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Necessary and Sufficient Condition for Adjoint Uniqueness of the Graph $(\bigcup_{i\in A} P_i)\bigcup(\bigcup_{j\in B} U_j)$

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Abstract For a graph G, let $h(G;x)=h(G)$ and $[G]_h$ denote the adjoint polynomial and the adjoint equivalence class of G, respectively. In this paper, a new application of $[G]_h$ is given. Making use of $[G]_h$, we give a necessary and suffcient condition for adjoint uniqueness of the graph H such that $H\neq G$, where $H=(\bigcup_{i\in A} P_i)\bigcup(\bigcup_{j\in B} U_j)$, $A\subseteq A'=\{1,2,3,5\}\bigcup\{2n|n\in N, n\geq 3\}$, $B\subseteq B'=\{7,2n|n\in N, n\geq 5\}$ and $G=aP_1\bigcup a_0P_2\bigcup a_1P_3\bigcup a_2P_5\bigcup(\bigcup_{i=3}^{n}a_iP_{2i})$.

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1. Introduction

All the graphs considered here are simple and finite. Undefined notation and terminology can be found in [1]. For a graph G, let $\overline{G}$, $V(G)$, $E(G)$, $p(G)$ and $q(G)$, respectively, be the complement, vertex set, edge set, order and size of G. An ideal subgraph G_0 of graph G is a spanning subgraph of G such that every component of G_0 is a complete graph[2]. The adjoint polynomial of G is defined as follows

Definition 1.1[2] *Let G be a graph with p vertices and q edges. The polynomial*

$$h(G;x)=\sum_{i=0}^{p} b_i(G)x^{p-i}$$

is called the adjoint polynomial of G, where $b_i(G)$ is the number of ideal subgraphs with $p-i$ components.

From Definition 1.1, it is not difficult to get that $b_1(G)=q(G)$[2]. Thus $q(G)=p(H)$ and $p(G)=p(H)$ if $h(G;x)=h(H;x)$.

Two graphs G and H are said to be adjointly equivalent, denoted by $G\overset{h}{\sim}H$, if $h(G;x)=h(H;x)$. Clearly, "$\overset{h}{\sim}$" is an equivalence relation on the family of all graphs. Let $[G]_h=\{H\overset{h}{\sim}$

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$G\}$. A graph G is adjointly unique if $h(H;x) = h(G;x)$ implies that $G\cong H$. It has been well-known that $\overline{G}$ is chromatically unique if and only if G is adjointly unique[2].

Definition 1.2[2] *Let G be a graph and $h_1(G;x)$ the polynomial with a nonzero constant term such that $h(G;x) = x^{\alpha(G)}h_1(G;x)$. If $h_1(G;x)$ is an irreducible polynomial over the rational number field, then G is called irreducible graph.*

For convenience, we simply denote $h(G;x)$ by $h(G)$ and $h_1(G;x)$ by $h_1(G)$, respectively. Let $\beta(G)$ denote the minimum real root of $h(G)$. For two graphs G and H, $G\bigcup H$ denotes the disjoint union of G and H, and mH stands for the disjoint union of m copies.

Now we define some classes of graphs, which will be used throughout the paper.

(1) C_n (resp. P_n) denotes the cycle (resp. the path) of order n, and write $\mathcal{C} = \{C_n|n\geq 3\}$, $\mathcal{P} = \{P_n|n\geq 2\}$.

(2) D_n $(n \geq 4)$ denotes the graph obtained from C_3 and P_{n-2} by identifying a vertex C_3 with an end–vertex of P_{n-2}.

(3) T_{l_1,l_2,l_3} denotes a tree with a vertex v of degree 3 such that $T_{l_1,l_2,l_3} - v = P_{l_1}\bigcup P_{l_2}\bigcup P_{l_3}$, and write $\mathcal{T}_1 = \{T_{1,1,n}|n\geq 1\}$.

(4) Let P_{n-2} be the path with vertex sequence $x_1, x_2, \ldots, x_{n-2}$. U_n denotes the graph obtained from P_{n-2} by adding pendant edges at vertices x_2 and x_{n-3}, and write $\mathcal{U} = \{U_n|n\geq 6\}$.

(5) K_n denotes the complete graph with order n and $K_4^- = K_4 - e$, where $e\in E(K_4)$.

