*Journal of Mathematical Research & Exposition*
Nov., 2008, Vol. 28, No. 4, pp. 905–910
DOI:10.3770/j.issn:1000-341X.2008.04.021
Http://jmre.dlut.edu.cn

# Further Improvements and Applications on a Theorem Due to Reich

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**Abstract** In this paper, by using new analysis techniques, we have studied iterative construction problem for finding zeros of accretive mappings in uniformly smooth Banach spaces, and improved a theorem due to Reich. As its application, we have deduced a strong convergence theorem of fixed points for continuous pseudo-contractions.

**Keywords** accretive mapping; pseudocontraction; regularization iteration algorithm; range condition; Reich's inequality.

**Document code** A
**MR(2000) Subject Classification** 47H05; 47H09; 47H10; 58C15
**Chinese Library Classification** O177.91

## 1. Introduction

Let $E$ be a real Banach space with dual $E^*$. We denote by $J$ the normalized duality mapping from $E$ to $2^{E^*}$ defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if $E^*$ is strictly convex, then $J$ is single-valued. In the sequel, we shall denote the single-valued normalized duality map by $j$. A mapping $A : E \to 2^E$ is called accretive, if for all $x, y \in D(A)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x - y)\rangle \geq 0 \tag{1}$$

for all $u \in Ax$ and all $v \in Ay$. The mapping $A$ is called strongly accretive if right side of (1) is replaced by $k\|x - y\|^2$, where $k \in (0, 1)$. It is well known that the mapping $A$ is accretive if and only if for all $x, y \in D(A)$ and $\forall s > 0$ there holds

$$\|x - y\| \leq \|x - y + s(u - v)\| \tag{2}$$

for all $u \in Ax$ and all $v \in Ay$. An accretive mapping $A$ is called m-accretive if range of $(I + rA)$ is $E$ for all $r > 0$, i.e., $R(I + rA) = E$. The class that is closely related to the class of accretive mappings is the class of pseudocontractions. Let $K$ be a nonempty subset of $E$. The mapping

**Received date**: 2006-10-20; **Accepted date**: 2007-07-13
**Foundation item**: the National Natural Science Foundation of China (No. 10471033).

$T : K \to K$ is called pseudocontractive if $A = I - T$ is accretive. Accordingly, the mapping $T : K \to K$ is called strongly pseudocontractive if $A = I - T$ is stongly accretive.

Interest of accretive mappings stems mainly from their firm connection with equations of evolution (for example, heat, wave, or Schrödinger equations). Thanks to mathematicians' efforts, since 1967, the theory of the accretive operators has been well developed, and research on the zeros of the accretive operators has been attracting a lot of excellent mathematicians. In 1974, Bruck[1] adopted a regularization iteration algorithm to construct zeros of maximal monotone operators in a Hilbert space. In 1980, Reich[3] developed the main results of Bruck[1] into uniformly smooth Banach spaces more general than Hilbert spaces. To be specific, Reich proved the following convergence theorem.

**Theorem R1** *Let $E$ be a uniformly smooth Banach space and $A \subset E \times E$ be an m-accretive mapping with $A^{-1}0 \neq \emptyset$. Suppose that $\{\lambda_n\}$ and $\{\theta_n\}$ are two positive sequences satisfying the following control conditions: (i) $\theta_n \to 0$ $(n \to \infty)$; (ii) $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$; (iii) $\frac{\theta_{n-1}}{\theta_n} - 1 = o(\lambda_n \theta_n)$; (iv) $b(\lambda_n) = o(\theta_n)$, where $b : R^+ \to R^+$ is the function appearing in the Reich inequality (RI) in the Section 2. Let a sequence $\{x_n\} \subset D(A)$ be generated from arbitrary $x_0 \in D(A)$ by*

$$x_{n+1} \in x_n - \lambda_n (Ax_n + \theta_n x_n), \quad n \geq 0. \tag{3}$$

*If both $\{x_n\}$ and $\{u_n\}$ are bounded for all $u_n \in Ax_n$, then $x_n \to x^* \in A^{-1}(0)$, $n \to \infty$.*

In the light of [1] iterative algorithm (3) is called a regularization iteration algorithm.

