Journal of Mathematical Research & Exposition
Nov., 2008, Vol. 28, No. 4, pp. 935–944
DOI:10.3770/j.issn:1000-341X.2008.04.025
Http://jmre.dlut.edu.cn

Solvability of 4-Point Boundary Value Problems at Resonance for Fourth-Order Ordinary Differential Equations

ZHANG Zu Feng[1,2], *WEI Zhang Zhi*[1]
(1. Department of Mathematics, Suzhou University, Ahnui 234000, China;
2. Department of Mathematics, Huazhong University of Science and Technology, Hubei 430074, China)
(E-mail: jshyzzf@sohu.com)

Abstract In this paper, we consider the following fourth order ordinary differential equation

$$x^{(4)}(t) = f(t, x(t), x'(t), x''(t), x'''(t)), \quad t \in (0,1) \tag{E}$$

with the four-point boundary value conditions:

$$x(0) = x(1) = 0,\ \alpha x''(\xi_1) - \beta x'''(\xi_1) = 0,\ \gamma x''(\xi_2) + \delta x'''(\xi_2) = 0, \tag{B}$$

where $0 < \xi_1 < \xi_2 < 1$. At the resonance condition $\alpha\delta + \beta\gamma + \alpha\gamma(\xi_2 - \xi_1) = 0$, an existence result is given by using the coincidence degree theory. We also give an example to demonstrate the result.

Keywords fourth order equation; resonance; coincidence degree.

Document code A
MR(2000) Subject Classification 34B15
Chinese Library Classification O175.6

1. Introduction

Boundary value problems for ordinary differential equations play a very important role in both theory and applications. They are used to describe a large number of physical, biological and chemical phenomena. In [1], Chen studied the equation $x^{(4)}(t) = f(t, x(t)),\ 0 < t < 1$ with the four-point boundary value conditions:

$$x(0) = x(1) = 0,\ \alpha x''(\xi_1) - \beta x'''(\xi_1) = 0,\ \gamma x''(\xi_2) + \delta x'''(\xi_2) = 0,$$

at non-resonance case: $\alpha\delta + \beta\gamma + \alpha\gamma(\xi_2 - \xi_1) \neq 0$ by means of upper and lower solutions method and fixed point theorems. In this work, by using coincidence degree theory, an existence result for Equations (B) and (E) is established under non-linear growth restriction on f at resonance case: $\alpha\delta + \beta\gamma + \alpha\gamma(\xi_2 - \xi_1) = 0$.

In recent years, much research work has been done on boundary value problems at resonance case. We refer readers to [2–5] for some recent results.

Received date: 2006-09-23; **Accepted date**: 2007-01-19
Foundation item: the Master's Research Fund of Suzhou University (No. 2008yss19).

Now we recall some notations and an abstract existence result.

Let $Y,\ Z$ be real Banach spaces, $L:\mathrm{dom}L\subset Y\to Z$ be a Fredholm map of index 0 and $P:Y\to Y,\ Q:Z\to Z$ be continuous projectors such that $\mathrm{Im}P=\mathrm{Ker}L,\ \mathrm{Ker}Q=\mathrm{Im}L$ and $Y=\mathrm{Ker}L\oplus\mathrm{Ker}P,\ Z=\mathrm{Im}L\oplus\mathrm{Im}Q$. It follows that $L|_{\mathrm{dom}L\cap\mathrm{Ker}P}:\mathrm{dom}L\cap\mathrm{Ker}P\to\mathrm{Im}L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of Y such that $\mathrm{dom}L\cap\Omega\neq\emptyset$, the map $N:Y\to Z$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N:\overline{\Omega}\to Y$ is compact.

The theorem A we use is Theorem IV.13 of [6].

Theorem A *Let L be a Fredholm operator of index 0 and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) $Lx\neq\lambda Nx$ *for every* $(x,\lambda)\in[(\mathrm{dom}L\backslash\mathrm{Ker}L)\cap\partial\Omega]\times(0,1)$;

(ii) $Nx\notin\mathrm{Im}L$ *for every* $x\in\mathrm{Ker}L\cap\partial\Omega$;

(iii) $\deg(\wedge QN|_{\mathrm{Ker}L},\Omega\cap\mathrm{Ker}L,0)\neq0$,

where $Q:Z\to Z$ *is a projection as above with* $\mathrm{Im}L=\mathrm{Ker}Q,\ \wedge:\mathrm{Im}Q\to\mathrm{Ker}L$ *is a linear isomorphism. Then the equation* $Lx=Nx$ *has at least one solution in* $\mathrm{dom}L\cap\overline{\Omega}$.

We use the classical spaces $C^3[0,1]$ and $L^1[0,1]$. For the $x\in C^3[0,1]$, we use the norm $\|x\|_\infty=\max_{t\in[0,1]}|x(t)|,\ \|x\|=\max\{\|x\|_\infty,\|x'\|_\infty,\|x''\|_\infty,\|x'''\|_\infty\}$, and denote the norm in $L^1[0,1]$ by $\|\cdot\|_1$. We also use the Sobolev space $W^{4,1}(0,1)$ defined by $W^{4,1}(0,1)=\{x:[0,1]\to R|x,x',x'',x'''$ are absolutely continuous on $[0,1]$ with $x^{(4)}(t)\in L^1[0,1]\}$ with its usual norm.

2. Existence result for BVP (E) and (B)

In view of BVP (E), (B) and the resonance condition: $\alpha\delta+\beta\gamma+\alpha\gamma(\xi_2-\xi_1)=0$, we discuss existence of solutions to the BVP (E), (B) subject to the following case:

$$\alpha\neq0,\ \gamma\neq0,\ \alpha\delta+\beta\gamma+\alpha\gamma(\xi_2-\xi_1)=0.$$

Let $Y=C^3[0,1]$, $Z=L^1[0,1]$. Define L to be the linear operator from $\mathrm{dom}L\subset Y\to Z$ with

$$\mathrm{dom}L=\{x\in W^{4,1}(0,1):x(0)=x(1)=0,\alpha x''(\xi_1)-\beta x'''(\xi_1)=0,\ \gamma x''(\xi_2)+\delta x'''(\xi_2)=0\}$$

and $Lx=x^{(4)},\ x\in\mathrm{dom}L$. We define $N:Y\to Z$ by setting

$$Nx=f(t,x(t),x'(t),x''(t),x'''(t)),\quad t\in(0,1).$$

Then the BVP (E), (B) can be written as $Lx=Nx$.

