Journal of Mathematical Research & Exposition
Nov., 2008, Vol. 28, No. 4, pp. 855–863
DOI:10.3770/j.issn:1000-341X.2008.04.014
Http://jmre.dlut.edu.cn

Maschke-Type Theorem for Two-Sided Weak Smash Products

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Abstract This paper mainly gives a Maschke-type theorem for two-sided weak smash products over semisimple weak Hopf algebras.

Keywords weak Hopf algebras; weak H-bimodule algebras; two-sided weak smash products; Maschke-type theorem.

Document code A
MR(2000) Subject Classification 16W30
Chinese Library Classification O153.3

1. Introduction and preliminaries

Weak bialgebra and weak Hopf algebras given in [1] are generalizations of ordinary bialgebra and Hopf algebras in the following sense: the defining axioms are the same, but the multiplicativity of the counit and the comultiplicativity of unit are replaced by weaker axioms.

Many results of classical Hopf algebra theory can be generalized to weak Hopf algebras. For examples, a duality theorem for weak module algebras[2] was given by Nikshych; a Maschke-type theorem for weak module algebras[3] was given by Zhang, which extends the famous Maschke theorem[4] for usual module algebras given by Cohen and Fishman; the fundamental theorem of weak Doi-Hopf modules[5] was given by Zhang and Zhu, which not only extends the fundamental theorem of weak Hopf modules given in [1], but also extends the fundamental theorem of relative Hopf modules given in [6].

In this paper, we firstly introduce the concept of two-sided weak smash products and give some examples of two-sided weak smash products. Then, we give the Maschke-type theorem of two-sided weak smash products over semisimple weak Hopf algebras, which extends Theorem 1 given in [3].

We always work over a fixed field k and follow Montgomery's book[7] for terminologies on coalgebras and comodules.

We recall some concepts and results used in this paper.

Received date: 2006-11-16; **Accepted date**: 2007-11-22
Foundation item: the National Natural Science Foundation of China (No. 10571153); the Educational Ministry Science Technique Research Key Foundation of China (No. 108154); the Grant for Young Teachers of Nanjing Agricultural University (No. KJ06025).

Definition 1.1[1] *Let H be both an algebra and a coalgebra. If H satisfies the conditions (1.1)–(1.3) below, then it is called a weak bialgebra given in [1]. If it satisfies the conditions (1.1)–(1.4) below, then it is called a weak Hopf algebra with weak antipode S_H.*

For any $x, y, z \in H$,

(1.1): $\Delta_H(xy) = \Delta_H(x)\Delta_H(y)$.

(1.2): (a) $\Delta_H^2(1_H) = (\Delta_H(1_H)\otimes 1_H)(1_H\otimes\Delta_H(1_H))$; (b) $\Delta_H^2(1_H) = (1_H\otimes\Delta_H(1_H))(\Delta_H(1_H)\otimes 1_H)$, *where* $\Delta_H^2 = (\Delta_H \otimes id_H)\circ\Delta_H$.

(1.3): (a) $\varepsilon_H(xyz) = \Sigma\varepsilon_H(xy_1)\varepsilon_H(y_2z)$; (b) $\varepsilon_H(xyz) = \Sigma\varepsilon_H(xy_2)\varepsilon_H(y_1z)$.

(1.4): (a) $\Sigma x_1S_H(x_2) = \Sigma\varepsilon_H(1_1x)1_2$; (b) $\Sigma S_H(x_1)x_2 = \Sigma 1_1\varepsilon_H(x1_2)$; (c) $\Sigma S_H(x_1)x_2S_H(x_3) = S_H(x)$, *where* $\Delta_H(1_H) = \Sigma 1_1 \otimes 1_2$.

For any weak bialgebra H, define the maps $\sqcap_H^L, \sqcap_H^R : H \to H$ by the formulas

$$\sqcap_H^L(x) = \Sigma\varepsilon_H(1_1x)1_2; \sqcap_H^R(x) = \Sigma 1_1\varepsilon_H(x1_2).$$

Denote the image $\sqcap_H^L(H)$ by H^L and the image $\sqcap_H^R(H)$ by H^R.