(6) Let $C_4(P_2)$ be the graph obtained from C_4 and P_2 by identifying a vertex of C_4 with an end–vertex of P_2, and let $K_{1,n-1}$ be the star with order n.

(7) $C_3(P_2, P_2)$ denotes the graph obtained from C_3 by adding a pendant edge at any two vertices of C_3, respectively.

By the adjoint equivalence class $[G]_h$ of a graph G, the necessary and sufficient condition for adjoint uniqueness of G can be determined. In this paper, a new application of $[G]_h$ is given. Making use of $[G]_h$, we establish a necessary and suffcient condition for adjoint uniqueness of the graph H such that $H\not\cong G$, where $H = (\bigcup_{i\in A} P_i)\bigcup(\bigcup_{j\in B} U_j)$, $A \subseteq A' = \{1, 2, 3, 5\}\bigcup\{2n|n \in N, n \geq 3\}$, $B \subseteq B' = \{7, 2n|n \in N, n \geq 5\}$ and $G = aP_1\bigcup a_0P_2\bigcup a_1P_3\bigcup a_2P_5\bigcup(\bigcup_{i=3}^{n} a_iP_{2i})$.

2. Basic lemmas

Lemma 2.1[2] *Let G be a graph with k components $G_1, G_2, \ldots, G_k$. Then*

$$h(G) = \prod_{i=1}^{k} h(G_i).$$

For an edge $e = v_1v_2$ of a graph G, the graph $G{*}e$ is defined as follows: The vertex set of $G{*}e$ is $(V(G) - \{v_1, v_2\})\bigcup\{v\}(v\notin G)$, and the edge set of $G{*}e$ is $\{e'|e' \in E(G)$, e' is not incident with v_1 or $v_2\}\bigcup\{uv|u\in N_G(v_1)\bigcap N_G(v_2)\}$, where $N_G(v)$ is the set of vertices of G which are adjacent to v.

Lemma 2.2[4] *Let G be a graph with $e\in E(G)$. Then*

$$h(G;x) = h(G-e;x) + h(G{*}e;x),$$

where $G-e$ denotes the graph obtained by deleting the edge e from G.

Lemma 2.3[5] *Let G be a connected graph. Then*

(1) $\beta(G)=-4$ if and only if

$$G\in\mathcal{G}_1=\{T_{1,2,5},T_{2,2,2},T_{1,3,3},K_{1,4},C_4(P_2),C_3(P_2,P_2),K_4^-,D_8\}\cup\mathcal{U}.$$

(2) $\beta(G)>-4$ if and only if

$$G\in\mathcal{G}_2=\{P_1,T_{1,2,i}(2\le i\le 4),D_i(4\le i\le 7)\}\cup\mathcal{P}\cup\mathcal{C}\cup\mathcal{T}_1.$$

(3) $\beta(G)\ge-3$ if and only if

$$G\in\mathcal{G}_3=\{P_1,P_2,P_3,P_4,P_5,C_3,T_{1,1,1}\}.$$

Lemma 2.4 *(1)*[2] *$h(P_{2n+1})=h(P_n\bigcup C_{n+1})$, where $n\ge3$.*

(2)[5] *$h(U_n)=x^3(x+4)h(P_{n-4})$, $h(U_6)=h(2K_1\bigcup K_4^-)$, $h(U_8)=h(C_3\bigcup K_{1,4})$, $h(U_9)=h(K_1\bigcup K_{1,3}\bigcup K_4^-)$, and $h(U_{2m+1})=h(U_{m+2}\bigcup C_{m-1})$, where $m\ge5$.*

(3) $h(P_1\bigcup U_m)=h(K_{1,4}\bigcup P_{m-4})$, where $m\ge6$.

(4) For $n\ge2$, $m\ge6$ and $m\ne n+4$, $h(P_n\bigcup U_m)=h(U_{n+4}\bigcup P_{m-4})$.