Lemma 1 *If $\alpha\neq0,\ \gamma\neq0,\ 0<\xi_1<\xi_2<1$, then there exists $l\in\{0,1,2\}$ such that* $\alpha\gamma\int_{\xi_1}^{\xi_2}\int_0^s\tau^l\mathrm{d}\tau\mathrm{d}s+\alpha\delta\int_0^{\xi_2}\tau^l\mathrm{d}\tau+\beta\gamma\int_0^{\xi_1}\tau^l\mathrm{d}\tau\neq0.$

Proof Suppose the assertion fails to be true, then

$$\frac{\alpha\gamma}{2}(\xi_2^2-\xi_1^2)+\alpha\delta\xi_2+\beta\gamma\xi_1=0,$$

$$\frac{\alpha\gamma}{3}(\xi_2^3-\xi_1^3)+\alpha\delta\xi_2^2+\beta\gamma\xi_1^2=0,$$

$$\frac{\alpha\gamma}{4}(\xi_2^4-\xi_1^4)+\alpha\delta\xi_2^3+\beta\gamma\xi_1^3=0.$$

Thus we have

$$\begin{pmatrix} \frac{1}{2}(\xi_2^2-\xi_1^2) & \xi_2 & \xi_1 \\ \frac{1}{3}(\xi_2^3-\xi_1^3) & \xi_2^2 & \xi_1^2 \\ \frac{1}{4}(\xi_2^4-\xi_1^4) & \xi_2^3 & \xi_1^3 \end{pmatrix} \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1}$$

Next we shall show

$$\begin{vmatrix} \frac{1}{2}(\xi_2^2-\xi_1^2) & \xi_2 & \xi_1 \\ \frac{1}{3}(\xi_2^3-\xi_1^3) & \xi_2^2 & \xi_1^2 \\ \frac{1}{4}(\xi_2^4-\xi_1^4) & \xi_2^3 & \xi_1^3 \end{vmatrix} \neq 0, \tag{2}$$

if not, then

$$\begin{vmatrix} \frac{1}{2}(\xi_2^2-\xi_1^2) & \xi_2 & \xi_1 \\ \frac{1}{3}(\xi_2^3-\xi_1^3) & \xi_2^2 & \xi_1^2 \\ \frac{1}{4}(\xi_2^4-\xi_1^4) & \xi_2^3 & \xi_1^3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & \xi_1 \\ \frac{1}{6}(\xi_1^3-\xi_2^3) & \xi_2^2-\xi_1\xi_2 & \xi_1^2 \\ \frac{1}{4}(\xi_1^4-\xi_2^4) & \xi_2^3-\xi_1^2\xi_2 & \xi_1^3 \end{vmatrix} = 0. \tag{3}$$

From (3) and $0<\xi_1<\xi_2<1$, we have $(\xi_1-\xi_1)^2=0$, which contradicts $\xi_1<\xi_2$. Hence (2) holds, then from (1), we have $\alpha\gamma=0$, which is a contradiction from $\alpha\neq 0,\ \gamma\neq 0$. Therefore Lemma 1 holds. □

Lemma 2 *If $\alpha\neq 0,\ \gamma\neq 0,\ \alpha\delta+\beta\gamma+\alpha\gamma(\xi_2-\xi_1)=0$, then $L:\mathrm{dom}L\subset Y\to Z$ is a Fredholm operator of index 0. Furthermore, the linear continuous projector operator $Q:Z\to Z$ can be defined by*

$$Qy=\frac{1}{\alpha\gamma\int_{\xi_1}^{\xi_2}\int_0^s \tau^l\mathrm{d}\tau\mathrm{d}s+\alpha\delta\int_0^{\xi_2}\tau^l\mathrm{d}\tau+\beta\gamma\int_0^{\xi_1}\tau^l\mathrm{d}\tau}\Big(\alpha\gamma\int_{\xi_1}^{\xi_2}\int_0^s y(\tau)\mathrm{d}\tau\mathrm{d}s+ \\ \alpha\delta\int_0^{\xi_2}y(\tau)\mathrm{d}\tau+\beta\gamma\int_0^{\xi_1}y(\tau)\mathrm{d}\tau\Big)t^l$$

and the linear operator $K_P:\mathrm{Im}L\to\mathrm{dom}L\cap\mathrm{Ker}P$ can be written as

$$K_Py=-\frac{t(t-1)}{2}\Big(\int_0^{\xi_1}\int_0^s y(\tau)\mathrm{d}\tau\mathrm{d}s-\frac{\beta}{\alpha}\int_0^{\xi_1}y(\tau)\mathrm{d}\tau\Big)+ \\ \int_0^t\int_0^{s_3}\int_0^{s_2}\int_0^{s_1}y(\tau)\mathrm{d}\tau\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3- \\ t\int_0^1\int_0^{s_3}\int_0^{s_2}\int_0^{s_1}y(\tau)\mathrm{d}\tau\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3.$$

Also

$$\|K_Py\|\le\Delta_1\|y\|_1\quad\text{for all } y\in\mathrm{Im}L,$$

here $\Delta_1=\max\{\frac{5}{2}+|\frac{\beta}{2\alpha}|,\ 2+|\frac{\beta}{\alpha}|\}$.