Note that H is an ordinary bialgebra if and only if $\Delta_H(1_H) = 1_H \otimes 1_H$, and if and only if ε_H is an algebra map.

By [1], we have the following conclusions.

- (W1) H^L and H^R are subalgebras of H and $\Delta_H(1_H) = \Sigma 1_1 \otimes 1_2 \in H^R \otimes H^L$.
- (W2) For any $x \in H^L, y \in H^R$, $xy = yx$.
- (W3) For any $x \in H^L, y \in H^R$, $\Delta_H(x) = \Sigma 1_1x \otimes 1_2$, $\Delta_H(y) = \Sigma 1_1 \otimes y1_2$.
- (W4) $\sqcap_H^R \circ S_H = \sqcap_H^R \circ \sqcap_H^L = S_H \circ \sqcap_H^L$.
- (W5) For any $h \in H, \Sigma\sqcap_H^L(h_1)\otimes h_2 = \Sigma S_H(1_1)\otimes 1_2h$ and $\Sigma h_1 \otimes \sqcap_H^R(h_2) = \Sigma h1_1 \otimes S_H(1_2)$.

So $\Sigma \sqcap_H^L(1_1) \otimes 1_2 = \Sigma S_H(1_1) \otimes 1_2, \Sigma 1_1 \otimes \sqcap_H^R(1_2) = \Sigma 1_1 \otimes S_H(1_2)$.

- (W6) For any $h \in H$, $\Sigma \sqcap_H^R(h_1) \otimes h_2 = \Sigma 1_1 \otimes h1_2$; $\Sigma h_1 \otimes \sqcap_H^L(h_2) = \Sigma 1_1h \otimes 1_2$.

According to [3], we have the following results:

- (W7) $S_H^{-1} \circ \sqcap_H^R = \sqcap_H^L \circ S_H^{-1}; S_H^{-1} \circ \sqcap_H^L = \sqcap_H^R \circ S_H^{-1}$.
- (W8) $\Sigma \sqcap_H^L(h)S_H(x_1) \otimes x_2 = \Sigma S_H(x_1) \otimes x_2 \sqcap_H^L(h)$, for any $h, x \in H$.

Let H be a finite-dimensional weak Hopf algebra. Then, by [1], the weak antipode S_H is a bijection with inverse S_H^{-1}. Hence, by [8], we have

- (W9) $\Sigma h_2S_H^{-1}(h_1) = \sqcap_H^R S_H^{-1}(h)$, for any $h, x \in H$.
- (W10) S_H^{-1} is both an anti-algebra map and an anti-coalgebra map.

Definition 1.2 *Let H be a weak bialgebra. A k-algebra A is called a weak left H-module algebra in [2] if A is a left H-module via $h \otimes a \mapsto h \cdot a$ such that for any $a, b \in A, h \in H$,*

(1.5) $h \cdot (ab) = \Sigma(h_1 \cdot a)(h_2 \cdot b)$

(1.6) $h \cdot 1_A = \sqcap_H^L(h) \cdot 1_A$.

Similarly, we can define a weak right H-module algebra.

Definition 1.3 *Let H be a weak Hopf algebra, and A a weak left H-module algebra. A weak smash product $A\#H$ in [2] can be defined as follows:*

$A\#H = A\otimes_{H^L} H$(relative tensor product) as a k-module and its multiplication is given by the formula

$$(a\#h)(b\#g) = \Sigma a(h_1\cdot b)\#h_2 g$$

for any $a, b\in A, g, h\in H$.

Note here that H is a left H^L-module via its multiplication, and A is a right H^L-module via defining: for any $a\in A, x\in H^L$,

$$a\leftharpoonup x = S_H^{-1}(x)\cdot a = a(x\cdot 1_A).$$

By [2], we know that $A\#H$ is an algebra with unit $1_A\#1_H$.