Proof of (3) We have, from (2) of the lemma, that

$$h(P_1\bigcup U_m)=x^4(x+4)h(P_{m-4})=h(K_{1,4}\bigcup P_{m-4}).$$

Proof of (4) It follows, from (2) of the lemma, that

$$h(P_n\bigcup U_m)=(x^4(x+4)h(P_{m-4}))h(P_n)=(x^4(x+4)h(P_n))h(P_{m-4})=h(U_{n+4}\bigcup P_{m-4}).$$

Lemma 2.5 *(1)*[6] *For $n\ge2$, $\beta(P_n)>\beta(P_{n+1})>\beta(C_{n+1})>\beta(C_{n+2})$.*

(2)[5] *$\beta(C_4)=\beta(D_4)=\beta(P_7)$, $\beta(T(1,2,2))=\beta(D_5)=\beta(P_{11})$, $\beta(T(1,2,3))=\beta(D_6)=\beta(P_{17})$, $\beta(T(1,2,4))=\beta(D_7)=\beta(P_{29})$.*

(3)[2,5] *For $m\ge4$ and $l\ge1$, $(h_1(C_m),h_1(P_{2l}))=1$ and $\beta(P_{2m-1})=\beta(C_m)=\beta(T_{1,1,m-2})$.*

3. The chromaticity of graphs

Lemma 3.1 *Let $G=aP_1\bigcup a_0P_2\bigcup a_1P_3\bigcup a_2P_5$, where a and a_i $(i=0,1,2)$ are nonnegative integers. Then*

$$[G]_h=\mathcal{G}_4=\{(a-r)P_1\bigcup(a_0+r)P_2\bigcup a_1P_3\bigcup(a_2-r)P_5\bigcup rT_{1,1,1}|0\le r\le\min\{a,a_2\}\}.$$

Proof Obviously, $\mathcal{G}_4\subseteq[G]_h$. Now, we need only prove $[G]_h\subseteq\mathcal{G}_4$.

Let an any graph $H\in[G]_h$ and $H=\bigcup_i H_i$. Then $h(H)=h(G)$. By (3) of Lemma 2.3, we have that $H_i\in\mathcal{G}_3$. By Lemma 2.2 and calculation, it follows that

$$h_1(P_4)=h_1(C_3)=x^2+3x+1,\ h_1(P_5)=h_1(P_2)h_1(T_{1,1,1})=(x+1)(x+3),$$

$$h_1(P_3)=x+2,\ h_1(C_3)\nmid h_1(P_i)\ (i=2,3,5)\text{ and }h_1(C_3)\nmid h_1(T(1,1,1)).$$

This implies $h_1(C_3)\nmid h_1(G)$. Simultaneously, $h_1(C_3)\nmid h_1(H)$. Hence H contains no C_3 and P_4 as its components. Without loss of generality, let

$$H = bP_1 \bigcup b_0 P_2 \bigcup b_1 P_3 \bigcup b_2 P_5 \bigcup sT_{1,1,1}.$$

Comparing the common factors $x+3$, $x+2$ and $x+1$ of $h(H)$ with those of $h(G)$, we have $b_2 + s = a_2$, $b_1 = a_1$ and $b_0 + b_2 = a_0 + a_2$, respectively. So $b_2 = a_2 - s$ and $b_0 = a_0 + s$. Hence

$$H = bP_1 \bigcup (a_0 + s) P_2 \bigcup a_1 P_3 \bigcup (a_2 - s) P_5 \bigcup sT_{1,1,1} \text{ and } 0 \leq s \leq a_2.$$

Note that $h^s(P_2) h^s(T_{1,1,1}) = h^s(P_1) h^s(P_5)$. Then we eliminate the common factors of $h(H)$ and $h(G)$. So, we obtain $b + s = a$, that is, $b = a - s$ and $0 \leq s \leq a$. Thus

$$H = (a-s) P_1 \bigcup (a_0 + s) P_2 \bigcup a_1 P_3 \bigcup (a_2 - s) P_5 \bigcup sT_{1,1,1} \text{ and } 0 \leq s \leq \{a, a_2\},$$

which imply $H \in \mathcal{G}_4$ and $[G]_h \subseteq \mathcal{G}_4$. This completes the proof of the lemma. □

Theorem 3.1 *Let* $G = aP_1 \bigcup a_0 P_2 \bigcup a_1 P_3 \bigcup a_2 P_5 \bigcup (\bigcup_{i=3}^n a_i P_{2i})$, *where* a *and* a_i $(0 \leq i \leq n)$ *are nonnegative integers. Then*

$$[G]_h = \mathcal{G}_5 = \{(a-r) P_1 \bigcup (a_0 + r) P_2 \bigcup a_1 P_3 \bigcup (a_2 - r) P_5 \bigcup rT_{1,1,1} \bigcup (\bigcup_{i=3}^{n} a_i P_{2i}) | 0 \leq r \leq \min\{a, a_2\}\}.$$

Proof Obviously, $\mathcal{G}_5 \subseteq [G]_h$, so we should prove $[G]_h \subseteq \mathcal{G}_5$.

