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Strongly Regular (α,β)-Families and Translation Strongly Regular (α,β)-Geometries

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Abstract In this paper, we introduce the concept of a strongly regular (α,β)-family. It generalizes the concept of an SPG-family in [4] and [5]. We provide a method of constructing strongly regular (α,β)-geometries from strongly regular (α,β)-families. Furthermore, we prove that each strongly regular (α,β)-geometry constructed from a strongly regular (α,β)-regulus translation is isomorphic to a translation strongly regular (α,β)-geometry; while $t-r>\beta$, the converse is also true.

Keywords projective space; strongly regular (α,β)-regulus; strongly regular (α,β)-geometry.

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1. Introduction

A partial linear space of order (s,t) is a connected incidence structure $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathbf{I})$, with $\mathcal{P}$ a finite non-empty set of elements called points, $\mathcal{L}$ a family of subsets of $\mathcal{P}$ called lines and $\mathbf{I}$ an incidence relation satisfying the following axioms.

(1) Any two distinct points are incident with at most one line;

(2) Each line is incident with exactly $s+1$ points $(s\geq 1)$;

(3) Each point is incident with exactly $t+1$ lines $(t\geq 1)$.

For a partial linear space $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathbf{I})$, the graph Γ with vertex set $\mathcal{P}$ such that two distinct vertices are adjacent if they are collinear in $\mathcal{S}$ is called the point graph of $\mathcal{S}$.

An anti-flag in a partial linear space $\mathcal{S}$ is a pair (p,L) with $p\in\mathcal{P}$ and $L\in\mathcal{L}$ such that p is not incident with L. For a given anti-flag (p,L) of $\mathcal{S}$, the incidence number of (p,L), denoted by $i(p,L)$, is the number of points collinear with p and incident with L. A (finite) (α,β)-geometry of order (s,t) is a partial linear space $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathbf{I})$ of order (s,t) for some s and t, such that for any anti-flag (p,L) of $\mathcal{S}$ we have $i(p,L)=\alpha$ or $i(p,L)=\beta$, and each of these two cases occurs. For an anti-flag (p,L), L is called an α-line with respect to p if $i(p,L)=\alpha$, similarly for a β-line.

An (α,β)-geometry is *strongly regular* if there exist integers p and r such that the following conditions are satisfied.

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(1) If points x and y are collinear, then there exist p lines on x that are α-lines with respect to y;

(2) If points x and y are not collinear, then there exist r lines on x that are α-lines with respect to y.

It is known that if $\alpha \neq \beta$, the point graph of a strongly regular (α,β)-geometry is strongly regular[7].

If $i(p, L) = \alpha$ for every anti-flag (p, L) of $\mathcal{S}$ of order (s, t), then $\mathcal{S}$ is called a *partial geometry* of order (s, t) and denoted by $\mathrm{pg}(s, t, \alpha)$[1]. It is clear that the point graph of a partial geometry is strongly regular. The point graph of a $(0, \alpha)$-geometry is not necessarily strongly regular. The $(0, \alpha)$-geometry of order (s, t) having a strongly regular point graph is called a semipartial geometry of order (s, t) and denoted by $\mathrm{spg}(s, t, \alpha, \mu)$[2]. Here μ is the number of vertices adjacent to two non-adjacent vertices.

Let $\mathcal{R} = \{\mathrm{PG}^{(0)}(m, q), \mathrm{PG}^{(1)}(m, q), \ldots, \mathrm{PG}^{(t)}(m, q)\}$, $t \geq 1$, be $t+1$ mutually disjoint $\mathrm{PG}(m, q)$ in a projective space $\mathrm{PG}(n, q)$, which generates $\mathrm{PG}(n, q)$. Now embed $\mathrm{PG}(n, q)$ in $\mathrm{PG}(n+1, q)$ as a hyperplane Π at infinity. We define the following incidence structure $\mathcal{S}(\mathcal{R}) = (\mathcal{P}, \mathcal{L}, \mathbf{I})$. The point set $\mathcal{P}$ consists of all points of $\mathrm{PG}(n+1, q) \setminus \Pi$. The line set $\mathcal{L}$ is the set of all $(m+1)$-dimensional subspaces of $\mathrm{PG}(n+1, q)$ intersecting Π in an element of $\mathcal{R}$ and not contained in Π. The incidence relation $\mathbf{I}$ is the incidence relation inherited from $\mathrm{PG}(n+1, q)$. The geometry $\mathbf{S}(\mathcal{R})$ is called the generalized linear representation of $\mathcal{R}$. In the case $m = 0$ the geometry is called the linear representation of $\mathcal{R}$. Suppose that $\mathcal{H}$ is a point set in $\mathrm{PG}(n, q)$. Then the linear representation of $\mathcal{H}$ is usually denoted by $\mathrm{T}_n^*(\mathcal{H})$. For more properties, we refer to [3].

