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Dynamics of the Arithmetic Function Ω_k

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Abstract In this paper, we generalize the results of Goldring W. in 2006 and study dynamics of the arithmetic function Ω_k.

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1. Introduction

A classical problem of number theory is to study dynamics of arithmetic functions. There are many results in the literature concerning various functions[1−11]. Goldring[3] established dynamics of a type of arithmetic function w. Let A_3 be the set of all positive integers pqr, where p, q, r are primes and possibly two, but not all three of them are equal. For any $n = pqr \in A_3$, define a function w by $w(n) = P(p+q)P(p+r)P(q+r)$, where $P(m)$ is the largest prime factor of m. It is clear that if $n = pqr \in A_3$, then $w(n) \in A_3$. For any $n \in A_3$, define $w^0(n) = n$, $w^i(n) = w(w^{i-1}(n))\,(i = 1,\ 2,\ \ldots)$. Goldring[3] proved that any element $n \in A_3$ is w-periodic, i.e., there exists an integer $i \geq 0$ such that $w^i(n) = 20$. For recent progress one may see[1−2].

In this paper, we generalize the result of Goldring[3] and study dynamics of the arithmetic function Ω_k.

In what follows we shall try to be consistent in our use of the following notations.

Definition 1 *Let $\overline{A_k} := \{n \in \mathbf{Z}_+ | \Omega(n) = k\}$, where $\Omega(n)$ is the total number of prime factors of n.*

For element $n \in \overline{A_k}$, let $n = p_1p_2p_3\cdots p_k$, where $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_k$ and all of them are primes. Then $p_1, p_2, \ldots, p_k$ are not all equal. We define an arithmetic function Ω_k: $\overline{A_k} \to \overline{A_k}$ by

$$\Omega_k(n) = P(p_1+p_2)P(p_2+p_3)\cdots P(p_k+p_1).$$

Definition 2 *Since $\Omega_k(\overline{A_k}) \subseteq \overline{A_k}$, we define the Ω_k-orbit of n by a sequence $\Delta(n)$ such as*

$$\Delta(n) = [\ n,\ \ \Omega_k(n),\ \ \Omega_k^2(n),\ \ldots, \Omega_k^i(n), \ldots\ ]$$

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where $\Omega_k^0(n) = n$, $\Omega_k^i(n) = \Omega_k(\Omega_k^{i-1}(n))$, $i = 1, 2, \ldots$. And let the set

$$A_k = \overline{A_k} \setminus \{n \in \overline{A_k} \mid \omega(\Omega_k^i(n)) = 1 \text{ for some } i\},$$

where $\omega(n)$ is the number of distinct prime factors of n.

An element n of A_k is Ω_k-periodic if its Ω_k-orbit is periodic, i.e. there exists a non-negative integer s and a positive integer t such that $\Omega_k^s(n) = \Omega_k^{s+t}(n)$. Only the periodic of one type of function Ω_k appears in our paper, so we can just call it simply.

The smallest integer s which satisfies the above condition is called the index of periodicity of n, denoted $\mathrm{ind}(n)$. The positive integer t is called the periodicity of n and the smallest periodicity t is denoted by $\mathrm{ord}(n)$. For example , if $a \in A_k$ is periodic, take $s = \mathrm{ind}(a)$, $t = \mathrm{ord}(a)$, then

$$\Delta(a) = [\ a, \Omega_k(a), \ldots, \Omega_k^{s-1}(a),\ \overline{\Omega_k^s(a), \ldots, \Omega_k^{s+t-1}(a)}\ ].$$

Definition 3 *The array $\overline{b_1, b_2, \ldots, b_t}$ $(i = 1, 2, \ldots, t)$ is called a circular array of A_k if $b_1, b_2, b_3, \ldots, b_t \in A_k$ and these numbers satisfy $\Omega_k(b_s) = b_{s+1}$, $s = 1, 2, \ldots, t-1$, $\Omega_k(b_t) = b_1$.*

In general, we regard all arrays such as $\overline{b_i, b_{i+1}, \ldots, b_t, b_1, \ldots, b_{i-1}}, i = 1, 2, \ldots, t$ as equal array, denoted by $b_i^{\Omega_k}$, where b_i is any element in this circular array.

