n\Big) = 1 + \sum_{n=1}^{\infty} \beta_n z^n, \tag{11}$$

where the coefficients β_n's are given by

$$\beta_n = \sum_{\sigma(n)} \frac{\alpha_1^{k_1} \cdots \alpha_n^{k_n}}{k_1! \cdots k_n!}. \tag{12}$$

In this example, $f(z) = \exp(z) = e^z$ is an entire function. Thus by Corollary 2 we see that the RHS of (11) is absolutely convergent for $|z| < \rho$, provided that $\sum_{n=1}^{\infty} \alpha_n z^n$ is analytic within the circle $|z| < \rho$.

As a simple instance, consider the generating function for the Bell number–sequence $\{B_n\}$:

$$\exp(e^z - 1) = \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n!} z^n\Big) = 1 + \sum_{n=1}^{\infty} B_n \frac{z^n}{n!}, \tag{13}$$

where the numbers $B_n/n!$ are given by (12)

$$\frac{1}{n!} B_n = \sum_{\sigma(n)} \frac{(1/1!)^{k_1} \cdots (1/n!)^{k_n}}{k_1! \cdots k_n!}. \tag{14}$$

Since $\sum z^n/n!$ converges absolutely for $|z| < \infty$, we see that the RHS of (13) is also absolutely convergent for $|z| < \infty$.

Example 2 Let $\alpha \in R$ and consider the expansion (Corollary 2 of [1])

$$\Big(1 + \sum_{n=1}^{\infty} \alpha_n z^n\Big)^{\alpha} = 1 + \sum_{n=1}^{\infty} \beta_n z^n, \tag{15}$$

where β_n's are given by

$$\beta_n = \sum_{\sigma(n)} (\alpha)_k \frac{\alpha_1^{k_1} \cdots \alpha_n^{k_n}}{k_1! \cdots k_n!}, \tag{16}$$

$(\alpha)_k$ denoting the falling factorial, namely $(\alpha)_k = \alpha(\alpha-1)\cdots(\alpha-k+1)$, $k \geq 1$ and $(\alpha)_0 = 1$. In this example, $f(z) = z^\alpha$ is an analytic function of z for $|z| < 1$. Thus by the convergence theorem (§2), or its Corollary 3, we see that the RHS of (15) is absolutely convergent for z such that

$$|z| < \sup\{\rho_0/(1+s^2)^{1/2} | 0 < \rho_0 < \rho\}, \tag{17}$$

where $s = (\sum_{n=1}^\infty |\alpha_n|^2 \rho_0^{2n})^{1/2}$, and $\sum_{n=1}^\infty \alpha_n z^n$ is assumed to be analytic within the circle $|z| < \rho$.

Evidently, the binomial expansion theorem

$$(1+z)^\alpha = 1 + \sum_{n=1}^\infty \binom{\alpha}{n} z^n, \quad |z| < 1 \tag{18}$$

is the simplest instance implied by (15)–(17), with $\alpha_1 = 1$, $\alpha_n = 0$ $(n \geq 2)$, $s = \rho_0$ and $\rho = +\infty$.

Example 3 For $\alpha \in R$, consider the series expansion

$$(1-\log(1-z))^\alpha = \Big(1 + \sum_{n=1}^\infty \frac{1}{n} z^n\Big)^\alpha = 1 + \sum_{n=1}^\infty \beta_n z^n \tag{19}$$

where $\sum_{n=1}^\infty \frac{1}{n} z^n$ is analytic for $|z| < 1 = \rho$. Comparing this with (15) and using (17), we see that the RHS of (19) converges absolutely for z such that

$$|z| < \sup\{\rho/(1+s^2)^{1/2} | 0 < \rho_0 < 1\} \tag{20}$$

where $s = (\sum_{n=1}^\infty (\frac{1}{n})^2 \rho_0^{2n})^{1/2}$ $(0 < \rho_0 < 1)$. The RHS of (20) may be computed by letting $\rho_0 \to 1$, so that (20) may be rewritten as

$$|z| < 1\Big/\Big(1 + \sum_{n=1}^\infty \frac{1}{n^2}\Big)^{1/2} = 1\Big/\Big(1 + \frac{\pi^2}{6}\Big)^{1/2} \approx 0.615. \tag{21}$$

On the other hand it is easily seen that for$|z| > 0$ required inequality $|\log(1-z)| < 1$ leads to the exact relation $|z < (e-1)/e|$. This yields the exact radius of convergence $\bar{r} = (e-1)/e \approx 0.632$ for the power series (19). From this we see that the RHS of (21) is an approximate value of $\bar{r}$ with a relative error $(0.632 - 0.615)/0.632 \simeq 0.027$. This special example may also suggest the fact that the RHS of (4) or of (10) just gives a kind of lower approximation to $\bar{r} = 1/\overline{\lim}_n |\beta_n|^{1/n}$ for the convergence of the power series (1). As mentioned in Remark of §2, better approximations to $\bar{r}$ may possibly be obtained by some sharper estimates of β_n via (2).

References

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