A strongly regular (α,β)-regulus is a collection $\mathcal{R}$ of m-dimensional subspaces of $\mathrm{PG}(n, q)$, $|\mathcal{R}| >1$, satisfying

(1) $\Sigma_i \cap \Sigma_j = \emptyset$, for every $\Sigma_i, \Sigma_j \in \mathcal{R}$, $\Sigma_i \neq \Sigma_j$.

(2) If an $(m+1)$-dimensional subspace contains some $\Sigma_i \in \mathcal{R}$, then it has a point in common with α or β subspaces of $\mathcal{R}\backslash\Sigma_i$. Such an $(m+1)$-dimensional subspace that meets α elements of $\mathcal{R}\backslash\Sigma_i$ is said to be an α-secant to $\mathcal{R}$ at Σ_i, similarly for a β-secant.

(3) If a point of $\mathrm{PG}(n, q)$ is contained in an element Σ of $\mathcal{R}$, then it is contained in a constant number p of α-secant $(m+1)$-dimensional subspaces on elements of $\mathcal{R}\backslash\Sigma$.

(4) If a point of $\mathrm{PG}(n, q)$ is contained in no element of $\mathcal{R}$, then it is contained in a constant number r of α-secant $(m+1)$-dimensional subspaces of $\mathcal{R}$.

In [7] it was proved that a generalized linear representation $S(\mathcal{R})$ constructed from a strongly regular (α,β)-regulus R is a strongly regular (α,β)-geometry.

In [4] and [5] De Clerck and De Winter et al. introduced a theory of elation and translation semipartial geometries and provided a group theoretical characterization of semipartial geometries constructed from SPG-reguli. In this paper, we will introduce the concept of strongly regular (α,β)-families, which generalizes the concept of SPG-families in [4] and [5]. We will construct elation and translation strongly regular (α,β)-geometries from strongly regular (α,β)-families. Furthermore, we will show that when $t - r > \beta$, the theory of translation strongly regular (α, β)-geometries is equivalent to the theory of strongly regular (α, β)-reguli.

2. Strongly regular (α, β)-families

In this paper, let I denote the set $\{0, 1, 2, \ldots, t\}$, $t \geq 1$.

Definition 2.1 *Let G be a finite group. Except in a few mentioned cases we will use a multiplicative notation for G. Suppose that $J = \{S_0, S_1, \ldots, S_t\}$ is a set of $t+1$ $(t \geq 1)$ subgroups of order $s+1$ of G $(s \geq 1)$ with $S_i \cap S_j = \{id\}$ whenever $i \neq j$, for $i, j \in I$. Then the pair (G, J) is called a geometric family. We say the pair (G, J) is a strongly regular (α, β)-family with parameters $(s, t, \alpha, \beta, p, r)$ if the following conditions are satisfied.*

(1) There exist two integers α and β $(0 \leq \alpha < \beta < s+1)$ such that for each S_i $(i \in I)$ and each $g \notin S_i$, there either exists a unique set $\{j_1, j_2, \ldots, j_\alpha\} \subset I$ with the property that $S_i g \cap S_{j_k} \neq \emptyset$, $\forall\ k \in \{1, 2, \ldots, \alpha\}$, or exists a unique set $\{j_1, j_2, \ldots, j_\beta\} \subset I$ with the property that $S_i g \cap S_{j_k} \neq \emptyset$, $\forall k \in \{1, 2, \ldots, \beta\}$, and both of these two cases occur. The pair (S_i, g) is called an α-pair in the former case, while (S_i, g) is called a β-pair in the latter case.

