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Extended Hyperbolic Function Rational Expansion Algorithm with Symbolic Computation to Construct Solitary Wave Solutions of Discrete mKdV Lattice

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Abstract With the aid of Maple, the extended hyperbolic function rational expansion method is used to construct explicit and exact travelling solutions for the discrete mKdV lattice. As a result, many solutions are obained which include kink-shaped solitary wave solutions, bell-shaped solitary wave solutions and singular solitary wave solutions.

Keywords The discrete mKdV lattice; Hyperbolic function expansion method; Solitary wave solution.

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1. Introduction

Differential-difference equations have played a crucial role in the modelling of many phenomena in different fields, which include condensed matter physics, mechanical engineering, vibrations in lattices, pulses in biological chains. DDEs also encounter such systems in numerical simulation of soliton dynamics in high energy physics where they arise as approximations of continuum models.

Many work have been done on DDEs, including investigations of integrability criteria, the computation of densities, generalized and master symmetries, and recursion operators[1]. Notable is the work by Levi and colleagues[2], Yamilov and co-workers[3−7], where the classification of DDEs (into canonical forms), integrability tests, and connections between integrable PDEs and DDEs are analyzed in detail.

Since the work of Fermi, Pasta and Ulam[8], the investigation of exact solutions of the DDEs have been the focus of many nonlinear studies[9]. Unlike difference equations which are fully discrete, DDEs are semi-discrete with some (or all) of their spacial variables discrete while time is usually kept continuous. So there are more difficulties in finding the exact solutions for DDEs. We know that there has been considerable work done on finding exact solutions of PDEs, various

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direct methods have become increasingly attractive partly due to the availability of computer symbolic like Maple or Mathematica, such as algebraic method[10], rational expansion method[11], tanh method[12], the generalized Riccati equation method[13] and so on.

In this paper, hyperbolic function rational expansion method is presented to uniformly construct more new exact solutions for DDEs. And the discrete mKdV lattice is chosen to illustrate the method.

2. Summary of the hyperbolic function rational expansion method

For a given differential-difference DDE

$$\Delta\left(u_{n+p_1}(x),\ldots,u_{n+p_l}(x),u'_{n+p_1}(x),\ldots,u'_{n+p_l},\ldots,u^{(k)}_{n+p_1}(x),\ldots,u^{(k)}_{n+p_l}(x)\right)=0, \tag{2.1}$$

where $x=(x_1,\ldots,x_m)$, $u=(u_1,\ldots,u_q)$, $n=(n_1,\ldots,n_s)$ and $m,q,s,p_1,\ldots,p_l$ are integers, and $u^{(k)}(x)$ denotes the collection of mixed derivative terms of order k. Firstly, we introduce the following travelling wave transform

$$u_{i,n}=U_{i,n}(\xi_n),\quad \xi_n=\sum_{i=1}^{s}d_in_i+\sum_{j=1}^{m}\lambda_jx_j+\delta, \tag{2.2}$$

where d_i $(i=1,\ldots,s),\lambda_j(j=1,\ldots,m),\delta$, are constants to be determined later. Substituting (2.2) into Eq.(2.1) yields an ordinary differential equation (ODE)

$$H\left(U_{n+p_1},\ldots,U_{n+p_l},U'_{n+p_1},\ldots,U'_{n+p_l},\ldots,U^{(k)}_{n+p_1},\ldots,U^{(k)}_{n+p_l}\right)=0. \tag{2.3}$$

Step 1 We seek the solutions of Eq.(2.3) in the form

$$U_{i,n}(\xi_n)=a_0+\sum_{j=1}^{n_i}\frac{a_{ij}\mathrm{sech}^j(\xi_n)}{(\mu_1\tanh(\xi_n)+\mu_2)^j}+\sum_{j=1}^{n_i}\frac{b_{ij}\tanh^j(\xi_n)}{(\mu_1\tanh(\xi_n)+\mu_2)^j}. \tag{2.4}$$