2. Two-sided weak smash products

Definition 2.1 *Let A be a weak left H-module algebra and B be a weak right H-module algebra. A two-sided weak smash product $A\#H\#B$ is defined as follows:*

$A\#H\#B = A\otimes_{H^L} H\otimes_{H^R} B$(relative tensor product) as a k-module and its multiplication is given by the formula

$$(a\otimes h\otimes b)(a'\otimes h'\otimes b') = \Sigma a(h_1\rightharpoonup a')\otimes h_2h'_1\otimes(b\leftharpoonup h'_2)b'$$

for any $a, a'\in A, h, h'\in H, b, b'\in B$.

Note here that H is both a left H^L-module and a right H^R-module via its multiplication, and A is a right H^L-module via defining:

$$a\leftharpoonup x = S_H^{-1}(x)\cdot a = a(x\cdot 1_A)$$

for any $a\in A, x\in H^L$, and B is a left H^R-module via defining:

$$y\rightharpoonup b = (1_B\cdot y)b$$

for any $y\in H^R, b\in B$.

Then, by the above multiplication, $A\#H\#B$ is an algebra with unit $1_A\#1_H\#1_B$.

As a matter of fact, for any $a\in A, h\in H, b\in B$, we have

$$\begin{aligned}
(1_A\#1_H\#1_B)(a\#h\#b) &= \Sigma(1_1\cdot a)\#1_2h_1\#(1_B\cdot h_2)b\\
&= \Sigma(1_1\cdot a)\#1_2h_1\#(1_B\cdot\sqcap_H^R(h_2))b\\
&\overset{\text{(W5)}}{=} \Sigma(1_1\cdot a)\#1_2h1'_1\#(1_B\cdot S_H(1'_2))b\quad(\Delta_H(1_H) = \Sigma 1'_1\otimes 1'_2)\\
&\overset{\text{(W1)}}{=} \Sigma(1_1\cdot a)\cdot 1_2\#h\#1'_1\cdot((1_B\cdot S_H(1'_2))b)\\
&\overset{\text{(1.5)}}{=} \Sigma(1_1\cdot a)(1_2\cdot 1_A)\#h\#(1_B\cdot 1'_1)(1_B\cdot S_H(1'_2))b\\
&\overset{\text{(W5)}}{=} \Sigma a\#h\#(1_B\cdot 1'_1)(1_B\cdot\sqcap_H^R(1'_2))b\\
&= \Sigma a\#h\#(1_B\cdot 1'_1)(1_B\cdot 1'_2)b\\
&= a\#h\#b,\\
(a\#h\#b)(1_A\#1_H\#1_B) &= \Sigma a(h_1\cdot 1_A)\#h_21'_1\#(b\cdot 1'_2) = \Sigma a(h_1\cdot 1_A)\#h_2\#1'_1\cdot(b\cdot 1'_2)\\
&= \Sigma a(h_1\cdot 1_A)\#h_2\#(1_B\cdot 1'_1)(b\cdot 1'_2)
\end{aligned}$$

$$
\begin{aligned}
&= \Sigma a(h_1 \cdot 1_A)\#h_2\#b = \Sigma a(\sqcap_H^L(h_1) \cdot 1_A)\#h_2\#b \\
&\overset{\text{(W5)}}{=} \Sigma a(S_H(1_1) \cdot 1_A)\#1_2 h\#b = \Sigma (a(\sqcap_H^L(1_1) \cdot 1_A)) \cdot 1_2\#h\#b \\
&= \Sigma a(\sqcap_H^L(1_1) \cdot 1_A)(1_2 \cdot 1_A)\#h\#b = \Sigma a(1_1 \cdot 1_A)(1_2 \cdot 1_A)\#h\#b \\
&= a\#h\#b.
\end{aligned}
$$

It is easy to prove that the associativity of $A\#H\#B$ is satisfied, so $A\#H\#B$ is an algebra with unit $1_A\#1_H\#1_B$.