(2) For each $g \in \bigcup_{i \in I} S_i \backslash \{id\}$, there exists an integer p such that there exists a unique set $\{j_1, j_2, \ldots, j_p\}$ with the property that (S_{j_k}, g) is an α-pair, $\forall k \in \{1, 2, \ldots, p\}$.

(3) For each $g \in G \backslash \bigcup_{i \in I} S_i$, there exists an integer r such that there exists a unique set $\{j_1, j_2, \ldots, j_r\}$ with the property that (S_{j_k}, g) is an α-pair, $\forall k \in \{1, 2, \ldots, r\}$.

This definition generalizes the concept of an SPG-family in [4] and [5]. We will introduce a new method of constructing strongly regular (α, β)-geometries from strongly regular (α, β)-families. At first, we will introduce the concept of a (right) coset geometry[5].

Definition 2.2 *Let G be a group and let $J = \{S_i | i \in I\}$ be a set of subgroups of G. Then the right coset geometry $S(G, J)$ is the incidence geometry with as points, the elements of G, as lines the right cosets $S_i g$, $i \in I$ and $g \in G$, and for which the incidence relation is containment.*

The proof of the following lemma can be found in [5] and is included here for completeness.

Lemma 2.3 *Let (G, J) be a strongly regular (α, β)-family. Then*

(1) $|S_i g \cap S_j h| \in \{0, 1\}$, $i \neq j$, $\forall g, h \in G$;

(2) $G = \langle S_0, S_1, \ldots, S_t >\rangle$.

Proof (1) Suppose that $y, z \in S_i g \cap S_j h$. This implies the existence of elements $s_{i_y}, s_{i_z} \in S_i$ and $s_{j_y}, s_{j_z} \in S_j$, for which $y = s_{i_y} g = s_{j_y} h$ and $z = s_{i_z} g = s_{j_z} h$. Hence $yz^{-1} = s_{i_y} s_{i_z}^{-1} = s_{j_y} s_{j_z}^{-1}$, yielding $yz^{-1} \in S_i \cap S_j$, that is, $y = z$.

(2) At first we will prove $r < t+1$. Suppose $r = t+1$. Without loss of generality, assume that $s_0 \in S_0 \backslash \{id\}$ and (S_1, s_0) is a β-pair. If each element of $S_1 s_0$ would belong to a certain S_j, then we would find for each $s_1 \in S_1$ a $j \in I$ such that $s_1 s_0 = s_j$, with $s_j \in S_j$. From (1) it follows easily that distinct elements of S_1 determine distinct indices j, yielding $\beta \geq s+1$, a contradiction. Hence there exists a $g \in S_1 s_0$ such that $g \in G \backslash \bigcup_{i \in I} S_i$. Clearly, (S_1, g) is a β-pair, a contradiction.

Thus for each $g \in G\backslash \bigcup_{i\in I} S_i$, there exists a $S_i \in J$ for which (S_i, g) is a β-pair. Then there exists a set S_j $(j \neq i)$ satisfying $S_i g \cap S_j \neq \emptyset$. Hence there exist $s_i \in S_i, s_j \in S_j$ with $s_i g = s_j$, and so $g = s_i^{-1} s_j \in \langle S_0, S_1, \ldots, S_t\rangle$. □

Theorem 2.4 *The right coset geometry $S(G,J)$ is a strongly regular (α,β)-geometry with parameters (s,t,α,β,p,r).*

Proof From Lemma 2.3 we see that $S(G,J)$ is a partial linear space of order (s,t).

(1) Let $S_i h$ $(i \in I)$ be any line of $S(G,J)$ and let g be any point of $S(G,J)$ not contained in $S_i h$. Since $g \notin S_i h$, $hg^{-1} \notin S_i$. From the definition of strongly regular (α,β)-families we have that there exists a unique set $\{j_1, j_2, \ldots, j_\delta\} \subset I$ with the property that $S_i hg^{-1} \cap S_{j_k} \neq \emptyset$, $\forall\ k \in \{1,2,\ldots,\delta\}$, $\delta = \alpha$ or β. Consequently, there exist elements $s_{i_k} \in S_i$ and $s_{j_k} \in S_j$, for $k \in \{1,2,\ldots,\delta\}$, such that $s_{i_k} h = s_{j_k} g$, $\delta = \alpha$ or β. Hence we have constructed δ points of $S_i h$ collinear with g in $S(G,J)$. On the other hand it is clear that each point of $S_i h$ collinear with g must be constructed as above. This implies that $S(G,J)$ is a (α,β)-geometry.