From the properties of hyperbolic function, we have

$$U_{i,n+r}(\xi_{n+r})=a_0+\sum_{j=1}^{n_i}+\frac{a_{ij}\mathrm{sech}^j(\xi_n)}{[\mu_1\tanh(\xi_n)\cosh(rd)+\mu_1\sinh(rd)+\mu_2\cosh(rd)+\mu_2\tanh(\xi_n)\sinh(rd)]^j}+$$
$$\sum_{j=1}^{n_i}\frac{b_{ij}\left[\tanh(\xi_n)\cosh(rd)+\sinh(rd)\right]^j}{[\mu_1\tanh(\xi_n)\cosh(rd)+\mu_1\sinh(rd)+\mu_2\cosh(rd)+\mu_2\tanh(\xi_n)\sinh(rd)]^j}, \tag{2.5}$$

where r is an arbitrary integer, $a_{ij},b_{ij},d_i,\lambda_j,\mu_1,\mu_2$ are constants to be determined later, and n_i can be determined by homogeneous balance principle.

Step 2 Substituting (2.4) and (2.5) into Eq.(2.3) and then setting all the coefficients of $\mathrm{sech}^i(\xi_n)\tanh^j(\xi_n)$ $(i=0,1,j=0,1,\ldots)$ to be zero yields a set of algebraic equations with respect to a_0, a_{ij},b_{ij},c_{ij},d_i, λ_j,μ_1,μ_2. The algebraic equations are too tedious, so we omit it.

Step 3 With the help of Maple, we solve the over-determined nonlinear algebraic equations for a_0, $a_{ij},b_{ij},d_i,\lambda_j,\mu_1,\mu_2$.

Step 4 Substituting the obtained conclusions in Step 3 into Eq.(2.4) gives the explicit and exact travelling solutions of Eq.(2.1).

3. Exact solutions of the discrete mKdV lattice

We consider the discrete mKdV lattice[14,15]

$$\dot{u}_n(t) = \left(\alpha - u_n^2(t)\right)\left(u_{n+1}(t) - u_{n-1}(t)\right), \tag{3.1}$$

where α is a constant. According to the above method, we seek more travelling wave solutions of Eq.(3.1).

We make the following travelling transformation

$$u_n = U_n(\xi_n), \quad \xi_n = dn + \lambda t + \delta, \tag{3.2}$$

where d, λ, δ are constants to be determined later, and thus Eq.(3.1) becomes

$$\lambda U_n'(\xi_n) - \left(\alpha - U_n^2(\xi_n)\right)\left(U_{n+1}(\xi_{n+1}) - U_{n-1}(\xi_{n-1})\right) = 0. \tag{3.3}$$

By the homogeneous balance principle, we may assume the solutions of Eq.(3.3) in the form

$$U_n = a_0 + \frac{b_1 \mathrm{sech}(\xi_n) + b_2 \tanh(\xi_n)}{\mu_1 \tanh(\xi_n) + \mu_2}, \tag{3.4a}$$

$$U_{n+1} = a_0 + \frac{b_1 \mathrm{sech}(\xi_n) + b_2 \left[\tanh(\xi_n)\cosh(d) + \sinh(d)\right]}{\mu_1 \tanh(\xi_n)\cosh(d) + \mu_1 \sinh(d) + \mu_2 \cosh(d) + \mu_2 \tanh(\xi_n)\sinh(d)}, \tag{3.4b}$$

$$U_{n-1} = a_0 + \frac{b_1 \mathrm{sech}(\xi_n) + b_2 \left[\tanh(\xi_n)\cosh(d) + \sinh(d)\right]}{\mu_1 \tanh(\xi_n)\cosh(d) - \mu_1 \sinh(d) + \mu_2 \cosh(d) - \mu_2 \tanh(\xi_n)\sinh(d)}. \tag{3.4c}$$

Substituting (3.4) into Eq.(3.3), collecting coefficients of $\mathrm{sech}^i(\xi_n)\tanh^j(\xi_n)$ $(i = 0, 1, j = 0, 1, 2 \ldots)$, and setting them to be zero, we get a set of over-determined algebraic equations with respect to a_0, b_1, $b_2, \lambda, d, \mu_1, \mu_2$.