Let an any graph $H \in [G]_h$ and $H = \bigcup_{k \in A} H_k$, where H_k is a connected graph. From $h(H) = h(G)$ and (2) of Lemma 2.3, we have $H_k \in \mathcal{G}_2$. If $a_i = 0$ for $3 \leq i \leq n$, from Lemma 3.1, we know that the theorem holds.

If $a_{i_0} \neq 0$ for some $i_0 \in [3, n]$, from (1) of Lemma 2.5, it follws that $\beta(G) = \beta(P_{2i_0})$. By (2) and (3) of Lemma 2.5, we obtain that

$$H_k \in \{P_i, P_{2l}, C_3, T_{1,1,1}) | 1 \leq i \leq 5, l \geq 3\}.$$

So there exists a number $k \in A$ such that $\beta(H_k) = \beta(H) = \beta(G) = \beta(P_{2i_0})$ and $H_k \cong P_{2i_0}$. Eliminating the common factor $h(P_{2i_0})$ of $h(H)$ and $h(G)$ and repeating the above process until $P_{2i} \not\subseteq G$ for $3 \leq i \leq n$, we have that

$$\bigcup_{i \in A_1} H_k \cong \bigcup_{i=3}^{n} a_i P_{2i} \text{ and } G' = aP_1 \bigcup a_0 P_2 \bigcup a_1 P_3 \bigcup a_2 P_5. \tag{3.1}$$

Let $H' = \bigcup_{i \in A - A_1} H_k$. Eliminating the common factor $h(\bigcup_{i=3}^n a_i P_{2i})$ of $h(H)$ and $h(G)$, we have $h(H') = h(G')$, that is, $H' \in [G']_h$. From (3.1) and Lemma 3.1, it follows that

$$H = (a-r) P_1 \bigcup (a_0 + r) P_2 \bigcup a_1 P_3 \bigcup (a_2 - r) P_5 \bigcup rT_{1,1,1} \bigcup (\bigcup_{i=3}^{n} a_i P_i), 0 \leq r \leq \min\{a, a_2\}.$$

Hence $H \in \mathcal{G}_5$, that is, $[G]_h \subseteq \mathcal{G}_5$. This completes the proof of the theorem. □

Corollary 3.1 *Let* $G \in \{aP_1 \bigcup a_0 P_2 \bigcup a_1 P_3 \bigcup (\bigcup_{i=3}^n a_i P_{2i}), a_0 P_2 \bigcup a_1 P_3 \bigcup a_2 P_5 \bigcup (\bigcup_{i=3}^n a_i P_{2i})\}$, *where* a *and* a_i $(0 \leq i \leq n)$ *are nonnegative integers. Then* G *is adjoint uniqueness.*

Lemma 3.2 *Let* $H = \bigcup_{k=1}^s H_k$ *and* $G = a_0 P_2 \bigcup a_1 P_3 \bigcup a_2 P_5 \bigcup (\bigcup_{i=3}^n a_i P_{2i})$. *If* $h_1(H) = h_1(G)$, *then* $\alpha(H) \geq \alpha(G)$. *Furthermore,* *(1) If* $\alpha(H)=\alpha(G)$, *then* $H\cong G$ *and* $s=\sum_{i=0}^{n} a_i$.

(2) If $\alpha(H)>\alpha(G)$, *then* $H\cong(a-r)P_1\bigcup(a_0+r)P_2\bigcup a_1P_3\bigcup(a_2-r)P_5\bigcup rT_{1,1,1}\bigcup(\bigcup_{i=3}^{n}a_iP_{2i})$ *and* $s=a+\sum_{i=0}^{n} a_i$, *where* $a=\alpha(H)-\alpha(G)$ *and* $0\le r\le \min\{a_1,a_2\}$.

Proof Suppose that $\alpha(H)<\alpha(G)$. From $h_1(H)=h_1(G)$ and $\alpha(G)-\alpha(H)=b$, we have that $x^b h(H)=h(G)$, that is, $h(bP_1\bigcup H)=h(G)$. By Corollary 3.1, we obtain that $bP_1\bigcup H\cong G$ which is a contradiction. Hence $\alpha(H)\ge\alpha(G)$.