An element n of A_k is said to lie in circular array $b_i^{\Omega_k}$ ultimately if there exists an integer $j \geqslant 0$ such that $\Omega_k^j(n) \in b_i^{\Omega_k}$. The whole circular array in A_k is denoted by $A_k^{\Omega_k}$ and the cardinality of $A_k^{\Omega_k}$ is denoted by $|A_k^{\Omega_k}|$. The result of Goldring[1] can be formulated as $A_3^{\Omega_3} = \{20^{\Omega_3}\}$.

The main results in our paper are as follows:

Theorem 1 *Every element of A_k is periodic and each lies in some one circular array ultimately. When $k \geqslant 5$, A_k has $\frac{1}{2}(k-2)(k-3)$ circular arrays properly, that is to say, $A_k^{\Omega_k} = \{(2^a 3^b 5^c)^{\Omega_k} | a + b + c = k, a \geqslant 1, b \geqslant 2, c \geqslant 1, a, b, c \in \mathbb{Z}\}$. In addition $A_4^{\Omega_4} = \{60^{\Omega_4}, 90^{\Omega_4}\}$.*

Theorem 2 *If $n \in A_k$ and $P(n) = p > 3$, then $P(\Omega_k^i(n)) \leqslant p + 2$ for any integer $i \geqslant 0$.*

2. Proofs of Theorems

Lemma 1 *If t, p, q are prime and $t \leqslant q < p$, then $P(t + q) \leqslant p$. The equality holds if and only if $t = 2$, $q = p - 2$.*

Proof If t, q are both odd primes, then $P(t + q) < p$. If $2 = t \leqslant q < p$ and $q + 2$ is composite, then $P(t+q) = P(2+q) \leqslant q < p$. If $2 = t \leqslant q < p$ and $q+2$ is prime, then $P(t+q) = 2+q \leqslant p$. The equality holds if and only if $t = 2$, $q = p - 2$. This completes the proof. □

Lemma 2 *If $n \in A_k$ and $P(n) > 5$, then there exists a positive integer $1 \leqslant i \leqslant 2k$ such that $P(\Omega_k^i(n)) < P(n)$.*

Proof By directly verification we know that Lemma 2 is true for n with $P(n) = 7$. For $n \in A_k$, let $n = p_1, p_2, \ldots, p_k$, where $p_1, p_2, \ldots, p_k$ are not all equal primes and $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_k$. Let $P(n) = p > 7$ and the exponent of p is m, i.e., $p^m \| n$.

Then we consider the following three cases.

Case 1 Both $p + 2$ and $p - 2$ are composite numbers.

If p_i, $p_j < p$, then p_i, $p_j < p - 2$. Thus $P(p_i + p_j) < p - 2 < p$ by Lemma 1. Since $p + 2$

is composite, we know that $P(p_i + p) < p$ for any prime $p_i < p$. Hence the exponent of p in $P(\Omega_k(n))$ is less than m. Now let function Ω_k act $m-1$ times on $\Omega_k(n)$ continually, we have $p \nmid P(\Omega_k^m(n))$, hence $P(\Omega_k^m(n)) < p$.

Case 2 $p+2$ is composite and $p-2$ is prime.

Noting that $p > 7$ and $p,\ p-2$ are both primes, we have $p-4$ is composite. Assume $(p-2)^s \,\|n$. Similarly to Case 1, we have $p-2 \nmid P(\Omega_k^s(n))$, and the exponent of p in $P(\Omega_k^s(n))$ is not larger than $m+s$. Now let function Ω_k act $m+s$ times on $\Omega_k^s(n)$ continually, we have $p \nmid P(\Omega_k^{m+2s}(n))$, hence $P(\Omega_k^{m+2s}(n)) < p,\ \ m+2s \leqslant 2k$.

Case 3 $p+2$ is prime.

Noting that $p > 7$ and $p,\ p+2$ are both primes, we know that $p-2,\ p+4$ are both composite numbers and the exponent of $p+2$ in $\Omega_k^m(n)$ is not larger than m and $p \nmid P(\Omega_k^m(n))$. Now let Ω_k act m times on $\Omega_k^m(n)$, we obtain that $p+2 \nmid P(\Omega_k^{2m}(n))$ and prime factor p does not appear any more during this process, hence $P(\Omega_k^{2m}(n)) < p,\ 2m \leqslant 2k$.