Proof It is clear that

$$\mathrm{Ker}L=\{x\in\mathrm{dom}L: x(t)=d(t^3+\frac{3\beta-3\alpha\xi_1}{\alpha}t^2-\frac{\alpha+3\beta-3\alpha\xi_1}{\alpha}t),\ d\in R\}.$$

We now show that

$$\mathrm{Im}L=\{y\in Z:\alpha\gamma\int_{\xi_1}^{\xi_2}\int_0^s y(\tau)\mathrm{d}\tau\mathrm{d}s+\alpha\delta\int_0^{\xi_2}y(\tau)\mathrm{d}\tau+\beta\gamma\int_0^{\xi_1}y(\tau)\mathrm{d}\tau=0\}. \tag{4}$$

Since the problem
$$x^{(4)}(t)=y \tag{5}$$
has solution $x(t)$ which satisfies
$$x(0)=x(1)=0,\ \ \alpha x''(\xi_1)-\beta x'''(\xi_1)=0,\ \ \gamma x''(\xi_2)+\delta x'''(\xi_2)=0,$$
if and only if
$$\alpha\gamma\int_{\xi_1}^{\xi_2}\int_0^s y(\tau)\mathrm{d}\tau\mathrm{d}s+\alpha\delta\int_0^{\xi_2}y(\tau)\mathrm{d}\tau+\beta\gamma\int_0^{\xi_1}y(\tau)\mathrm{d}\tau=0. \tag{6}$$

In fact, if (5) has solution $x(t)$ such that $x(0)=x(1)=0,\ \alpha x''(\xi_1)-\beta x'''(\xi_1)=0,\ \gamma x''(\xi_2)+\delta x'''(\xi_2)=0$, then from (5), we have
$$x(t)=x(0)+x'(0)t+\frac{1}{2}x''(0)t^2+\frac{1}{6}x'''(0)t^3+\int_0^t\int_0^{s_3}\int_0^{s_2}\int_0^{s_1}y(\tau)\mathrm{d}\tau\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3.$$
Thus
$$\alpha x''(\xi_1)-\beta x'''(\xi_1)=\alpha\Big(x''(0)+x'''(0)\xi_1+\int_0^{\xi_1}\int_0^s y(\tau)\mathrm{d}\tau\mathrm{d}s\Big)-\beta\Big(x'''(0)+\int_0^{\xi_1}y(\tau)\mathrm{d}\tau\Big)=0, \tag{7}$$
$$\gamma x''(\xi_2)+\delta x'''(\xi_2)=\gamma\Big(x''(0)+x'''(0)\xi_2+\int_0^{\xi_2}\int_0^s y(\tau)\mathrm{d}\tau\mathrm{d}s\Big)+\delta\Big(x'''(0)+\int_0^{\xi_2}y(\tau)\mathrm{d}\tau\Big)=0. \tag{8}$$
Then from (7), (8) and $\alpha\delta+\beta\gamma+\alpha\gamma(\xi_2-\xi_1)=0$, we can obtain
$$\alpha\gamma\int_{\xi_1}^{\xi_2}\int_0^s y(\tau)\mathrm{d}\tau\mathrm{d}s+\alpha\delta\int_0^{\xi_2}y(\tau)\mathrm{d}\tau+\beta\gamma\int_0^{\xi_1}y(\tau)\mathrm{d}\tau=0.$$
On the other hand, if (6) holds, set
$$\begin{aligned}x(t)=&d\Big(t^3+\frac{3\beta-3\alpha\xi_1}{\alpha}t^2-\frac{\alpha+3\beta-3\alpha\xi_1}{\alpha}t\Big)-\\&\frac{t(t-1)}{2}\Big(\int_0^{\xi_1}\int_0^s y(\tau)\mathrm{d}\tau\mathrm{d}s-\frac{\beta}{\alpha}\int_0^{\xi_1}y(\tau)\mathrm{d}\tau\Big)+\\&\int_0^t\int_0^{s_3}\int_0^{s_2}\int_0^{s_1}y(\tau)\mathrm{d}\tau\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3-\\&t\int_0^1\int_0^{s_3}\int_0^{s_2}\int_0^{s_1}y(\tau)\mathrm{d}\tau\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3,\end{aligned}$$
where d is an arbitrary constant, then $x(t)$ is a solution of (5), $x(0)=x(1)=0,\ \alpha x''(\xi_1)-\beta x'''(\xi_1)=0,\ \gamma x''(\xi_2)+\delta x'''(\xi_2)=0$. Hence (4) holds.

For $y\in Z$, take the projector
$$\begin{aligned}Qy=&\frac{1}{\alpha\gamma\int_{\xi_1}^{\xi_2}\int_0^s\tau^l\mathrm{d}\tau\mathrm{d}s+\alpha\delta\int_0^{\xi_2}\tau^l\mathrm{d}\tau+\beta\gamma\int_0^{\xi_1}\tau^l\mathrm{d}\tau}\Big(\alpha\gamma\int_{\xi_1}^{\xi_2}\int_0^s y(\tau)\mathrm{d}\tau\mathrm{d}s+\\&\alpha\delta\int_0^{\xi_2}y(\tau)\mathrm{d}\tau+\beta\gamma\int_0^{\xi_1}y(\tau)\mathrm{d}\tau\Big)t^l.\end{aligned}$$
Let $y_1=y-Qy$. Since $\alpha\gamma\int_{\xi_1}^{\xi_2}\int_0^s y_1(\tau)\mathrm{d}\tau\mathrm{d}s+\alpha\delta\int_0^{\xi_2}y_1(\tau)\mathrm{d}\tau+\beta\gamma\int_0^{\xi_1}y_1(\tau)\mathrm{d}\tau=0$, we have $y_1\in\mathrm{Im}L$. Hence $Z=\mathrm{Im}L+\mathrm{Im}Q$. Since $\mathrm{Im}L\cap\mathrm{Im}Q=\{0\}$, we have $Z=\mathrm{Im}L\oplus\mathrm{Im}Q$. Thus
$$\dim\ \mathrm{Ker}L=\dim\ \mathrm{Im}Q=\mathrm{codim}\ \mathrm{Im}L=1.$$

Hence L is a Fredholm operator of index 0.