(1) If $\alpha(H)=\alpha(G)$, then $h(H)=h(G)$. By Corollary 3.1, it follows that $H\cong G$ and $s=\sum_{i=0}^{n} a_i$.

(2) If $\alpha(H)>\alpha(G)$, from $h_1(H)=h_1(G)$ and $\alpha(H)-\alpha(G)=a$, we have $h(H)=h(aP_1\bigcup G)$. From Theorem 3.1, we get that the result holds. □

Lemma 3.3[5] $\bigcup_{n\in A} U_n$ *is adjointly unique for* $A=\{7,2n|n\in N,n\ge5\}$, *where* N *is the set of positive integers.*

Theorem 3.2 *Let* $A=\{1,2,3,5\}\bigcup\{2n|n\in N,n\ge3\}$ *and* $B=\{7,2n|n\in N,n\ge5\}$. *Let* $G=(\bigcup_{i\in A_1} P_i)\bigcup(\bigcup_{j\in B_1} U_j)$, *where* $A_1\subseteq A$, $B_1\subseteq B$ *and* N *is the set of positive integers. Then* G *is adjointly unique if and only if*

(1) $A=\emptyset$.

(2) $A=\{1,3,2b|b\ge1,b\ne2\}$ *and* $B=\emptyset$.

(3) $A=\{3,5,2b|b\ge1,b\ne2\}$ *and* $B=\emptyset$.

(4) $A=\{3,5,2b|b\ge1,b\ne2\}$ *and* $B=\{j|j=i+4,i\in A\backslash\{2,5\}\}$.

Proof The necessity of the theorem follows from Lemma 2.4. Now, we prove the sufficiency of the theorem.

(1) If $A=\emptyset$, then $A_1=\emptyset$. From Lemma 3.3, we obtain that the result holds.

(2) and (3) From Corollary 3.1, we obtain that the results hold.

(4) Suppose that any graph $H=\bigcup_{k=1}^{s}H_k$ satisfies $h(H)=h(G)$. It follows, from Lemma 2.1, that

$$\prod_{k\in S} h(H_k)=\prod_{i\in A_1} h(P_i)\prod_{j\in B_1} h(U_j). \tag{3.2}$$

By Lemma 2.3, we have $H_k\in\mathcal{G}_1\bigcup\mathcal{G}_2$.

By Lemma 2.2 and calculation, we have

$$h_1(T(2,2,2))=h_1^2(P_2)h_1(K_{1,4}).$$

$$h_1(T(1,3,3))=h_1(P_2)h_1(P_3)h_1(K_{1,4}),$$

$$h_1(D_8)=h_1(T(1,2,5))=h_1(P_2)h_1(P_4)h_1(K_{1,4}),$$

$$h_1(C_3(P_2,P_2))=h_1(C_4(P_2))=h_1(K_4^-)=h_1(P_2)h_1(K_{1,4}).$$

Since $h_1(K_{1,4})=x+4$, eliminating all the factors $x+4$ and x in the two sides of (3.2), we obtain, from Lemma 2.4 and $j=i+4$, that

$$\prod_{k\in S_1} h_1(H_k^{'})=\prod_{i\in A_1} h_1(P_i)\prod_{j\in B_2} h_1(P_{j-4})=\prod_{i\in S_2} h_1(P_i)\ \text{ and }\ |S_1|\le|S|, \tag{3.3}$$

where $S_2 = \{3,5,2l|l\geq 1, l\neq 2\}$, $|S_2| = |A_1| + |B_1|$ and $H'_k\in\{T(1,2,i)(2\leq i\leq 4), D_i(4 \leq i \leq 7)\}\bigcup\mathcal{P}\bigcup\mathcal{C}\bigcup\mathcal{T}_1$. Obviously, $H'_k\neq P_1$ for $k\in S_1$. From Lemma 3.2, it follows that

$$\sum_{k\in S_1}\alpha(H_k)\geq\sum_{i\in S_2}\alpha(P_i). \tag{3.4}$$

Claim. $\bigcup_{k\in S_1} H'_k\cong\bigcup_{i\in S_2} P_i$.

To prove the claim, we distinguish the following two cases from (3.4):

Case 1 $\sum_{k\in S_1}\alpha(H_k) = \sum_{i\in S_2}\alpha(P_i)$. By (3.3), we have $\prod_{k\in S_1} h(H'_k) = \prod_{i\in S_2} h(P_i)$. From Corollary 3.1, we get that the claim holds.