Remark 2.2 (1) By [2], $A = H^L$ is a weak left H-module algebra via the trivial action $h \cdot h^L = \sqcap_H^L(hh^L)$. Then $A\#H\#B = H^L\#H\#B \cong H\#B$ with muitiplication given by

$$(h\#b)(h'\#b') = \Sigma hh'_1\#(b \cdot h'_2)b'$$

and hence $H\#B$ is an algebra with unit $1_H\#1_B$.

(2) By [1], $B = H^R$ is a weak right H-module algebra via the trivial action $h^R \cdot h = \sqcap_H^R(h^R h)$. Then $A\#H\#B = A\#H\#H^R \cong A\#H$ (the weak smash product in Definition 1.3) with muitiplication given by

$$(a\#h)(a'\#h') = \Sigma a(h_1 \cdot a')\#h_2 h'$$

and so $A\#H$ is an algebra with unit $1_A\#1_H$.

Example 2.3 (1) Let B be a weak right H-module algebra with the right H-module action "$\leftharpoondown$". On the opposite algebra B^{op}, we may define the left H-module action "$\rightharpoonup$" as follows: for any $h \in H, b \in B^{op}$,

$$h \rightharpoonup b = b \leftharpoondown S_H(h).$$

Since for any $h \in H$,

$$
\begin{aligned}
\sqcap_H^L(h) \rightharpoonup 1_B = 1_B \leftharpoondown S_H(\sqcap_H^L(h)) &\overset{\text{(W4)}}{=} 1_B \leftharpoondown \sqcap_H^R S_H(h) \\
&= 1_B \leftharpoondown S_H(h) = h \rightharpoonup 1_B,
\end{aligned}
$$

it is easy to prove that $(B^{op}, \rightharpoonup)$ is a weak left H-module algebra. Then, by Definition 2.1, $B^{op}\#H\#B$ is a two-sided weak smash product with multiplication as follows: for any $b, b' \in B^{op}, h, h' \in H, p, p' \in B$,

$$(b\#h\#p)(b'\#h'\#p') = \Sigma(b' \leftharpoondown S_H(h_1))b\#h_2 h'_1\#(p \leftharpoondown h'_2)p'.$$

(2) Let A be a weak left H-module algebra with the left H-module action "$\rightharpoonup$". On the opposite algebra A^{op}, we may define the right H-module action "$\leftharpoondown$" as follows: for any $h \in H, a \in A$,

$$a \leftharpoondown h = S_H(h) \rightharpoonup a.$$

It is easy to prove that $(A^{op}, \leftharpoondown)$ is a weak right H-module algebra. Then, by Definition 2.1, $A\#H\#A^{op}$ is a two-sided weak smash product with multiplication as follows: for any $a, a' \in A, h, h' \in H, p, p' \in A^{op}$,

$$(a\#h\#p)(a'\#h'\#p') = \Sigma a(h_1 \rightharpoonup a')\#h_2 h'_1\#p'(S_H(h'_2) \rightharpoonup p).$$

(3) Let H be a weak bialgebra, and A a weak (H,H)-bimodule algebra (that is, A is both an (H,H)-bimodule and a weak left H-module algebra and a weak right H-module algebra). Then, by Definition 2.1, $A\#H\#A$ is a two-sided weak smash product with the following multiplication: for any $a, a', p, p' \in A, h, h' \in H$,

$$(a\#h\#p)(a'\#h'\#p') = \Sigma a(h_1 \rightharpoonup a')\#h_2h'_1\#(p \leftharpoonup h'_2)p'.$$

(4) Let H be a finite-dimensional weak Hopf algebra. Define the actions as follows: for any $h \in H, f \in H^*$,

$$f \rightharpoonup h = \Sigma\langle f, h_2\rangle h_1;\ \ h \leftharpoonup f = \Sigma\langle f, h_1\rangle h_2.$$