Taking $P : Y \to Y$ as

$$Px = \frac{x'''(0)}{6}\Big(t^3 + \frac{3\beta - 3\alpha\xi_1}{\alpha}t^2 - \frac{\alpha + 3\beta - 3\alpha\xi_1}{\alpha}t\Big),$$

one can write the generalized inverse $K_P : \mathrm{ImL} \to \mathrm{dom}L \cap \mathrm{Ker}P$ of L as

$$\begin{aligned} K_P y = &-\frac{t(t-1)}{2}\Big(\int_0^{\xi_1}\int_0^s y(\tau)\mathrm{d}\tau\mathrm{d}s - \frac{\beta}{\alpha}\int_0^{\xi_1} y(\tau)\mathrm{d}\tau\Big) + \\ &\int_0^t\int_0^{s_3}\int_0^{s_2}\int_0^{s_1} y(\tau)\mathrm{d}\tau\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3 - \\ &t\int_0^1\int_0^{s_3}\int_0^{s_2}\int_0^{s_1} y(\tau)\mathrm{d}\tau\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3. \end{aligned} \tag{9}$$

In fact, for $y(t) \in \mathrm{Im}L$, we have

$$(LK_P)y(t) = [K_P y(t)]^{(4)} = y(t),$$

and for $x \in \mathrm{dom}L \cap \mathrm{Ker}P$, we know

$$\begin{aligned} (K_P L)x(t) = &-\frac{t(t-1)}{2}\Big(\int_0^{\xi_1}\int_0^s x^{(4)}(\tau)\mathrm{d}\tau\mathrm{d}s - \frac{\beta}{\alpha}\int_0^{\xi_1} x^{(4)}(\tau)\mathrm{d}\tau\Big) + \\ &\int_0^t\int_0^{s_3}\int_0^{s_2}\int_0^{s_1} x^{(4)}(\tau)\mathrm{d}\tau\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3 - \\ &t\int_0^1\int_0^{s_3}\int_0^{s_2}\int_0^{s_1} x^{(4)}(\tau)\mathrm{d}\tau\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3. \end{aligned}$$

In view of $x \in \mathrm{dom}L \cap \mathrm{Ker}P$, $x(0) = x(1) = 0$, $\alpha x''(\xi_1) - \beta x'''(\xi_1) = 0$ and $Px = \frac{x'''(0)}{6}(t^3 + \frac{3\beta-3\alpha\xi_1}{\alpha}t^2 - \frac{\alpha+3\beta-3\alpha\xi_1}{a}t) = 0$, we have

$$(K_P L)x(t) = x(t).$$

This shows that $K_P = (L|_{\mathrm{dom}L\cap\mathrm{Ker}P})^{-1}$. From (9), we have

$$\|K_P y\|_\infty \leqslant \frac{1}{8}(\|y\|_1 + |\frac{\beta}{\alpha}|\|y\|_1) + 2\|y\|_1 = (\frac{17}{8} + |\frac{\beta}{8\alpha}|)\|y\|_1$$

and

$$\|(K_P y)'\|_\infty \leqslant \frac{5}{2}\|y\|_1 + |\frac{\beta}{2\alpha}|\|y\|_1$$

and

$$\|(K_P y)''\|_\infty \leqslant 2\|y\|_1 + |\frac{\beta}{\alpha}|\|y\|_1$$

and

$$\|(K_P y)'''\|_\infty \leqslant \|y\|_1.$$

Thus $\|K_P y\| = \max\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty, \|x'''\|_\infty\} \le \Delta_1\|y\|_1$.

This completes the proof of Lemma 2. □

Lemma 3 *Let $f : [0,1]\times R^4 \to R$ be a continuous function. Assume that the following conditions are satisfied:*

(A_1) There exist functions $u_1,\ u_2,\ u_3,\ u_4,\ e$ in $L^1[0,1]$ and a constant $\theta\in[0,1)$, such that

$$\begin{aligned}|f(t,x_1,x_2,x_3,x_4)|\leq & u_1(t)|x_1|+u_2(t)|x_2|+u_3(t)|x_3|+u_4(t)|x_4|+\\ & e(t)(|x_1|^\theta+|x_2|^\theta+|x_3|^\theta+|x_4|^\theta)+r(t)\end{aligned}$$

for all $(x,y,z,u)\in R^4,\ t\in[0,1]$;

(A_2) There exists a positive constant Δ such that $\|u_1\|_1+\|u_2\|_1+\|u_3\|_1+\|u_4\|_1<\frac{1}{\Delta}$. Then there exist nonnegative functions $\overline{u}_1,\ \overline{u}_2,\ \overline{u}_3,\ \overline{u}_4,\ \overline{r}$ in $L^1[0,1]$ such that

$$|f(t,x_1,x_2,x_3,x_4)|\leq \overline{u}_1(t)|x_1|+\overline{u}_2(t)|x_2|+\overline{u}_3(t)|x_3|+\overline{u}_4(t)|x_4|+\overline{r}(t). \tag{10}$$

Proof This lemma is similar to the Lemma 3.2 of [2] and the proof is similar too, so we omit it. □

For brevity, we use $u_1,\ u_2,\ u_3,\ u_4,\ r$ to denote $\overline{u}_1,\ \overline{u}_2,\ \overline{u}_3,\ \overline{u}_4,\ \overline{r}$, respectively. Then (10) can be written as

$$|f(t,x_1,x_2,x_3,x_4)|\leq u_1(t)|x_1|+u_2(t)|x_2|+u_3(t)|x_3|+u_4(t)|x_4|+r(t).$$

From now on, we always assume that $\rho(t)\triangleq t^3+\frac{3\beta-3\alpha\xi_1}{\alpha}t^2-\frac{\alpha+3\beta-3\alpha\xi_1}{\alpha}t,\ t\in[0,1]$.