(2) Suppose that g and h are two collinear points of $S(G,J)$. This is equivalent to $gh^{-1} \in \bigcup_{i\in I} S_i$. Without loss of generality, assume that $gh^{-1} \in S_0$. From the definition of strongly regular (α,β)-families it follows that there exists a unique set $\{j_1, j_2, \ldots, j_p\} \subset I$, with the property that (S_{j_k}, gh^{-1}) is an α-pair, $\forall\ k \in \{1,2,\ldots,p\}$. This yields p lines of $S(G,J)$ on g that are α-lines with respect to h. On the other hand, each α-line of $S(G,J)$ on g with respect to h must be constructed as above.

(3) Suppose that g and h are two non-collinear points of $S(G,J)$. This is equivalent to $gh^{-1} \notin \bigcup_{i\in I} S_i$. From the definition of strongly regular (α,β)-families it follows that there exists a unique set $\{j_1, j_2, \ldots, j_r\} \subset I$, with the property that (S_{j_k}, gh^{-1}) is an α-pair, $\forall\ k \in \{1,2,\ldots,r\}$. This yields r lines of $S(G,J)$ on g that are α-lines with respect to h. On the other hand, each α-line of $S(G,J)$ on g with respect to h must be constructed as above.

Thus we conclude that $S(G,J)$ is a strongly regular(α,β)-geometry with parameters (s,t,p,r). □

A strongly regular (α,β)-geometry constructed as above will be called an elation strongly regular (α,β)-geometry. Whenever G is abelian it will be called a translation strongly regular (α,β)-geometry.

3. Translation strongly regular (α,β)-geometries

In this section, we suppose that (G,J) is a strongly regular (α,β)-family with parameters (s,t,α,β,p,r) and G is abelian. Let K be the set of all endomorphisms σ of G satisfying $S_i^\sigma \subset S_i$ $(\forall i \in I)$. From the fact that G is abelian it follows that $(K,+,\cdot)$, with the usual addition and multiplication of endomorphisms, is a ring. This ring will be called the kernel of the translation strongly regular (α,β)-geometry $S(G,J)$.

Similarly to the proofs of Theorems 2.4 and 2.5, we may obtain the following results.

Lemma 3.1 *If $t - r > \beta$, then $(K,+,\cdot)$ is a field.*

Theorem 3.2 *Each translation strongly regular (α,β)-geometry with $t-r>\beta$ is isomorphic to a strongly regular (α,β)-geometry constructed from a strongly regular (α,β)-regulus $\mathcal{R}$ in* $\mathrm{PG}(n,q)$*, where* $\mathrm{GF}(q)$ *is a subfield of the kernel.*

Let $\mathcal{P}=\mathrm{PG}(n,q)$ be a Desarguesian projective space of dimension at least 2. Let Π be a fixed hyperplane of $\mathcal{P}$ and p be a fixed point of $\mathcal{P}$. Then, $\mathrm{Persp}(\Pi,p)$ is the set of all automorphisms of $\mathcal{P}$ fixing Π point wise and fixing all hyperplanes containing p. It follows that $\mathrm{Persp}(\Pi,p)$ constitutes a group. The elements of $\mathrm{Persp}(\Pi,p)$ are called perspectivities with axis Π and center p. More specifically the elements of $\mathrm{Persp}(\Pi,p)$ are called elations if $p\in\Pi$, and homologies if $p\notin\Pi$. Furthermore, it is clear that the set of all perspectivities of $\mathcal{P}$ with a given axis or with a given center forms a group. Finally notice that the set of all elations with a given axis Π constitutes a group acting regularly on the point set of $\mathcal{P}\backslash\Pi$.

The following observation is important in the theory of Desarguesian projective spaces.