**Question** Can the boundedness of $\{x_n\}$ be dropped?

Recently, Chidume and Zegeye[5] studied this question. They proved that the conclusion of Theorem R1 still holds without boundedness of $\{x_n\}$ if real sequences $\{\lambda_n\}$ and $\{\theta_n\}$ satisfy an additional condition: there exists a positive constant $d$ such that $\frac{\lambda_n}{\theta_n} \leq d$ for all $n \geq 0$. We note that this restriction on $\{\lambda_n\}$ and $\{\theta_n\}$ is not convenient.

The purpose of this paper is to prove that the conclusion of Theorem R1 indeed holds without boundedness of $\{x_n\}$.

## 2. Preliminaries

Let $E$ be a real normed linear space with $\dim E \geq 2$. The modulus of smoothness of $E$ is defined by

$$\rho_E(\tau) = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}, \quad \tau > 0.$$

If $\rho_E(\tau) > 0$, $\forall \tau > 0$, then $E$ is said to be smooth. If there exists constant $c > 0$ and a real number $1 < q < \infty$, such that $\rho_E(\tau) \leq c\tau^q$, then $E$ is said to be q-uniformly smooth. A Banach space $E$ is called uniformly smooth if $\lim_{\tau \to 0} \rho_E(\tau)/\tau = 0$. Typical examples of such spaces are the Lebesgue $L_p$, the sequence $l_p$ and the Sobolev $W_p^m$ spaces for $1 < p < \infty$.

Uniformly smooth Banach spaces enjoy very nice geometrical properties. In 1978, Reich[2] established the following famous inequality, which is called Reich inequality in the light of [2].

**Theorem R2**[2] *Let $E$ be a real uniformly smooth Banach space. Then there exists a continuous* *increasing function* $b : R^+ \to R^+$ *which satisfies:*

*(i)* $b(ct) \le cb(t),\ \forall c \ge 1$*;*

*(ii)* $b(0) = 0$*;*

*(iii)* $\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x)\rangle + \max\{\|x\|, 1\}\,\|y\|\, b(\|y\|),\ \forall x, y \in E.$ (RI)

Let $K \subseteq E$ be a closed convex subset, and $P : E \to K$ be a mapping of $E$ into $K$. Then $P$ is said to be sunny if

$$P(Px + t(x - Px)) = Px$$

for $x \in E$ and $t \ge 0$. A mapping $P$ is said to be a retraction if $P^2 = P$. $K \subseteq E$ is said to be a sunny nonexpansive retract if there exists a sunny nonexpansive retraction $P : E \to K$.

We need following famous convergence theorem due to Reich.

**Theorem R3**[4] *Let* $E$ *be a real uniformly smooth Banach space, and let* $A \subset E \times E$ *be an accretive mapping satisfying the range condition:*

$$\overline{D(A)} \subset R(I + rA)$$

*for all* $r > 0$, *where* $\overline{D(A)}$ *is a closed convex set of* $E$. *If* $A^{-1}(0) \ne \emptyset$, *then, for all* $x \in \overline{D(A)}$, *the limit* $\lim_{t\to\infty} J_t(x)$ *exists and belongs to* $A^{-1}(0)$. *In this case, by putting* $Px = \lim_{t\to\infty} J_t(x)$, *then* $P : \overline{D(A)} \to A^{-1}(0)$ *is a sunny nonexpansive retraction.*

**Lemma Weng X**[6] *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two non-negative real sequences such that*

$$a_{n+1} \le (1 - t_n)a_n + b_n,\ \forall n \ge n_0,$$

*where* $0 \le t_n < 1$, $\sum_{n=1}^{\infty} t_n = \infty$ *and* $b_n = o(t_n)$. *Then* $a_n \to 0 (n \to \infty)$. *For the rest of this paper, let* $x^* \in A^{-1}(0)$ *be such that* $J_t(0) \to x^*$ *as* $t \to \infty$, *which is guaranteed by Theorem R2.*

## 3. Main results

**Theorem 1** *Let* $E$ *be a real uniformly smooth Banach space and* $A \subset E \times E$ *be an accretive mapping satisfying the range condition*

$$\overline{D(A)} \subset R(I + rA),\ \forall r > 0,$$

*where* $\overline{D(A)}$ *is a closed convex subset of* $E$. *Let* $z \in \overline{D(A)}$ *and* $x_1 \in D(A)$ *be arbitrary, and let a sequence* $\{x_n\}$ *be defined iteratively by*

$$x_{n+1} \in x_n - \lambda_n(Ax_n + \theta_n(x_n - z)),\ n \ge 1, \tag{4}$$

*where* $\{\lambda_n\}$ *and* $\{\theta_n\}$ *satisfies conditions (i)–(iv) in Theorem R1.*

Suppose there exists a constant $C \ge 1$ such that

$$\|u_n\| \le C(1 + \|x_n\|),\ u_n \in Ax_n,\ n \ge 1. \tag{5}$$

If $A^{-1}(0) \ne \emptyset$, then $\{x_n\}_{n\ge1}$ converges strongly to $x^* \in A^{-1}(0)$.