It is easy to prove that $(H, \rightharpoonup)$ is a weak left H^*-module algebra, and $(H, \leftharpoonup)$ is a weak right H^*-module algebra. So $H\#H^*\#H$ is a two-sided weak smash product with the following multiplication: for any $h, h', g, g' \in H, \beta, \beta' \in H^*$,

$$\begin{aligned}(h\#\beta\#g)(h'\#\beta'\#g') &= \Sigma h(\beta_1 \rightharpoonup h')\#\beta_2\beta'_1\#(g \leftharpoonup \beta'_2)g' \\ &= \Sigma hh'_1\langle\beta_1, h'_2\rangle\#\beta_2\beta'_1\#\langle\beta'_2, g_1\rangle g_2g'.\end{aligned}$$

3. Maschke-type theorem for two-sided weak smash products

Let H be a weak bialgebra, and $(A, \rightharpoonup)$ a weak left H-module algebra, and $(B, \leftharpoonup)$ a weak right H-module algebra, and $A\#H\#B$ be a two-sided weak smash product.

It is easy to show that the following maps are injective algebra maps:

$$\begin{aligned}&i_1 : A \to A\#H\#B, a \mapsto a\#1_H\#1_B \\ &i_2 : H \to A\#H\#B, h \mapsto 1_A\#h\#1_B \\ &i_3 : B \to A\#H\#B, b \mapsto 1_A\#1_H\#b \\ &i_4 : A\#H \to A\#H\#B, a\#h \mapsto a\#h\#1_B \\ &i_5 : H\#B \to A\#H\#B, h\#b \mapsto 1_A\#h\#b.\end{aligned}$$

So $A, B, H, A\#H, H\#B$ are thought as subalgebras of the two-sided weak smash product $A\#H\#B$.

Denote $a\#h\#b$ by ahb.

Lemma 3.1 *Let H be a finite-dimensional weak Hopf algebra with a bijective weak antipode S_H, and $A\#H\#B$ a two-sided weak smash product. Then, for any $a \in A,\ g, h \in H,\ b \in B$,*

(1) $g(ahb) = \Sigma(g_1 \rightharpoonup a)g_2hb$;

(2) $(ahb)g = \Sigma h_2g_2(S_H^{-1}(h_1g_1) \rightharpoonup a)(b \leftharpoonup g_3)$.

Proof (1) For any $a \in A,\ g, h \in H,\ b \in B$,

$$\begin{aligned}g(ahb) &= (1_A\#g\#1_B)(a\#h\#b) = \Sigma(g_1 \rightharpoonup a)\#g_2h_1\#(1_B \leftharpoonup h_2)b \\ &= \Sigma(g_1 \rightharpoonup a)\#g_2h_1\#(1_B \leftharpoonup \sqcap^R_H(h_2))b \\ &\overset{(\mathrm{W5})}{=} \Sigma(g_1 \rightharpoonup a)\#g_2h1_1\#(1_B \leftharpoonup S_H(1_2))b\end{aligned}$$

$$
\begin{aligned}
&= \Sigma(g_1 \rightharpoonup a)\#g_2h\#(1_B \leftharpoonup 1_1)(1_B \leftharpoonup S_H(1_2))b \\
&= \Sigma(g_1 \rightharpoonup a)\#g_2h\#(1_B \leftharpoonup 1_1)(1_B \leftharpoonup \sqcap_H^R(1_2))b \\
&= \Sigma(g_1 \rightharpoonup a)\#g_2h\#(1_B \leftharpoonup 1_1)(1_B \leftharpoonup 1_2)b \\
&= \Sigma(g_1 \rightharpoonup a)\#g_2h\#b = \Sigma(g_1 \rightharpoonup a)g_2hb.
\end{aligned}
$$