Lemma 3.3[8] *Let* $\mathrm{PG}(n,q)$ *be a Desarguesian projective space of dimension at least* 2*. Then the group of all homologies with axis a fixed hyperplane* Π *and with center a fixed point* $p\notin\Pi$ *is isomorphic to the multiplicative group of* $\mathrm{GF}(q)$*.*

Suppose that p_1, p_2 and p_3 are two by two distinct points on a projective line $\mathrm{PG}(1,q)$, and that p_4 is a fourth point on that line. If one chooses homogeneous coordinates in such a way that the coordinates of p_1, p_2 and p_3, are $(0,1)$, $(1,0)$ and $(1,1)$, respectively, then the *cross ratio* $(p_1,p_2;p_3,p_4)$ of the 4-tuple (p_1,p_2,p_3,p_4) is the affine coordinate of the point p_4 with respect to the chosen coordinates if $p_4\neq p_1$, and equals ∞ if $p_4=p_1$.

We have the following interesting property.

Lemma 3.4[8] *Every linear automorphism of* $\mathrm{PG}(1,q)$ *preserves the cross ratio.*

Now we may get the following result.

Theorem 3.5 *Every strongly regular (α,β)-geometry constructed from a strongly regular (α,β)-regulus $\mathcal{R}$ in* $\mathrm{PG}(n,q)$ *is isomorphic to a translation strongly regular (α,β)-geometry $S(G,J)$, with G the group of all elations of* $\mathrm{PG}(n+1,q)$ *with axis* $\mathrm{PG}(n,q)$*, and with the elements of* $\mathrm{GF}(q)$ *in the kernel.*

Proof Let $S(R)$ be a strongly regular (α,β)-geometry with parameters (s,t,p,r) constructed from a strongly regular (α,β)-regulus R in $\Pi:=\mathrm{PG}(n,q)$. Let G be the group of all elations of $\mathrm{PG}(n+1,q)$ with axis Π. Then G is a regular group of automorphisms of $S(R)$. Choose a point x of $S(R)$ and denote by $L_0,L_1,\ldots,L_t$ the lines of $S(R)$ through x. Define $S_i:=\{g\in G|L_i^g=L_i\}$ $(i\in I)$, which are subgroups of G of order $s+1$. Clearly $S_i\cap S_j=\{id\}$ for all $i\neq j$, with $i,j\in I$.

(1) Consider S_i and $g\notin S_i$ $(g\in G)$. Then the point x^g is not on L_i, and so x^g is collinear with δ distinct points $x^{h_1},x^{h_2},\ldots,x^{h_\delta}$ on the line L_i, with $\{h_1,h_2,\ldots,h_\delta\}\subset S_i$, where $\delta=\alpha$ or β. This implies that x is collinear with $x^{h_kg^{-1}}$ in $S(R)$, $k=1,2,\ldots,\delta$, with $\delta=\alpha$ or β. Hence there exists a $j_k\in I$ such that $h_kg^{-1}\in S_{j_k}$, that is, $S_{j_k}g\cap S_i\neq\emptyset$, $k=1,2,\ldots,\delta$, with $\delta=\alpha$ or β. Conversely, it is now obvious that every S_l for $S_lg\cap S_i\neq\emptyset$ gives rise to a point L_i collinear with x^g.

(2) Suppose that $g \in \bigcup_{i\in I} S_i \backslash \{id\}$. Without loss of generality, assume that $g \in S_0$. Then x is collinear with x^g in $S(R)$. So there are p distinct lines $L_{j_1}, L_{j_2}, \dots, L_{j_p}$ such that $i(x^g, L_{j_k}) = \alpha$ $(j_k \neq 0)$, $k = 1, 2, \dots, p$. Similarly as in (1) we know that (S_{j_k}, g) is an α-pair, $k \in \{1, 2, \dots, p\}$. And for any $j \in \{0, 1, \dots, t\} \backslash \{j_1, j_2, \dots, j_p, 0\}$, we have that (S_j, g) is a β-pair.