**Proof** Without loss of generality, we may assume that $z = 0$. Otherwise, we consider $D(\tilde{A}) = D(A) - z$, $\tilde{A}x = A(x + z)$, $x \in D(\tilde{A})$. By the range condition, we know that the equation

$$0 \in (I + \theta_n^{-1}A)x \tag{6}$$

has a unique solution $y_n \in D(A)$, $\forall n \geq 1$, i.e., $0 \in \theta_n y_n + Ay_n$, $\forall n \geq 1$, and hence

$$0 \in \lambda_n \theta_n y_n + \lambda_n A y_n, \quad n \geq 1. \tag{7}$$

By Theorem R3, we conclude that $y_n = J_{\theta_n^{-1}}(0) \to P(0) = x^* \in A^{-1}(0)$, $(n \to \infty)$, in particular $\{y_n\}$ is bounded. Since $0 \in \theta_n y_n + Ay_n$, $n \geq 1$, there exists $v_n \in Ay_n$ such that $\theta_n y_n + v_n = 0$. Therefore $\{v_n\}$ is also bounded. Set

$$M = \max\left\{\sup_{n\geq 0}\{\|y_n\|\}, \sup_{n\geq 0}\{\|v_n\|\}\right\}. \tag{8}$$

By using the linear growth condition (5), we obtain that $\frac{\|u_n\|}{1+\|x_n\|} \leq C$ for all $u_n \in Ax_n$. Thus, for each $v_n \in Ay_n$, we have

$$\frac{\|u_n - v_n\|}{1+\|x_n\|} \leq \frac{\|u_n\| + \|v_n\|}{1+\|x_n\|} \leq C + \frac{M}{1+\|x_n\|}. \tag{9}$$

As $A$ is accretive, for $v_n \in Ay_n$, it follows from (2) and (7) that

$$\begin{aligned}\|y_n - y_{n-1}\| &\leq \|y_n - y_{n-1} + \frac{1}{\theta_{n-1}}(v_n - v_{n-1})\| \\ &= \|(y_n + \frac{1}{\theta_n}v_n) - (y_{n-1} + \frac{1}{\theta_{n-1}}v_{n-1}) + \frac{1}{\theta_{n-1}}v_n - \frac{1}{\theta_n}v_n\| \\ &= |\frac{1}{\theta_{n-1}} - \frac{1}{\theta_n}|\|v_n\| = |\frac{1}{\theta_{n-1}} - \frac{1}{\theta_n}|\theta_n\|y_n\| \\ &= |\frac{\theta_n}{\theta_{n-1} - 1}|\|y_n\|. \end{aligned} \tag{10}$$

By using Theorem R2 (i), (iii), algorithm (4) and (7)–(10), we have, for all $u_n \in Ax_n$, $v_n \in Ay_n$ that

$$\begin{aligned}\|\frac{x_{n+1} - y_n}{1+\|x_n\|}\|^2 =&\|(1-\lambda_n\theta_n)\frac{(x_n - y_n)}{1+\|x_n\|} - \frac{\lambda_n}{1+\|x_n\|}(u_n - v_n)\|^2 \\ \leq&(1-\lambda_n\theta_n)^2\frac{\|x_n - y_n\|^2}{(1+\|x_n\|)^2} - \frac{2\lambda_n(1-\lambda_n\theta_n)}{(1+\|x_n\|)^2}\langle u_n - v_n, j(x_n - y_n)\rangle + \\ &\max\{\frac{\|x_n - y_n\|}{1+\|x_n\|}, 1\}\lambda_n\frac{\|u_n - v_n\|}{1+\|x_n\|}b(\lambda_n\frac{\|u_n - v_n\|}{1+\|x_n\|}) \\ \leq&(1-\lambda_n\theta_n)\frac{\|x_n - y_n\|^2}{(1+\|x_n\|)^2} + (M+1)\lambda_n(C + \frac{M}{1+\|x_n\|})^2 b(\lambda_n). \end{aligned} \tag{11}$$