Theorem 1 *Let $f:[0,1]\times R^4\to R$ be a continuous function. Assume that (A_1) of Lemma 3 is satisfied and the following conditions hold:*

(A_3) There exists constant $M>0$ such that for $x\in domL$, if $|x'''(t)|>M$ for all $t\in(0,1)$, then

$$\begin{aligned}&\alpha\gamma\int_{\xi_1}^{\xi_2}\int_0^s f(\tau,x(\tau),x'(\tau),x''(\tau),x'''(\tau))\mathrm{d}\tau\mathrm{d}s+\\ &\alpha\delta\int_0^{\xi_2} f(\tau,x(\tau),x'(\tau),x''(\tau),x'''(\tau))\mathrm{d}\tau+\\ &\beta\gamma\int_0^{\xi_1} f(\tau,x(\tau),x'(\tau),x''(\tau),x'''(\tau))\mathrm{d}\tau\neq 0;\end{aligned}$$

(A_4) There exists constant $M^*>0$ such that for all $d\in R$, if $|d|>M^*$, then either

$$\begin{aligned}d\Big(&\alpha\gamma\int_{\xi_1}^{\xi_2}\int_0^s f(\tau,\mathrm{d}\rho(\tau),\mathrm{d}\rho'(\tau),\mathrm{d}\rho''(\tau),\mathrm{d}\rho'''(\tau))\mathrm{d}\tau\mathrm{d}s+\\ &\alpha\delta\int_0^{\xi_2} f(\tau,\mathrm{d}\rho(\tau),\mathrm{d}\rho'(\tau),\mathrm{d}\rho''(\tau),\mathrm{d}\rho'''(\tau))\mathrm{d}\tau+\\ &\beta\gamma\int_0^{\xi_1} f(\tau,\mathrm{d}\rho(\tau),\mathrm{d}\rho'(\tau),\mathrm{d}\rho''(\tau),\mathrm{d}\rho'''(\tau))\mathrm{d}\tau\Big)<0\end{aligned}$$

or else

$$\begin{aligned}d\Big(&\alpha\gamma\int_{\xi_1}^{\xi_2}\int_0^s f(\tau,\mathrm{d}\rho(\tau),\mathrm{d}\rho'(\tau),\mathrm{d}\rho''(\tau),\mathrm{d}\rho'''(\tau))\mathrm{d}\tau\mathrm{d}s+\\ &\alpha\delta\int_0^{\xi_2} f(\tau,\mathrm{d}\rho(\tau),\mathrm{d}\rho'(\tau),\mathrm{d}\rho''(\tau),\mathrm{d}\rho'''(\tau))\mathrm{d}\tau+\\ &\beta\gamma\int_0^{\xi_1} f(\tau,\mathrm{d}\rho(\tau),\mathrm{d}\rho'(\tau),\mathrm{d}\rho''(\tau),\mathrm{d}\rho'''(\tau))\mathrm{d}\tau\Big)>0.\end{aligned}$$

Then, the BVP (E), (B) with $\alpha \neq 0,\ \gamma \neq 0,\ x(0) = x(1) = 0,\ \alpha x''(\xi_1) - \beta x'''(\xi_1) = 0,\ \gamma x''(\xi_2) + \delta x'''(\xi_2) = 0,\ \alpha\delta + \beta\gamma + \alpha\gamma(\xi_2 - \xi_1) = 0$ *has at least one solution in* $C^3[0,1]$ *provided that*

$$\|u_1\|_1 + \|u_2\|_1 + \|u_3\|_1 + \|u_4\|_1 < \frac{1}{\Delta_1 + \Delta_2},$$

where Δ_1 *is as in Lemma 2,* $\Delta_2 = \frac{1}{6}\|\rho(t)\|_{c^3[0,1]}$.

Proof Set

$$\Omega_1 = \{x \in \mathrm{dom}L \setminus \mathrm{Ker}L:\ Lx = \lambda Nx,\ \lambda \in [0,1]\}.$$

Then for $x \in \Omega_1,\ Lx = \lambda Nx$, thus $\lambda \neq 0,\ Nx \in \mathrm{Im}L = \mathrm{Ker}Q$. Hence

$$\alpha\gamma \int_{\xi_1}^{\xi_2} \int_0^s f(\tau, x(\tau), x'(\tau), x''(\tau), x'''(\tau))\mathrm{d}\tau \mathrm{d}s + \alpha\delta \int_0^{\xi_2} f(\tau, x(\tau), x'(\tau), x''(\tau), x'''(\tau))\mathrm{d}\tau +$$
$$\beta\gamma \int_0^{\xi_1} f(\tau, x(\tau), x'(\tau), x''(\tau), x'''(\tau))\mathrm{d}\tau = 0.$$

Thus from (A_3), there exists $t_0 \in [0,1]$ such that $|x'''(t_0)| \leq M$. In view of $x'''(0) = x'''(t_0) - \int_0^{t_0} x^{(4)}(t)\mathrm{d}t$, then

$$|x'''(0)| \leq M + \|x^{(4)}(t)\|_1 = M + \|Lx\|_1 \leq M + \|Nx\|_1. \tag{11}$$

Again for $x \in \Omega_1, x \in \mathrm{dom}L \setminus \mathrm{Ker}L$, then $(I - P)x \in \mathrm{dom}L \cap \mathrm{Ker}P,\ LPx = 0$. From Lemma 2, we have

$$\|(I-P)x\| = \|K_P L(I-P)x\| \leq \Delta_1 \|L(I-P)x\|_1 = \Delta_1 \|Lx\|_1 \leq \Delta_1 \|Nx\|_1. \tag{12}$$

From (11) and (12), we have

$$\begin{aligned}\|x\| &\leq \|Px\| + \|(I-P)x\| \\ &= \frac{|x'''(0)|}{6}\|t^3 + \frac{3\beta - 3\alpha\xi_1}{\alpha}t^2 - \frac{\alpha + 3\beta - 3\alpha\xi_1}{\alpha}t\| + \|(I-P)x\| \\ &\leq (M + \|Nx\|_1)\Delta_2 + \Delta_1\|Nx\|_1 \\ &= \Delta_2 M + (\Delta_1 + \Delta_2)\|Nx\|_1\ .\end{aligned}$$