(3) Suppose that $g \in G \backslash \bigcup_{i\in I} S_i$. Then x is not collinear with x^g in $S(R)$. So there are r distinct lines $L_{j_1}, L_{j_2}, \dots, L_{j_r}$ such that $i(x^g, L_{j_k}) = \alpha$, $k = 1, 2, \dots, r$. Similarly to (1), we know that (S_{j_k}, g) is an α-pair, $k \in \{1, 2, \dots, r\}$. And for any line $j \in \{0, 1, \dots, t\} \backslash \{j_1, j_2, \dots, j_r\}$, we have that (S_j, g) is a β-pair.

Thus we conclude that (G, J) is a strongly regular (α, β)-family. It is now easily seen that $\phi : G \to S(R) : g \to x^g$ determines an isomorphism between $S(G, J)$ and $S(R)$.

Let x be the point of $PG(n+1, q) \backslash \Pi$ corresponding to $id \in G$. From Lemma 3.3 it follows that the multiplicative group of $\mathrm{GF}(q)$ is isomorphic to the group of homologies of $\mathrm{PG}(n+1)$ with axis Π and center x. We will show that it can be seen as a group of automorphism of G. We will identity the elements of G with the corresponding points of $\mathrm{PG}(n+1, q) \backslash \Pi$. Let $\sigma \neq 1$ be a homology of $\mathrm{PG}(n+1, q)$ with axis Π and center x and let $g \neq h$ be elements of $G \backslash \{x\}$. First suppose that x, g and h are not collinear in $\mathrm{PG}(n+1, q)$. Define $g_\infty := \langle x, g\rangle \cap \Pi$, $h_\infty := \langle x, h\rangle \cap \Pi$ and $l_\infty := \langle g, h\rangle \cap \Pi$. Then the triangle with vertices g, g^σ, h_∞ and the triangle with vertices h, h^σ, g_∞ are perspective triangles with center l_∞. From Desargues Theorem it follows that x, gh and $g^\sigma h^\sigma$ are collinear, which implies that $g^\sigma h^\sigma$ and $(gh)^\sigma$ must coincide, that is, $g^\sigma h^\sigma = (gh)^\sigma$. The cross ratio $(x, g_\infty; g, g^{-1}) = -1$. From Lemma 3.4 it follows that $(x^\sigma, g_\infty^\sigma; g^\sigma, (g^{-1})^\sigma) = (x, g_\infty; g, g^{-1}) = -1$, implying $(g^{-1})^\sigma = (g^\sigma)^{-1}$. Finally suppose that x, g and h are collinear and that $h \neq g^{-1}$. Choose any l not on $\langle x, g\rangle$. Then x, gl and $l^{-1}h$ are not collinear in $\mathrm{PG}(n+1, q)$ since otherwise $gll^{-1}h = gh$ would be a point of $\langle x, gl\rangle$, implying that $h = g^{-1}$ as $\langle x, gh\rangle \cap \langle x, gl\rangle = \{x\}$. Hence $(gh)^\sigma = (gll^{-1}h)^\sigma = (gl)^\sigma(l^{-1}h)^\sigma = g^\sigma l^\sigma (l^{-1})^\sigma h^\sigma = g^\sigma h^\sigma$. It is now clear that the group of homologies of $\mathrm{PG}(n+1)$ with axis Π and center x can indeed be seen as a group of automorphisms of G.

For any fixed S_i, let g be an element in S_i and σ be an element in $\mathrm{GF}(q)$. Then we have $L_i^g = L_i$. Hence for any h on L_i, we know that g, hg are on the line L_i. On the other hand, x, g and g^σ are collinear in $S(R)$. So g^σ must be on the line L_i. Now it is easily seen that $hg^\sigma \in L_i$. Thus $L_i^{g^\sigma} = L_i$, that is, $g^\sigma \in S_i$. Thus we conclude that the elements of $\mathrm{GF}(q)$ are in the kernel of the corresponding translation strongly regular (α, β)-geometry. When $t - r > \beta$, $\mathrm{GF}(q)$ is a subfield of the kernel. □

Hence we have shown that whenever $t - r > \beta$, the theory of translation strongly regular (α, β)-geometries is equivalent to the theory of strongly regular (α, β)-reguli. Furthermore it follows that in this case K is the largest field such that there exists a strongly regular (α, β)-reguli R in $\mathrm{PG}(n, K)$ with the property that $S(G, J) \cong S(R)$.

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