This completes the proof.

Proof of Theorem 1 By Lemma 2, we need only to consider the element n in A_k whose largest prime factor $P(n) \leqslant 5$.

When $k = 4$, $|A_4^{\Omega_4}| = 2$. The elements of A_4 lying in the circular array $(2^2 \cdot 3 \cdot 5)^{\Omega_4}$ are 40, 60, 54, 225, respectively. The elements lying in $(2{\cdot}3^2{\cdot}5)^{\Omega_4}$ are 24, 36, 100, 90, 150, 250, 135, 375. Therefore, $A_4^{\Omega_4} = \{60^{\Omega_4},\ 90^{\Omega_4}\}$.

When $k = 5$, the elements of A_5 lying in the circular array $(2^2 \cdot 3^2 \cdot 5)^{\Omega_5}$ are 675, 162, 180, the elements lying in $(2 \cdot 3^3 \cdot 5)^{\Omega_5}$ are 1125, 1875, 270, 300, 500, 72, and the elements lying in $(2 \cdot 3^2 \cdot 5^2)^{\Omega_5}$ are 405, 1250, 750, 450, 108, 120, 200, 48, 80. Therefore, we have

$$|A_5^{\Omega_5}| = 3 = \frac{1}{2}(5-2)(5-3),$$

$$A_5^{\Omega_5} = \{(2^2 \cdot 3^2 \cdot 5)^{\Omega_5},\ (2 \cdot 3^3 \cdot 5)^{\Omega_5},\ (2 \cdot 3^2 \cdot 5^2)^{\Omega_5}\}.$$

When $k \geqslant 6$, let $n = 2^a 3^b 5^c$ and $a+b+c = k,\ a,b,c \geqslant 0,\ a,b,c \in \mathbb{Z}$.

If $a \geqslant 1,\ b \geqslant 2,\ c \geqslant 1$, then $\Delta(n) = [\ \overline{2^a 3^b 5^c,\ 2^a 3^{b-1} 5^c 7}\ ]$. There are $\frac{1}{2}(k-2)(k-3)$ circular arrays altogether. The whole of them is

$$\{(2^a 3^b 5^c)^{\Omega_k} \mid a+b+c = k, a \geqslant 1, b \geqslant 2, c \geqslant 1,\ a,b,c \in \mathbb{Z}\}. \tag{1}$$

For $n \in A_k$, at most one of a,b,c is 0. In the following section, we observe that if just one of a,b,c is 0, or $b = 1$, then n lies in some one circular array of (1) ultimately.

Then we consider the following cases.

Case 1 None of a,b,c is 0.

If $a = b = 1, c > 1$, then

$$\Delta(n) = [2 \cdot 3 \cdot 5^{k-2}, 2 \cdot 5^{k-2} \cdot 7, 3^2 \cdot 5^{k-3}, \overline{2 \cdot 3^2 \cdot 5^{k-3}, 2 \cdot 3 \cdot 5^{k-3} \cdot 7}].$$

If $a > 1,\ c \geqslant 2,\ b = 1$, i.e., $n = 2^{k-c-1} \cdot 3 \cdot 5^c$, then

$$\Delta(n) = [2^{k-c-1}{\cdot}3{\cdot}5^c,\ 2^{k-c-1}{\cdot}5^c{\cdot}7,\ 2^{k-c-2}{\cdot}3^2{\cdot}5^{c-1}{\cdot}7,\ \overline{2^{k-c-2} \cdot 3^3 \cdot 5^{c-1},\ 2^{k-c-2} \cdot 3^2 \cdot 5^{c-1} \cdot 7}\ ].$$

If $a \geqslant 1,\ c=1,\ b=1$, i.e., $n = 2^{k-2}\cdot 3\cdot 5$, then

$$\Delta(n) = [2^{k-2}\cdot 3\cdot 5,\ 2^{k-2}\cdot 5\cdot 7,\ 2^{k-3}\cdot 3^2\cdot 7,\ \overline{2^{k-4}\cdot 3^2\cdot 5^2,\ 2^{k-4}\cdot 3\cdot 5^2\cdot 7}\ ].$$

Case 2 $a=0$.