Then from Lemma 3, we obtain

$$\|x\| \leq \Delta_2 M + (\Delta_1 + \Delta_2)(\|u_1\|_1\|x\|_\infty + \|u_2\|_1\|x'\|_\infty + \|u_3\|_1\|x''\|_\infty + \|u_4\|_1\|x'''\|_\infty + \|r\|_1). \tag{13}$$

Thus from $\|x\|_\infty \leq \|x\|$ and (13), we obtain

$$\begin{aligned}\|x\|_\infty \leq &\frac{\Delta_1 + \Delta_2}{1 - (\Delta_1 + \Delta_2)\|u_1\|_1}\Big(\|u_2\|_1\|x'\|_\infty + \|u_3\|_1\|x''\|_\infty + \\ &\|u_4\|_1\|x'''\|_\infty + \|r\|_1 + \frac{\Delta_2 M}{\Delta_1 + \Delta_2}\Big).\end{aligned} \tag{14}$$

Thus from $\|x'\|_\infty \leq \|x\|$, (13) and (14), we obtain

$$\begin{aligned}\|x'\|_\infty \leq &\frac{\Delta_1 + \Delta_2}{1 - (\Delta_1 + \Delta_2)(\|u_1\|_1 + \|u_2\|_1)}\Big(\|u_3\|_1\|x''\|_\infty + \\ &\|u_4\|_1\|x'''\|_\infty + \|r\|_1 + \frac{\Delta_2 M}{\Delta_1 + \Delta_2}\Big).\end{aligned} \tag{15}$$

Thus from $\|x''\|_\infty \le \|x\|$, (13), (14) and (15), we obtain

$$\|x''\|_\infty \le \frac{\Delta_1+\Delta_2}{1-(\Delta_1+\Delta_2)(\|u_1\|_1+\|u_2\|_1+\|u_3\|_1)}\Big(\|u_4\|_1\|x'''\|_\infty+\|r\|_1+\frac{\Delta_2 M}{\Delta_1+\Delta_2}\Big). \tag{16}$$

And from $\|x'''\|_\infty \le \|x\|$, (13), (14), (15) and (16), it follows

$$\|x'''\|_\infty \le \frac{\Delta_1+\Delta_2}{1-(\Delta_1+\Delta_2)(\|u_1\|_1+\|u_2\|_1+\|u_3\|_1+\|u_4\|_1)}\Big(\|r\|_1+\frac{\Delta_2 M}{\Delta_1+\Delta_2}\Big). \tag{17}$$

From (17), we know $\|x'''\|_\infty$ is bounded. From (16), (15) and (14), we can obtain $\|x''\|_\infty$, $\|x'\|_\infty$ and $\|x\|_\infty$ are all bounded. In view of $\|x\|=\max\{\|x\|_\infty, \|x'\|_\infty\|x''\|_\infty, \|x'''\|_\infty\}$, we know Ω_1 is bounded.

Set

$$\Omega_2=\{x\in \mathrm{Ker}L: Nx\in \mathrm{Im}L\},$$

for $x\in\Omega_2,\ x\in \mathrm{Ker}L=\{x\in \mathrm{dom}L: x=d\rho(t)\}$ and $QNx=0$. Then

$$\begin{aligned}&\frac{1}{\alpha\gamma\int_{\xi_1}^{\xi_2}\int_0^s\tau^l\mathrm{d}\tau\mathrm{d}s+\alpha\delta\int_0^{\xi_2}\tau^l\mathrm{d}\tau+\beta\gamma\int_0^{\xi_1}\tau^l\mathrm{d}\tau}\Big(\alpha\gamma\int_{\xi_1}^{\xi_2}\int_0^s f(\tau,\mathrm{d}\rho(\tau),\mathrm{d}\rho'(\tau),\\&\mathrm{d}\rho''(\tau),\mathrm{d}\rho'''(\tau))\mathrm{d}\tau\mathrm{d}s+\alpha\delta\int_0^{\xi_2}f(\tau,d\rho(\tau),\mathrm{d}\rho'(\tau),\mathrm{d}\rho''(\tau),\mathrm{d}\rho'''(\tau))\mathrm{d}\tau+\\&\beta\gamma\int_0^{\xi_1}f(\tau,\mathrm{d}\rho(\tau),\mathrm{d}\rho'(\tau),\mathrm{d}\rho''(\tau),\mathrm{d}\rho'''(\tau))\mathrm{d}\tau\Big)=0.\end{aligned} \tag{18}$$

From (A_3), there exists $t_0\in[0,1]$ such that $|x'''(t_0)|\le M$. Hence $|d|\le\frac{M}{6}$. Then we can obtain $\|x\|\le M\Delta_2$. Thus Ω_2 is bounded.

Next, according to the condition (A_4), for any $d\in R$, if $|d|>M^*$, then either

$$\begin{aligned}&d\frac{1}{\alpha\gamma\int_{\xi_1}^{\xi_2}\int_0^s\tau^l\mathrm{d}\tau\mathrm{d}s+\alpha\delta\int_0^{\xi_2}\tau^l\mathrm{d}\tau+\beta\gamma\int_0^{\xi_1}\tau^l\mathrm{d}\tau}\Big(\alpha\gamma\int_{\xi_1}^{\xi_2}\int_0^s f(\tau,\mathrm{d}\rho(\tau),\mathrm{d}\rho'(\tau),\\&\mathrm{d}\rho''(\tau),\mathrm{d}\rho'''(\tau))\mathrm{d}\tau\mathrm{d}s+\alpha\delta\int_0^{\xi_2}f(\tau,\mathrm{d}\rho(\tau),\mathrm{d}\rho'(\tau),\mathrm{d}\rho''(\tau),\mathrm{d}\rho'''(\tau))\mathrm{d}\tau+\\&\beta\gamma\int_0^{\xi_1}f(\tau,\mathrm{d}\rho(\tau),\mathrm{d}\rho'(\tau),\mathrm{d}\rho''(\tau),\mathrm{d}\rho'''(\tau))\mathrm{d}\tau\Big)<0\end{aligned} \tag{19}$$