(2) By (1), for any $h \in H, a \in A$,

$$ha = \Sigma(h_1 \rightharpoonup a)h_2. \tag{A}$$

The following equality (B) holds. For any $a \in A, h \in H$,

$$\Sigma h_2(S_H^{-1}(h_1) \rightharpoonup a) = ah. \tag{B}$$

As a matter of fact, we have

$$
\begin{aligned}
\Sigma h_2(S_H^{-1}(h_1) \rightharpoonup a) &\overset{\text{(A)}}{=} \Sigma(h_2 \rightharpoonup (S_H^{-1}(h_1) \rightharpoonup a))h_3 \\
&= \Sigma(h_2S_H^{-1}(h_1) \rightharpoonup a)h_3 \overset{\text{(W9)}}{=} \Sigma(\sqcap_H^R(S_H^{-1}(h_1)) \rightharpoonup a)h_2 \\
&\overset{\text{(W7)}}{=} \Sigma(S_H^{-1}(\sqcap_H^L(h_1)) \rightharpoonup a)h_2 = \Sigma(S_H^{-1}(S_H(1_1) \rightharpoonup a))1_2h \\
&= \Sigma(1_1 \rightharpoonup a)1_2h \\
&= ah.
\end{aligned}
$$

By (A), it is easy to prove that (B) is equivalent to the following equality: for any $a \in A, h \in H$,

$$ah = \Sigma h_1(a \leftharpoonup h_2). \tag{C}$$

So

$$
\begin{aligned}
(ahb)g = (ah)(bg) &\overset{\text{(B,C)}}{=} \Sigma h_2(S_H^{-1}(h_1) \rightharpoonup a)g_1(b \leftharpoonup g_2) \\
&\overset{\text{(B)}}{=} \Sigma h_2\, \underbrace{g_2(S_H^{-1}(g_1) \rightharpoonup (S_H^{-1}(h_1) \rightharpoonup a))}(b \leftharpoonup g_3) \\
&\overset{\text{(W10)}}{=} \Sigma h_2g_2(S_H^{-1}(h_1g_1) \rightharpoonup a)(b \leftharpoonup g_3).
\end{aligned}
$$

Lemma 3.2 *Let H be a finite-dimensional weak Hopf algebra, and A a weak left H-module algebra, and B a weak right H-module algebra, and $A\#H\#B$ a two-sided weak smash product. Let $x \in \int_H^r$ (that is, for any $h \in H, xh = x\sqcap_H^R(h)$). Assume that V, W are left $A\#H\#B$-modules, and there is a left (A, B)-module map $\lambda : V \to W$, that is, $\lambda : V \to W$ is both a left A-module map and a left B-module map, and for any $h \in H, b \in B$,*

(3.2) $\Sigma b \leftharpoonup h_2 \otimes h_1 = \Sigma b \leftharpoonup h_1 \otimes h_2$.

Then the following map $\tilde{\lambda}$ is a left $A\#H\#B-$module map.

$$\tilde{\lambda} : V \to W,\ v \mapsto \Sigma S_H(x_1) \cdot \lambda(x_2 \cdot v).$$

Proof Firstly, we prove that $\tilde{\lambda} : V \to W$ is a left H-module map.

As a matter of fact, for any $h \in H$, by $xh = x \sqcap_H^R (h)$ we know

$$\Sigma x_1h_1 \otimes x_2h_2 = \Delta_H(xh) = \Delta_H(x \sqcap_H^R (h))$$

$$= \Sigma x_1 \sqcap_H^R (h)_1 \otimes x_2 \sqcap_H^R (h)_2 \overset{(W3)}{=} \Sigma x_1 1_1 \otimes x_2 \sqcap_H^R (h) 1_2$$
$$\overset{(W2)}{=} \Sigma x_1 1_1 \otimes x_2 1_2 \sqcap_H^R (h) \overset{(1.1)}{=} \Sigma x_1 \otimes x_2 \sqcap_H^R (h).$$

So we have

$$\Sigma x_1 h_1 \otimes x_2 h_2 \otimes h_3 = \Sigma x_1 \otimes x_2 \sqcap_H^R (h_1) \otimes h_2. \tag{D}$$