If $b, c \geqslant 3$, i.e., $n = 3^b\cdot 5^c$, then

$$\Delta(n) = [3^b\cdot 5^c,\ \overline{2^2\cdot 3^{b-1}\cdot 5^{c-1},\ 2^2\cdot 3^{b-2}\cdot 5^{c-1}\cdot 7}\ ].$$

If $b=1$, i.e., $n = 3\cdot 5^{k-1}$, then

$$\Delta(n) = [3\cdot 5^{k-1},\ 2^2\cdot 5^{k-2},\ 2\cdot 5^{k-3}\cdot 7^2,\ 3^2\cdot 5^{k-4}\cdot 7^2,\ 2\cdot 3^2\cdot 5^{k-4}\cdot 7,\ \overline{2\cdot 3^3\cdot 5^{k-4},\ 2\cdot 3^2\cdot 5^{k-4}\cdot 7}\ ].$$

If $b=2$, i.e., $n = 3^2\cdot 5^{k-2}$, then

$$\Delta(n) = [3^2\cdot 5^{k-2},\ 2^2\cdot 3\cdot 5^{k-3},\ 2^2\cdot 5^{k-3}\cdot 7,\ 2\cdot 3^2\cdot 5^{k-4}\cdot 7,\ \overline{2\cdot 3^3\cdot 5^{k-4},\ 2\cdot 3^2\cdot 5^{k-4}\cdot 7}\ ].$$

If $c=1$, i.e., $n = 3^{k-1}\cdot 5$, then

$$\Delta(n) = [3^{k-1}\cdot 5,\ 2^2\cdot 3^{k-2},\ \overline{2\cdot 3^{k-3}\cdot 5^2,\ 2\cdot 3^{k-4}\cdot 5^2\cdot 7}\ ].$$

If $c=2$, i.e., $n = 3^{k-2}\cdot 5^2$, then

$$\Delta(n) = [3^{k-2}\cdot 5^2,\ \overline{2^2\cdot 3^{k-3}\cdot 5,\ 2^2\cdot 3^{k-4}\cdot 5\cdot 7}\ ].$$

Case 3 $b=0$.

If $a, c \geqslant 3$, i.e., $n = 2^a\cdot 5^c$, then

$$\Delta(n) = [2^a\cdot 5^c,\ 2^{a-1}\cdot 5^{c-1}\cdot 7^2,\ 2^{a-2}\cdot 3^2\cdot 5^{c-2}\cdot 7^2,\ 2^{a-2}\cdot 3^3\cdot 5^{c-2}\cdot 7,\ \overline{2^{a-2}\cdot 3^4\cdot 5^{c-2},\ 2^{a-2}\cdot 3^3\cdot 5^{c-2}\cdot 7}\ ].$$

If $a=1$, i.e., $n = 2\cdot 5^{k-1}$, then

$$\Delta(n) = [2\cdot 5^{k-1},\ 5^{k-2}\cdot 7^2,\ 3^2\cdot 5^{k-3}\cdot 7,\ \overline{2\cdot 3^2\cdot 5^{k-3},\ 2\cdot 3\cdot 5^{k-3}\cdot 7}\ ].$$

If $a=2$, i.e., $n = 2^2\cdot 5^{k-2}$, then

$$\Delta(n) = [2^2\cdot 5^{k-2},\ 2\cdot 5^{k-3}\cdot 7^2,\ 3^2\cdot 5^{k-4}\cdot 7^2,\ 2\cdot 3^2\cdot 5^{k-4}\cdot 7,\ \overline{2\cdot 3^3\cdot 5^{k-4},\ 2\cdot 3^2\cdot 5^{k-4}\cdot 7}\ ].$$

If $c=1$, i.e., $n = 2^{k-1}\cdot 5$, then

$$\Delta(n) = [2^{k-1}\cdot 5,\ 2^{k-2}\cdot 7^2,\ 2^{k-3}\cdot 3^2\cdot 7,\ \overline{2^{k-4}\cdot 3^2\cdot 5^2,\ 2^{k-4}\cdot 3\cdot 5^2\cdot 7}\ ].$$

If $c=2$, i.e., $n = 2^{k-2}\cdot 5^2$, then

$$\Delta(n) = [2^{k-2}\cdot 5^2,\ 2^{k-3}\cdot 5\cdot 7^2,\ 2^{k-4}\cdot 3^2\cdot 7^2,\ 2^{k-5}\cdot 3^2\cdot 5^2\cdot 7,\ \overline{2^{k-5}\cdot 3^3\cdot 5^2,\ 2^{k-5}\cdot 3^2\cdot 5^2\cdot 7}\ ].$$

Case 4 $c=0$.