or else

$$\begin{aligned}&d\frac{1}{\alpha\gamma\int_{\xi_1}^{\xi_2}\int_0^s\tau^l\mathrm{d}\tau\mathrm{d}s+\alpha\delta\int_0^{\xi_2}\tau^l\mathrm{d}\tau+\beta\gamma\int_0^{\xi_1}\tau^l\mathrm{d}\tau}\Big(\alpha\gamma\int_{\xi_1}^{\xi_2}\int_0^s f(\tau,\mathrm{d}\rho(\tau),\mathrm{d}\rho'(\tau),\\&\mathrm{d}\rho''(\tau),\mathrm{d}\rho'''(\tau))\mathrm{d}\tau\mathrm{d}s+\alpha\delta\int_0^{\xi_2}f(\tau,\mathrm{d}\rho(\tau),\mathrm{d}\rho'(\tau),\mathrm{d}\rho''(\tau),\mathrm{d}\rho'''(\tau))\mathrm{d}\tau+\\&\beta\gamma\int_0^{\xi_1}f(\tau,\mathrm{d}\rho(\tau),\mathrm{d}\rho'(\tau),\mathrm{d}\rho''(\tau),\mathrm{d}\rho'''(\tau))\mathrm{d}\tau\Big)>0.\end{aligned} \tag{20}$$

If (19) holds, set

$$\Omega_3=\{x\in \mathrm{Ker}L: -\lambda x+(1-\lambda)\wedge QNx=0, \lambda\in[0,1]\},$$

here $\wedge: \mathrm{Im}Q\to \mathrm{Ker}L$ is the linear isomorphism given by $\wedge(d)=\mathrm{d}\rho(t),\ \forall\ d\in R,\ t\in[0,1]$. Since

$x = \mathrm{d}_0\rho(t) \in \Omega_3$, we have

$$\begin{aligned}\lambda d_0\rho(t) =&(1-\lambda)\frac{\rho(t)}{\alpha\gamma\int_{\xi_1}^{\xi_2}\int_0^s \tau^l \mathrm{d}\tau\mathrm{d}s + \alpha\delta\int_0^{\xi_2}\tau^l\mathrm{d}\tau + \beta\gamma\int_0^{\xi_1}\tau^l\mathrm{d}\tau}\Big(\alpha\gamma \\ &\int_{\xi_1}^{\xi_2}\int_0^s f(\tau, \mathrm{d}_0\rho(\tau), \mathrm{d}_0\rho'(\tau), \mathrm{d}_0\rho''(\tau), \mathrm{d}_0\rho'''(\tau))\mathrm{d}\tau\mathrm{d}s + \\ &\alpha\delta\int_0^{\xi_2} f(\tau, \mathrm{d}_0\rho(\tau), \mathrm{d}_0\rho'(\tau), \mathrm{d}_0\rho''(\tau), \mathrm{d}_0\rho'''(\tau))\mathrm{d}\tau + \\ &\beta\gamma\int_0^{\xi_1} f(\tau, \mathrm{d}_0\rho(\tau), \mathrm{d}_0\rho'(\tau), \mathrm{d}_0\rho''(\tau), \mathrm{d}_0\rho'''(\tau))\mathrm{d}\tau\Big)t^l.\end{aligned}$$

If $\lambda = 1$, then $d_0 = 0$. Otherwise, if $|d_0| > M^*$, in view of (19), we have

$$\begin{aligned}\lambda d_0^2 =&d_0(1-\lambda)\frac{1}{\alpha\gamma\int_{\xi_1}^{\xi_2}\int_0^s \tau^l \mathrm{d}\tau\mathrm{d}s + \alpha\delta\int_0^{\xi_2}\tau^l\mathrm{d}\tau + \beta\gamma\int_0^{\xi_1}\tau^l\mathrm{d}\tau}\Big(\alpha\gamma \\ &\int_{\xi_1}^{\xi_2}\int_0^s f(\tau, \mathrm{d}_0\rho(\tau), \mathrm{d}_0\rho'(\tau), \mathrm{d}_0\rho''(\tau), \mathrm{d}_0\rho'''(\tau))\mathrm{d}\tau\mathrm{d}s + \\ &\alpha\delta\int_0^{\xi_2} f(\tau, \mathrm{d}_0\rho(\tau), \mathrm{d}_0\rho'(\tau), \mathrm{d}_0\rho''(\tau), \mathrm{d}_0\rho'''(\tau))\mathrm{d}\tau + \\ &\beta\gamma\int_0^{\xi_1} f(\tau, \mathrm{d}_0\rho(\tau), \mathrm{d}_0\rho'(\tau), \mathrm{d}_0\rho''(\tau), \mathrm{d}_0\rho'''(\tau))\mathrm{d}\tau\Big)t^l < 0,\end{aligned}$$

which contradicts $\lambda d_0^2 \geq 0$. Thus $\Omega_3 \subset \{x \in \mathrm{Ker}L : \|x\| \leq 6M^*\Delta_2\}$ is bounded.

If (20) holds, then set

$$\Omega_3 = \{x \in \mathrm{Ker}L : \lambda x + (1-\lambda) \wedge QNx = 0, \lambda \in [0,1]\},$$

here $\wedge$ is as above. Similarly to the above argument, we can show that Ω_3 is bounded too.

In the following, we shall prove that all the conditions of Theorem A are satisfied. Let Ω be a bounded open subset of Y such that $\bigcup_{i=1}^3 \overline{\Omega}_i \subset \Omega$. By using the Ascoli-Arzela theorem, we can prove that $K_P(I-Q)N : \overline{\Omega} \to Y$ is compact. Thus N is L-compact on $\overline{\Omega}$. Then by the above argument we have:

(i) $Lx \neq \lambda Nx$ for every $(x,\lambda) \in [(\mathrm{dom}L \setminus \mathrm{Ker}L) \cap \partial\Omega] \times (0,1)$;

(ii) $Lx \notin \mathrm{Im}L$ for $x \in \mathrm{Ker}L \cap \partial\Omega$.