Multiplying both sides of (11) by $(1+\|x_n\|)^2$ gives

$$\begin{aligned}\|x_{n+1} - y_n\|^2 \leq&(1-\lambda_n\theta_n)\|x_n - y_n\|^2 + (M+1)[C(1+\|x_n\|) + M]^2\lambda_n b(\lambda_n) \\ \leq&(1-\lambda_n\theta_n)\|x_n - y_n\|^2 + 2(M+1[C^2(1+\|x_n\|)^2 + M^2]\lambda_n b(\lambda_n) \\ \leq&(1-\lambda_n\theta_n)\|x_n - y_n\|^2 + 2(M+1)[2C^2(1+\|x_n\|^2) + M^2]\lambda_n b(\lambda_n) \\ \leq&(1-\lambda_n\theta_n)\|x_n - y_n\|^2 + 2(M+1)(2C^2 + M^2)\lambda_n b(\lambda_n) + \\ &4(M+1)C^2\|x_n - y_n + y_n\|^2\lambda_n b(\lambda_n) \\ \leq&(1-\lambda_n\theta_n)\|x_n - y_n\|^2 + 2(M+1)(2C^2 + M^2)\lambda_n b(\lambda_n) + \\ &8(M+1)C^2\|x_n - y_n\|^2\lambda_n b(\lambda_n) + 8(M+1)C^2\|y_n\|^2\lambda_n b(\lambda_n) \end{aligned}$$

$$
\begin{aligned}
&\leq [(1-\lambda_n\theta_n)+8(M+1)C^2\lambda_n b(\lambda_n)]\|x_n-y_n\|^2+\\
&\ 2(M+1)(2C^2+M^2+4C^2M^2)\lambda_n b(\lambda_n).
\end{aligned} \tag{12}
$$

By Theorem R1 (iv), we can take $n_0$ large enough. When $n \geq n_0$, we obtain

$$
\frac{b(\lambda_n)}{\theta_n} \leq \frac{1}{16(M+1)C^2}. \tag{13}
$$

Using (12) in (11) and Theorem (iv), we have for all $n \geq n_0$ that

$$
\begin{aligned}
\|x_{n+1}-y_n\|^2 \leq &(1-\frac{1}{2}\lambda_n\theta_n)\|x_n-y_n\|^2+o(\lambda_n\theta_n)\\
\leq &(1-\frac{1}{2}\lambda_n\theta_n)\|x_n-y_{n-1}+y_{n-1}-y_n\|^2+o(\lambda_n\theta_n)\\
\leq &(1-\frac{1}{2}\lambda_n\theta_n)(\|x_n-y_{n-1}\|^2+2\|x_n-y_{n-1}\|\|y_{n-1}-y_n\|+\\
&\|y_{n-1}-y_n\|^2)+o(\lambda_n\theta_n)\\
\leq &(1-\frac{1}{2}\lambda_n\theta_n)[\|x_n-y_{n-1}\|^2+2\|x_n-y_{n-1}\||\frac{\theta_{n-1}}{\theta_n}-1|\|y_n\|+\\
&|\frac{\theta_{n-1}}{\theta_n}-1|^2\|y_n\|^2]+o(\lambda_n\theta_n)\\
= &(1-\frac{1}{2}\lambda_n\theta_n)[\|x_n-y_{n-1}\|+|\frac{\theta_{n-1}}{\theta_n}-1|M]^2+o(\lambda_n\theta_n).
\end{aligned} \tag{14}
$$

Squaring on both sides of (14), we have that

$$
\begin{aligned}
\|x_{n+1}-y_n\| &\leq (1-\frac{1}{4}\lambda_n\theta_n)\|x_n-y_{n-1}\|+M|\frac{\theta_{n-1}}{\theta_n}-1|+o(\lambda_n\theta_n)\\
&= (1-\frac{1}{4}\lambda_n\theta_n)\|x_n-y_{n-1}\|+o(\lambda_n\theta_n),
\end{aligned} \tag{15}
$$

where we have used conditions (iii),(iv) in Theorem 1 and (10).