If $a \geqslant 2,\ b \geqslant 3$, i.e., $n = 2^a\cdot 3^b$, then

$$\Delta(n) = [2^a\cdot 3^b,\ \overline{2^{a-1}\cdot 3^{b-1}\cdot 5^2,\ 2^{a-1}\cdot 3^{b-2}\cdot 5^2\cdot 7}\ ].$$

If $a=1$, i.e., $n = 2\cdot 3^{k-1}$, then

$$\Delta(n) = [2\cdot 3^{k-1},\ 3^{k-2}\cdot 5^2,\ \overline{2^2\cdot 3^{k-3}\cdot 5,\ 2^2\cdot 3^{k-4}\cdot 5\cdot 7}\ ].$$

If $b=1$, i.e., $n=2^{k-1}\cdot 3$, then

$$\Delta(n)=[2^{k-1}\cdot 3,\ 2^{k-2}\cdot 5^2,\ 2^{k-3}\cdot 5\cdot 7^2,\ 2^{k-4}\cdot 3^2\cdot 7^2,\ 2^{k-5}\cdot 3^2\cdot 5^2\cdot 7,\ \overline{2^{k-5}\cdot 3^3\cdot 5^2,\ 2^{k-5}\cdot 3^2\cdot 5^2\cdot 7}\ ].$$

If $b=2$, i.e., $n=2^{k-2}\cdot 3^2$, then

$$\Delta(n)=[2^{k-2}\cdot 3^2,\ 2^{k-3}\cdot 3\cdot 5^2,\ 2^{k-3}\cdot 5^2\cdot 7,\ 2^{k-4}\cdot 3^2\cdot 5\cdot 7,\ \overline{2^{k-4}\cdot 3^3\cdot 5,\ 2^{k-4}\cdot 3^2\cdot 5\cdot 7}\ ].$$

This completes the proof of Theorem 1. □

Proof of Theorem 2 For $n\in A_k$, let $n=p_1p_2p_3\cdots p_k$, where $p_1\geqslant p_2\geqslant\cdots\geqslant p_k$ and $P(n)=p>3$. Then

$$\Omega_k(n)=P_{12}P_{23}\cdots P_{k-1k}P_{k1},\quad \text{where}\ \ P_{ij}=P(p_i+p_j).$$

Noting that $P_{ij}\leqslant p+2$, $i,j=1,2\ldots,k$, we have $P(\Omega_k(n))\leqslant p+2$. We rearrange the prime factors of $\Omega_k(n)$ in descending order, so that

$$\Omega_k(n)=P_1P_2\cdots P_k,\quad \text{where}\ P_1\geqslant\cdots\geqslant P_k.$$

Now acting function Ω_k on it, we have

$$\Omega_k^2(n)=P_{12}^2P_{23}^2\cdots P_{k1}^2,\quad \text{where} P_{st}^2=P(P_s+P_t),\quad s,t=1,2,\ldots,k.$$

If $p+2$ is prime, noting that $p>3$, and $p,\ p+2$ are both primes, then $p+4$ is composite, and so $P_{st}^2\leqslant p+2$, $s,t=1,2,\ldots,k$, hence $P(\Omega_k^2(n))\leqslant p+2$. If $p+2$ is composite, $P(\Omega_k^2(n))\leqslant p+2$, obviously.

By induction on i, we have $P(\Omega_k^i(n))\leqslant p+2$ for any integer $i\geqslant 0$ if $p+2$ is prime, and $P(\Omega_k^i(n))\leqslant p$ if $p+2$ is composite.

This completes the proof of Theorem 2. □

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