By use of the Maple, solving the over-determined algebraic equations, we get the following results:

Case 1

$$b_1 = 0,\ a_0 = -\frac{\sqrt{\alpha}\mu_1 \tanh(d)}{\mu_2},\ \lambda = 2\,\alpha\,\tanh(d),\ b_2 = \frac{\sqrt{\alpha}\tanh(d)\left({\mu_1}^2 - {\mu_2}^2\right)}{\mu_2},$$

where μ_2, μ_1, d are arbitrary constants;

Case 2

$$b_2 = a_0 = \mu_1 = 0,\ \lambda = 2\,\alpha\,\sinh(d),\ b_1 = \sqrt{-\alpha}\mu_2 \sinh(d),$$

where μ_2, d are arbitrary constants;

Case 3

$$a_1 = a_2 = \mu_2 = 0,\ b_2 = -a_0\mu_1,\ b_1 = \sqrt{\alpha}\mu_1 \sinh(d),\ \lambda = 2\,\alpha\,\sinh(d),$$

where a_0, μ_1, d are arbitrary constants;

Case 4

$$b_1 = a_0 = \mu_1 = 0,\ \mu_2 = -\frac{b_2 \coth(d)}{\sqrt{\alpha}},\ \lambda = 2\,\alpha\,\tanh(d)\,,$$

where b_2, d are arbitrary constants;

Case 5

$$b_2 = a_0 = 0,\ \mu_1 = i\sqrt{3}\mu_2,\ \lambda = 2\,\alpha\,\sinh(d)\,,\ b_1 = 2\mu_2\sqrt{-\alpha}\sinh(d)\,,$$

where d, μ_2 are arbitrary constants;

Case 6

$$b_2 = a_0 = 0,\ \lambda = 2\,\alpha\,\sinh(d)\,,\ b_1 = \sqrt{\alpha\,({\mu_1}^2 - {\mu_2}^2)}\sinh(d)\,,$$

where μ_2, μ_1, d are arbitrary constants;

Case 7

$$a_0 = \mu_1 = 0,\ b_1 = ib_2,\ \lambda = 4\,\alpha\,\tanh\left(\frac{d}{2}\right),\ \mu_2 = \frac{b_2 \coth\left(\frac{d}{2}\right)}{\sqrt{\alpha}},$$

where b_2, d are arbitrary constants;

Case 8

$$\mu_1 = i\sqrt{3}\mu_2,\ b_2 = -4\,i\mu_2\sqrt{-\alpha}\tanh\left(\frac{d}{2}\right),\ a_0 = \sqrt{-3\alpha}\tanh\left(\frac{d}{2}\right),$$

$$b_1 = 2\,\mu_2\sqrt{-\alpha}\tanh\left(\frac{d}{2}\right),\ \lambda = 4\,\alpha\,\tanh\left(\frac{d}{2}\right),$$

where μ_2, d are arbitrary constants;

Case 9

$$d = \frac{i\pi}{2},\ \mu_1 = i\sqrt{3}\mu_2,\ \lambda = 4\,i\alpha,\ a_0 = \sqrt{3\alpha},\ b_2 = -4\,i\sqrt{\alpha}\mu_2,\ b_1 = 2\,\sqrt{\alpha}\mu_2,$$

where μ_2 is arbitrary constant;

Case 10

$$\lambda = 4\,\alpha\,\tanh\left(\frac{d}{2}\right),\ a_0 = -\frac{\sqrt{\alpha}\mu_1 \tanh\left(\frac{d}{2}\right)}{\mu_2},$$

$$b_1 = \sqrt{\alpha\,({\mu_1}^2 - {\mu_2}^2)}\tanh\left(\frac{d}{2}\right),\ b_2 = \frac{\sqrt{\alpha}\tanh\left(\frac{d}{2}\right)\left({\mu_1}^2 - {\mu_2}^2\right)}{\mu_2},$$

where μ_2, μ_1, d, are arbitrary constants;