Then, for any $g \in H, v \in V$,

$$\begin{aligned}
g \cdot \tilde{\lambda}(v) &= S_H(h) \cdot \tilde{\lambda}(v) = \Sigma S_H(h) S_H(x_1) \cdot \lambda(x_2 \cdot v) \\
&= \Sigma S_H(h_1) h_2 S_H(h_3) S_H(x_1) \cdot \lambda(x_2 \cdot v) \\
&= \Sigma S_H(h_1) \sqcap_H^L (h_2) S_H(x_1) \cdot \lambda(x_2 \cdot v) \\
&\overset{(W8)}{=} \Sigma S_H(h_1) S_H(x_1) \cdot \lambda(x_2 \sqcap_H^L (h_2) \cdot v) \\
&= \Sigma S_H(x_1 h_1) \cdot \lambda(x_2 h_2 S_H(h_3) \cdot v) \\
&\overset{(D)}{=} \Sigma S_H(x_1) \cdot \lambda(x_2 \sqcap_H^R (h_1) S_H(h_2) \cdot v) \\
&= \Sigma S_H(x_1) \cdot \lambda(x_2 S_H(h) \cdot v) \quad (\Sigma \sqcap_H^R (h_1) S_H(h_2) = S_H(h)) \\
&= \tilde{\lambda}(g \cdot v).
\end{aligned}$$

Secondly, we prove that $\tilde{\lambda} : V \to W$ is a left A-module map.

For any $a \in A$, then, by (B), we get

$$a S_H(x) = \Sigma S_H(x_1)(x_2 \rightharpoonup a) \tag{E}$$

and hence

$$\begin{aligned}
a \cdot \tilde{\lambda}(v) &= \Sigma a S_H(x_1) \cdot \lambda(x_2 \cdot v) \overset{(E)}{=} \Sigma S_H(x_1)(x_2 \rightharpoonup a) \cdot \lambda(x_3 \cdot v) \\
&= \Sigma S_H(x_1) \cdot \lambda((x_2 \rightharpoonup a) x_3 \cdot v) \overset{(C)}{=} \Sigma S_H(x_1) \cdot \lambda(x_2 a \cdot v) \\
&= \tilde{\lambda}(a \cdot v).
\end{aligned}$$

Then, we prove that $\tilde{\lambda} : V \to W$ is a left B-module map.

As a matter of fact, for any $b \in B, v \in V$,

$$\begin{aligned}
b \cdot \tilde{\lambda}(v) &= \Sigma b S_H(x_1) \cdot \lambda(x_2 \cdot v) = \Sigma S_H(x_2)(b \leftharpoonup S_H(x_1)) \cdot \lambda(x_3 \cdot v) \\
&= \Sigma S_H(x_2) \cdot \lambda((b \leftharpoonup S_H(x_1)) x_3 \cdot v) = \Sigma S_H(x_2) \cdot \lambda(x_3 (b \leftharpoonup S_H(x_1) x_4) \cdot v) \\
&\overset{(3.2)}{=} \Sigma S_H(x_1) \cdot \lambda(x_2 (b \leftharpoonup S_H(x_3) x_4) \cdot v) = \Sigma S_H(x_1) \cdot \lambda(x_2 (b \leftharpoonup \sqcap_H^R(x_3)) \cdot v) \\
&= \Sigma S_H(x_1) \cdot \lambda(x_3 (b \leftharpoonup \sqcap_H^R(x_2)) \cdot v) \overset{(W6)}{=} \Sigma S_H(x_1) \cdot \lambda(x_2 1_2 (b \leftharpoonup 1_1) \cdot v) \\
&= \Sigma S_H(x_1) \cdot \lambda(x_2 \underbrace{1_1 (b \leftharpoonup 1_2)} \cdot v) \overset{(C)}{=} \Sigma S_H(x_1) \cdot \lambda(x_2 b \cdot v) \\
&= \tilde{\lambda}(b \cdot v).
\end{aligned}$$

Lemma 3.3 *Let H be a finite-dimensional weak Hopf algebra. Then, by [1], the following are equivalent.*

(1) H is semisimple;

(2) There exists a normalized right integral $e \in \int_H^r$ such that $\sqcap_H^R(e) = 1_H$.