At last we will prove that (iii) of Theorem A is satisfied. Let

$$H(x,\lambda) = \pm\lambda x + (1-\lambda) \wedge QNx.$$

According to the above argument, we know $H(x,\lambda) \neq 0$ for $x \in \partial\Omega \cap \mathrm{Ker}L$. Thus, by the homotopy property of degree

$$\begin{aligned}&\deg(\wedge QN|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0) = \deg(H(\cdot,0), \Omega \cap \mathrm{Ker}L, 0) \\ &= \deg(H(\cdot,1), \Omega \cap \mathrm{Ker}L, 0) = \deg(\pm I, \Omega \cap \mathrm{Ker}L, 0) \neq 0.\end{aligned}$$

Then by Theorem A, $Lx = Nx$ has at least one solution in $\mathrm{dom}L \cap \overline{\Omega}$, so that the BVP (E), (B) has solution in $C^3[0,1]$. The proof is completed. $\square$

3. An example

Consider the problem

$$x^{(4)}(t)=t+1+t\cos x(t)+\frac{1}{2}\arctan x'(t)+\frac{1}{3}\sin x''(t)+\frac{1}{10}(t^2+1)x'''(t), \tag{21}$$

$$x(0)=x(1)=0,\ x''(\frac{1}{4})-\frac{1}{4}x'''(\frac{1}{4})=0,\ x''(\frac{1}{2})-\frac{1}{2}x'''(\frac{1}{2})=0. \tag{22}$$

Since $f(t,x_1,x_2,x_3,x_4)=t+1+t\cos x(t)+\frac{1}{2}\arctan x'(t)+\frac{1}{3}\sin x''(t)+\frac{1}{10}(t^2+1)x'''(t),\ \alpha=1,\ \beta=\frac{1}{4},\gamma=1,\ \delta=-\frac{1}{2},\ \xi_1=\frac{1}{4},\ \xi_2=\frac{1}{2}$, we have $\alpha\delta+\beta\gamma+\alpha\gamma(\xi_2-\xi_1)=0$, $\frac{1}{2}(\xi_2^2-\xi_1^2)\alpha\gamma+\alpha\delta\xi_2+\beta\gamma\xi_1=-\frac{3}{32}\neq 0$, $\Delta_1=\frac{21}{8},\ \Delta_2=1,\ \frac{1}{\Delta_1+\Delta_2}=\frac{8}{29}>\frac{1}{4}$. Let

$$u_1=u_2=u_3=0,\ u_4=\frac{1}{5},\ r(t)=5.$$

Then

$$|f(t,x_1,x_2,x_3,x_4)|\le\frac{1}{5}|x_4|+5,\quad \|u_1\|_1+\|u_2\|_1+\|u_3\|_1+\|u_4\|_1<\frac{1}{\Delta_1+\Delta_2}.$$

Since

$$\begin{aligned}
&\alpha\gamma\int_{\xi_1}^{\xi_2}\int_0^s f(\tau,x(\tau),x'(\tau),x''(\tau),x'''(\tau))\mathrm{d}\tau\mathrm{d}s+\alpha\delta\int_0^{\xi_2}f(\tau,x(\tau),x'(\tau),x''(\tau),x'''(\tau))\mathrm{d}\tau\\
&\quad\beta\gamma\int_0^{\xi_1}f(\tau,x(\tau),x'(\tau),x''(\tau),x'''(\tau))\mathrm{d}\tau\\
&\ =\int_{\frac{1}{4}}^{\frac{1}{2}}\int_0^s f(\tau,x(\tau),x'(\tau),x''(\tau),x'''(\tau))\mathrm{d}\tau\mathrm{d}s-\frac{1}{2}\int_0^{\frac{1}{2}}f(\tau,x(\tau),x'(\tau),x''(\tau),x'''(\tau))\mathrm{d}\tau+\\
&\quad\frac{1}{4}\int_0^{\frac{1}{4}}f(\tau,x(\tau),x'(\tau),x''(\tau),x'''(\tau))\mathrm{d}\tau\\
&\ =-\int_{\frac{1}{4}}^{\frac{1}{2}}\tau f(\tau,x'(\tau),x''(\tau),x'''(\tau))\mathrm{d}\tau,
\end{aligned}$$

as $|x'''(t)|>50$, f and $x'''(t)$ have the same sign. If we set $M=50$ and $M^*=\frac{50}{6}$, the conditions of Theorem 1 are satisfied. Hence from Theorem 1, there exists at least one solution $x(t)\in C^3[0,1]$ to (21) with the condition (22).

References

[1] CHEN Shihua, NI Wei, WANG Changping. *Positive solution of fourth order ordinary differential equation with four-point boundary conditions* [J]. Appl. Math. Lett., 2006, **19**(2): 161–168.

[2] DU Zengji, LIN Xiaojie, GE Weigao. *Solvability of a third-order nonlocal boundary value problem at resonance* [J]. Acta Math. Sinica (Chin. Ser.), 2006, **49**(1): 87–94.

[3] DU Zengji, LIN Xiaojie, GE Weigao. *On a third-order multi-point boundary value problem at resonance* [J]. J. Math. Anal. Appl., 2005, **302**(1): 217–229.

[4] LIU Bing, YU Jianshe. *Solvability of multi-point boundary value problem for second order ordinary differential equations* [J]. Appl. Math. Comput., 1998, **89**: 133–146.

[5] LIU Bing, YU Jianshe. *Solvability of multi-point boundary value problem at resonance. III* [J]. Appl. Math. Comput., 2002, **129**(1): 119–143.

[6] MAWHIN J. *Topological degree methods in nonlinear boundary value problems* [C]. In: NSFCBMS Regional Conference Series in Mathematics, American Mathematical Society. Providence: Rhode Island, USA, 1979.