Case 11

$$\lambda = 4\,i\alpha\,\sqrt{3},\ b_1 = \sqrt{3\alpha\,({\mu_2}^2 - {\mu_1}^2)},$$

$$b_2 = \frac{\sqrt{-3\alpha}\left({\mu_1}^2 - {\mu_2}^2\right)}{\mu_2},\ a_0 = -\frac{\sqrt{-3\alpha}\mu_1}{\mu_2},d = \frac{2\,i\pi}{3},$$

where μ_2 and μ_1 are arbitrary constants;

Substituting Case 1 $\sim$ Case 11 into (3.4a), we can obtain the following exact solutions of Eq.(3.1)

$$u(n,t)_1 = -\frac{\sqrt{\alpha}\mu_1 \tanh(d)}{\mu_2} + \frac{\sqrt{\alpha}\tanh(d)\left({\mu_1}^2 - {\mu_2}^2\right)\tanh(dn + 2\,\alpha\, \tanh(d)\,t + \delta)}{\mu_2\left(\mu_1 \tanh(dn + 2\,\alpha\,\tanh(d)\,t + \delta) + \mu_2\right)},$$

where μ_2, μ_1, d, δ are arbitrary constants;

$$u(n,t)_2 = \sqrt{-\alpha}\sinh(d)\,\mathrm{sech}\,(dn + 2\,\alpha\,\sinh(d)\,t + \delta)\,,$$

where μ_2, d, δ are arbitrary constants;

$$u(n,t)_3 = a_0 + \sqrt{\alpha}\sinh(d)\,\mathrm{csch}\,(dn + 2\,\alpha\sinh(d)\,t + \delta)\,,$$

where a_0, μ_1, d, δ are arbitrary constants;

$$u(n,t)_4 = -\sqrt{\alpha}\tanh(d)\tanh(dn + 2\,\alpha\,\tanh(d)\,t + \delta)\,,$$

where d, δ are arbitrary constants;

$$u(n,t)_5 = \frac{2\sqrt{-\alpha}\sinh(d)\,\mathrm{sech}\,(dn + 2\,\alpha\,\sinh(d)\,t + \delta)}{i\sqrt{3}\tanh(dn + 2\,\alpha\,\sinh(d)\,t + \delta) + 1},$$

where d, δ are arbitrary constants;

$$u(n,t)_6 = \frac{\sqrt{\alpha\,\left({\mu_1}^2 - {\mu_2}^2\right)}\sinh(d)\,\mathrm{sech}\,(dn + 2\,\alpha\sinh(d)\,t + \delta)}{\mu_1\tanh(dn + 2\,\alpha\sinh(d)\,t + \delta) + \mu_2},$$

where μ_2, μ_1, d, δ are arbitrary constants;

$$u(n,t)_7 = \sqrt{\alpha}\tanh\left(\frac{d}{2}\right)\left(i\mathrm{sech}\left(dn + 4\,\alpha\tanh\left(\frac{d}{2}\right)t + \delta\right) + \tanh\left(dn + 4\,\alpha\tanh\left(\frac{d}{2}\right)t + \delta\right)\right),$$

where d, δ are arbitrary constants;

$$u(n,t)_8 = \sqrt{-3\alpha}\tanh\left(\frac{d}{2}\right) + \frac{2\sqrt{-\alpha}\tanh\left(\frac{d}{2}\right)\mathrm{sech}\left(dn + 4\,\alpha\,\tanh\left(\frac{d}{2}\right)t + \delta\right)}{i\sqrt{3}\tanh\left(dn + 4\,\alpha\,\tanh\left(\frac{d}{2}\right)t + \delta\right) + 1} - \frac{4\,i\sqrt{-\alpha}\tanh\left(\frac{d}{2}\right)\tanh\left(dn + 4\,\alpha\,\tanh\left(\frac{d}{2}\right)t + \delta\right)}{i\sqrt{3}\tanh\left(dn + 4\,\alpha\,\tanh\left(\frac{d}{2}\right)t + \delta\right) + 1},$$