Lemma 3.4 *Let H be a finite-dimensional semisimple weak Hopf algebra, and $e \in \int_H^r$ be a normalized right integral. Let A be a weak left H-module algebra, and B a weak right H-module algebra, and $A\#H\#B$ a two-sided weak smash product. Let M be a left $A\#H\#B$-module, and W a left $A\#H\#B$-submodule of M, and (3.2) hold. If W is a direct summand of M as (A,B)-modules, then W is a direct summand of M as $A\#H\#B$-modules.*

Proof Let $\lambda : V \to W$ be a projection of left (A,B)-modules. That is, λ is both a left A-module map and a left B-module map, and for any $w \in W$, $\lambda(w) = w$.

Define

$$\tilde{\lambda} : V \to W, v \mapsto \Sigma S_H(e_1) \cdot \lambda(e_2 \cdot v).$$

Then, by Lemma 3.2, we know that $\tilde{\lambda}$ is a left $A\#H\#B$-module map. In the following, we have only to check that $\tilde{\lambda}$ is a projection.

As a matter of fact, for any $w \in W$,

$$\begin{aligned}\tilde{\lambda}(w) &= \Sigma S_H(e_1) \cdot \lambda(e_2 \cdot w) = \Sigma S_H(e_1) \cdot (e_2 \cdot w)\\ &= \Sigma S_H(e_1)e_2 \cdot w = \sqcap_H^R(e) \cdot w\\ &= 1_H \cdot w = w.\end{aligned}$$

By the above lemmas, we have

Theorem 3.5 (Maschke theorem) *Let H be a finite-dimensional semisimple weak Hopf algebra, and $e \in \int_H^r$ be a normalized right integral. Let A be a weak left H-module algebra, and B a weak right H-module algebra, and $A\#H\#B$ a two-sided weak smash product, and (3.2) hold. If A and B are semisimple algebras, then $A\#H\#B$ is also a semisimple algebra.*

By Remark 2.2 and Theorem 3.5, we obtain Theorem 1 given in [3].

Corollary 3.6 *Let H be a finite-dimensional semisimple weak Hopf algebra. Let A be a weak left H-module algebra, and B a weak right H-module algebra.*

(1) If A is semisimple, then $A\#H$ is also semisimple.

(2) If B is semisimple and (3.2) holds, then $H\#B$ is also semisimple.

Let H be a finite-dimensional semisimple weak Hopf algebra. If H is cocommutative, then it is easy to see that $H^L = H^R$. So, by Theorem 6.9 in [9], H^* is semisimple. Hence, according to Example 2.3 and Theorem 3.5, $H^{**} \cong H$ as Hopf algebras, we obtain

Corollary 3.7 *Let H be a finite-dimensional weak Hopf algebra. If H is commutative and cosemisimple, then the two-sided weak smash product $H\#H^*\#H$ is semisimple.*

Let H be a finite-dimensional weak Hopf algebra. Then, by [10], A is a weak left (right) H-comodule algebra if and only if A is a weak right (left) H^*-module algebra. It is obvious that by (W3) H^L is a weak left H-comodule algebra via Δ_H and H^R is a weak right H-comodule algebra via Δ_H, so H^L is a weak right H^*-module algebra and H^R is a weak left H^*-module algebra. According to Proposition 2.11 in [1], H^L and H^R are separable algebras, so we have

Corollary 3.8 *Let H be a finite-dimensional commutative and cosemisimple weak Hopf algebra.*

Then $H^R \# H^* \# H^L$ is semisimple.

Corollary 3.9 *Let H be a finite-dimensional cocommutative semisimple weak Hopf algebra, and A be a weak (H, H)-bimodule algebra. If A is semisimple, then $A \# H \# A$ is also semisimple.*

Acknowledgments This work was partly done during Professor Sonia Natale visited Nanjing Agricultural University. The author would like to thank her for fruitful discussions.

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