Noting that $\sum_{n=1}^{\infty}\lambda_n\theta_n=\infty$, we know from Lemma Weng X that $x_{n+1}-y_n \to 0$, $(n\to\infty)$. Thus $x_n \to P(0)=x^* \in A^{-1}(0)$, $(n\to\infty)$.

**Remark 1** If $\{u_n\}$, $u_n \in Ax_n$, is bounded, then there exists a constant $C \geq 1$ such that

$$
\|u_n\| \leq C \leq C + C\,\|x_n\| = C(1+\|x_n\|),
$$

which shows that Theorem R1 is still true without assuming the boundedness of $\{x_n\}$.

**Remark 2** If $A$ is Lipschitz continuous such that $A^{-1}(0)\neq\emptyset$, then

$$
\|u_n\| \leq L\,\|x_n-x^*\| = L\,\|x_n\| + L\,\|x^*\| \leq C(1+\|x_n\|),
$$

where $L \geq 1$ is the Lipschtz constant of $A$, $x^* \in A^{-1}(0)$, and $C = \max\{L\,\|x^*\|, L\} = L\max\{\|x^*\|, 1\}$. In particular, if $A$ is a single valued and bounded linear mapping, then

$$
\|Ax_n\| \leq \|A\|\,\|x_n\| \leq \|A\|\,(1+\|x_n\|).
$$

**Remark 3** If $A$ is a bounded mapping, then for large enough $r \geq 1$,

$$
M(r) = \sup\{\|u\| : u \in Ax, x \in D(A), \|x-x_1\| \leq 2r\} < \infty.
$$

If $\frac{b(\lambda_n)}{\theta_n} \leq \frac{2r}{\left(M(r)+\frac{3}{2}r\right)^2}$, $\forall n \geq 1$, by induction, we can prove that $\|x_n - x^*\| \leq r$, $\forall n \geq 1$. Therefore $\|u_n\| \leq C(1+\|x_n\|)$, $\forall u_n \in Ax_n$, $n \geq 1$.

As an application of Theorem 1, we have

**Theorem 2** *Let $E$ be a real uniformly smooth Banach space, $K$ be a closed convex subset of $E$, and $T : K \to K$ be a continuous pseudocontraction such that $F(T) \neq \emptyset$. Let $\{\lambda_n\}$ and $\{\theta_n\}$ be two positive sequences such that: (i) $\theta_n \to 0 \quad (n \to \infty)$; (ii) $\lambda_n(1+\theta_n) \leq 1$, $n \geq 1$; (iii) $\sum\limits_{n=1}^{\infty} \lambda_n\theta_n = \infty$; (iv) $\frac{\theta_{n-1}}{\theta_n} - 1 = o(\lambda_n\theta_n)$; (v) $b(\lambda_n) = o(\theta_n)$, where $b : R^+ \to R^+$ is the function appearing in the Reich inequality (RI). For arbitrary fixed vector $z \in K$ and arbitrary initial value $x_1 \in K$, let the sequence $\{x_n\}$ be defined iteratively by*

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n Tx_n + \lambda_n\theta_n(z - x_n),\ n \geq 1. \tag{16}$$

*If $\{x_n\}$ satisfies the linear growth condition:*

$$\|x_n - Tx_n\| \leq C(1+\|x_n\|), n \geq 1,\ C \geq 1,$$

*then $x_n \to x^* \in F(T)$, $(n \to \infty)$.*

**Proof** Putting $A = I - T$, we see that $A : K \to E$ is an accretive mapping such that $A^{-1}(0) = F(T) \neq \emptyset$ and satisfies the range condition $K \subseteq R(I + rA)$, $\forall r > 0$. (16) reduces to

$$x_{n+1} = x_n - \lambda_n(Ax_n + \theta_n(x_n - z)), n \geq 1.$$

By Theorem 1, we have that $x_n \to x^* \in A^{-1}(0) = F(T)$.

**Remark 4** If $K$ is a bounded closed convex subset of a uniformly smooth Banach space $E$ and $T : K \to K$ is a continuous pseudocontraction, then $F(T) \neq \emptyset$.

**Remark 5** We do not know whether or not Theorems 1 and 2 are true in more general Banach spaces.

**Remark 6** We do not know whether or not condition (iv) can be weakened to $\lambda_n = o(\theta_n)$.

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