where d, δ are arbitrary constants;

$$u(n,t)_9 = \sqrt{3\alpha} + \frac{2\sqrt{\alpha}\mathrm{sech}\left(\frac{i\pi n}{2} + 4\,i\alpha t + \delta\right)}{i\sqrt{3}\tanh\left(\frac{i\pi n}{2} + 4\,i\alpha t + \delta\right) + 1} - \frac{4\,i\sqrt{\alpha}\tanh\left(\frac{i\pi n}{2} + 4\,i\alpha t + \delta\right)}{i\sqrt{3}\tanh\left(\frac{i\pi n}{2} + 4\,i\alpha t + \delta\right) + 1},$$

where δ is arbitrary constant;

$$u(n,t)_{10} = -\frac{\sqrt{\alpha}\mu_1\tanh\left(\frac{d}{2}\right)}{\mu_2} + \frac{\sqrt{\alpha\,\left({\mu_1}^2 - {\mu_2}^2\right)}\tanh\left(\frac{d}{2}\right)\mathrm{sech}\left(dn + 4\,\alpha\,\tanh\left(\frac{d}{2}\right)t + \delta\right)}{\mu_1\tanh\left(dn + 4\,\alpha\,\tanh\left(\frac{d}{2}\right)t + \delta\right) + \mu_2} + \frac{\sqrt{\alpha}\tanh\left(\frac{d}{2}\right)\left({\mu_1}^2 - {\mu_2}^2\right)\tanh\left(dn + 4\,\alpha\,\tanh\left(\frac{d}{2}\right)t + \delta\right)}{\mu_2\left(\mu_1\tanh\left(dn + 4\,\alpha\,\tanh\left(\frac{d}{2}\right)t + \delta\right) + \mu_2\right)},$$

where $\mu_2, \mu_1, d, , \delta$ are arbitrary constants;

$$u(n,t)_{11} = -\frac{\sqrt{-3\alpha}\mu_1}{\mu_2} + \frac{\sqrt{3\alpha\,\left({\mu_2}^2 - {\mu_1}^2\right)}\mathrm{sech}\left(\frac{2\,i\pi}{3}n + 4\,i\alpha\,\sqrt{3}t + \delta\right)}{\mu_1\tanh\left(\frac{2\,i\pi}{3}n + 4\,i\alpha\,\sqrt{3}t + \delta\right) + \mu_2} +$$

$$\frac{\sqrt{-3\alpha}\left({\mu_1}^2-{\mu_2}^2\right)\tanh\left(\frac{2\,i\pi}{3}n+4\,i\alpha\sqrt{3}t+\delta\right)}{\mu_2\left(\mu_1\tanh\left(\frac{2\,i\pi}{3}n+4\,i\alpha\sqrt{3}t+\delta\right)+\mu_2\right)},$$

where μ_2, μ_1, δ are arbitrary constants;

Remark 1 The solutions $u(n,t)_1$ and $u(n,t)_4$ are kink-shaped solitary wave solution, solutions $u(n,t)_2, u(n,t)_5$ and $u(n,t)_6$ are bell-shaped solitary wave solutions, and $u(n,t)_3$ is singular solitary wave solution.

5. Conclusions

In this paper, we have derived many exact travelling wave solutions for the discrete mKdV lattice based upon the new hyperbolic function rational expansion method. These new travelling exact solution may be of great significance to explain some physical phenomenon. The paper shows that the method is sufficient to seek more new exact solitary wave solutions of DDEs. We need to find more general ansätzes for the further work about various extension and improved hyperbolic function method.

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