Case 2 $\sum_{k\in S_1}\alpha(H_k) > \sum_{i\in S_2}\alpha(P_i)$. Without loss of generality, let

$$\bigcup_{i\in S_2} P_i = a_0P_2\bigcup a_1P_3\bigcup a_2P_5\bigcup(\bigcup_{i=3}^{n} a_iP_{2i}),$$

where $|S_2| = \sum_{i=0}^{n} a_i$. Set $\sum_{k\in S_1}\alpha(H_k) - \sum_{i\in S_2}\alpha(P_i) = a > 0$. From (3.3), it follows that

$$h(\bigcup_{k\in S_1} H'_k) = h(aP_1\bigcup(\bigcup_{i\in S_2} P_{2i})).$$

By Lemma 3.2 and $H'_k\neq P_1$, we have that

$$\bigcup_{k\in S_1} H'_k\cong\bigcup(a_0+r)P_2\bigcup a_1P_3\bigcup(a_2-r)P_5\bigcup rT_{1,1,1}\bigcup(\bigcup_{i=3}^{n} a_iP_{2i}),$$

$a = r$ and $|S_1| = a + \sum_{i=0}^{n} a_i$. (3.5)

Hence $H_k\in\{K_{1,4}, P_1\}\bigcup\mathcal{P}\bigcup\mathcal{U}$. Obviously, for each component H_k, we get that $q(H_k) - p(H_k) = -1$ for $1\leq k\leq|S|$. Hence $q(H) - p(H) = -|S|$. Since $q(G) - p(G) = -|A| - |B| = -\sum_{i=0}^{n} a_i$ and $q(H) - p(H) = q(G) - p(G)$, we have

$$|S| = \sum_{i=0}^{n} a_i. \tag{3.6}$$

From (3.3), (3.5) and (3.6), it follows that $a = 0$, which contradicts $a > 0$. This completes the proof of the claim. □

By Claim and the above analysis, we have

$$H_k\in\{P_1, K_{1,4}\}\bigcup\mathcal{P}\bigcup\mathcal{U} \text{ and } |S| = |S_1| = |A_1| + |B_1|.$$

By $|S| = |S_1|$, we have $H_k\notin\{P_1, K_{1,4}\}$. Hence $H_k\in\mathcal{P}\bigcup\mathcal{U}$. Since H must have exactly $|B_1|$ components H_j such that $\beta(H_j) = -4$ for $1\leq j\leq|B_1|$, we have

$$\bigcup_{k\in S_3} H_k\cong\bigcup_{j\in B_1} U_j, \quad \text{where } |S_3| = |B_1|. \tag{3.7}$$

From (3.2) and (3.7), it follows that

$$\prod_{k\in S\setminus S_3} h(H_k) = \prod_{i\in A_1} h(P_i), \quad h(\bigcup_{k\in S\setminus S_3} H_k) = h(\bigcup_{i\in A_1} P_i).$$

We obtain, from Corollary 3.1, that

$$\bigcup_{k\in S-S_3} H_k \cong \bigcup_{i\in A_1} P_i. \tag{3.8}$$

By (3.7) and (3.8), it is easy to get that

$$\bigcup_{k\in S} H_k \cong (\bigcup_{i\in A_1} P_i)\bigcup(\bigcup_{j\in B_1} U_j), \quad H\cong G.$$

This completes the proof of the theorem. □

Corollary 3.2 *Let* $A=\{1,2,3,5\}\bigcup\{2n|n\in N, n\geq 3\}$ *and* $B=\{7,2n|n\in N, n\geq 5\}$. *Let* $G=(\bigcup_{i\in A_1} P_i)\bigcup(\bigcup_{j\in B_1} U_j)$, *where* $A_1\subseteq A$, $B_1\subseteq B$ *and* N *is the set of positive integers. Then* $\overline{G}$ *is chromatically unique if and only if*

(1) $A=\emptyset$.

(2) $A=\{1,3,2b|b\geq 1, b\neq 2\}$ *and* $B=\emptyset$.

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(4) $A=\{3,5,2b|b\geq 1, b\neq 2\}$ *and* $B=\{j|j=i+4, i\in A\backslash\{2,5\}\}$.

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