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A Reverse Hilbert's Type Inequality with Multi-Parameters

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Abstract By introducing some parameters and estimating the weight function, we obtain a reverse Hilbert's type inequality with the best constant factor. As its applications, we build its equivalent form and some particular results.

Keywords Hilbert's type inequality; Hölder's inequality; weight function.

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1. Introduction

If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\{a_n\}$, $\{b_n\}\geq 0$, such that $0<\sum_{n=1}^{\infty}a_n^p<\infty$ and $0<\sum_{n=1}^{\infty}b_n^q<\infty$, then we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\max\{m,n\}}<pq\{\sum_{n=1}^{\infty}a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}b_n^q\}^{\frac{1}{q}}, \tag{1.1}$$

where the constant factor pq is the best possible. Inequality (1.1) is Hilbert's type inequality[1,Th.342], and is important in analysis and its applications[2]. Recently, Kuang[3] gave a strengthened version of (1.1); Yang[4,5] considered a refinement of another Hilbert's type inequality.

In 2004, by introducing a parameter λ $(2-\min\{p,q\}<\lambda\leq 2)$, Yang[6,7] gave two generalizations of (1.1) and the extended equivalent forms as follows:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\max\{m^\lambda,n^\lambda\}}<k_\lambda(p)\big[\sum_{n=1}^{\infty}n^{(p-1)(2-\lambda)-1}a_n^p\big]^{1/p}\big[\sum_{n=1}^{\infty}n^{(q-1)(2-\lambda)-1}b_n^q\big]^{1/q}; \tag{1.2}$$

$$\sum_{n=1}^{\infty}n^{p+\lambda-3}\big[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\big]^p<\big[k_\lambda(p)\big]^p\sum_{n=1}^{\infty}n^{(p-1)(2-\lambda)-1}a_n^p; \tag{1.3}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\max\{m^\lambda,n^\lambda\}}<k_\lambda(p)\big[\sum_{n=1}^{\infty}n^{1-\lambda}a_n^p\big]^{1/p}\big[\sum_{n=1}^{\infty}n^{1-\lambda}b_n^q\big]^{1/q}; \tag{1.4}$$

$$\sum_{n=1}^{\infty}n^{(p-1)(\lambda-1)}\big[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\big]^p<\big[k_\lambda(p)\big]^p\sum_{n=1}^{\infty}n^{1-\lambda}a_n^p, \tag{1.5}$$

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where the constant factors $k_\lambda(p)=\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}$ and $\big[k_\lambda(p)\big]^p$ are all the best possible. Inequality (1.2) is equivalent to (1.3); (1.4) is equivalent to (1.5).

In 2005, Yang[8] built a reverse Hilbert's type inequality and its equivalent form as:

If $0<p<1, \frac{1}{p}+\frac{1}{q}=1, A=\{(\lambda,\alpha):\lambda,\alpha>0, 0<\phi_r\le 1(r=p,q), \phi_p+\phi_q=\lambda\alpha\}\ne\Phi$, and $a_n, b_n\ge 0$ satisfy $0<\sum_{n=1}^\infty n^{p(1-\phi_q)-1}a_n^p<\infty$ and $0<\sum_{n=1}^\infty n^{q(1-\phi_p)-1}b_n^q<\infty$, then for (λ,α), we have

$$\sum_{n=1}^\infty\sum_{n=1}^\infty\frac{a_m b_n}{(m^\alpha+n^\alpha)^\lambda}>\frac{1}{\alpha}B(\frac{\phi_p}{\alpha},\frac{\phi_q}{\alpha})\Big\{\sum_{n=1}^\infty[1-\theta_p(n)]n^{p(1-\phi_q)-1}a_n^p\Big\}^{\frac{1}{p}}\Big\{\sum_{n=1}^\infty n^{q(1-\phi_p)-1}b_n^q\Big\}^{\frac{1}{q}}; \quad (1.6)$$

$$\sum_{n=1}^\infty n^{p\phi_p-1}\Big[\sum_{n=1}^\infty\frac{a_m}{(m^\alpha+n^\alpha)^\lambda}\Big]^p>\Big[\frac{1}{\alpha}B(\frac{\phi_p}{\alpha},\frac{\phi_q}{\alpha})\Big]^p\sum_{n=1}^\infty[1-\theta_p(n)]n^{p(1-\phi_q)-1}a_n^p, \tag{1.7}$$

where $0<\theta_p(n)=O(\frac{1}{n^{\phi_p}})<1$, and the constant factors $\frac{1}{\alpha}B(\frac{\phi_p}{\alpha},\frac{\phi_q}{\alpha})$ and $\Big[\frac{1}{\alpha}B(\frac{\phi_p}{\alpha},\frac{\phi_q}{\alpha})\Big]^p$ are the best possible. Inequality (1.6) is equivalent to (1.7).

This paper gives another reverse Hilbert's type inequality as (1.6). As applications, we also consider the equivalent form and some particular results.

2. Some Lemmas

Lemma 2.1 *Suppose* $0<p<1, \frac{1}{p}+\frac{1}{q}=1, \lambda>0, 0<\phi_p\le 1, 0<\psi_q\le 1$ *and* $\phi_p+\psi_q=\lambda$. *Define the weight functions* $\omega_\lambda(p,n)$ *as:*

$$\omega_\lambda(p,n)=n^{\phi_p}\sum_{m=1}^\infty\frac{1}{\max\{m^\lambda,n^\lambda\}}(\frac{1}{m})^{1-\psi_q},\quad n\in N. \tag{2.1}$$

Then we have

$$\frac{\lambda}{\phi_p\psi_q}\Big[1-\theta_p(n)\Big]<\omega_\lambda(p,n)<\frac{\lambda}{\phi_p\psi_q}, \tag{2.2}$$

where $0<\theta_p(n)=\frac{\phi_p}{\lambda n^{\psi_q}}<1$.

Proof Since $0<\psi_q\le 1$, we have

$$\begin{aligned}\omega_\lambda(p,n)&<n^{\phi_p}\int_0^\infty\frac{1}{\max\{y^\lambda,n^\lambda\}}(\frac{1}{y})^{1-\psi_q}\mathrm{d}y\\&=n^{\phi_p}\Big[\int_0^n\frac{1}{n^\lambda}(\frac{1}{y})^{1-\psi_q}dy+\int_n^\infty\frac{1}{y^\lambda}(\frac{1}{y})^{1-\psi_q}\mathrm{d}y\Big]\\&=\frac{1}{\psi_q}+\frac{1}{\phi_p}=\frac{\lambda}{\phi_p\psi_q}.\end{aligned}$$

Define $\theta_p(n)$ as

$$\theta_p(n)=\frac{\phi_p\psi_q n^{\phi_p}}{\lambda}\int_0^1\frac{1}{\max\{y^\lambda,n^\lambda\}}(\frac{1}{y})^{1-\psi_q}\mathrm{d}y,\quad n\in N. \tag{2.3}$$

Since for $n\ge 1$,

$$\int_0^1\frac{1}{\max\{y^\lambda,n^\lambda\}}(\frac{1}{y})^{1-\psi_q}\mathrm{d}y=\int_0^1\frac{1}{n^\lambda}(\frac{1}{y})^{1-\psi_q}\mathrm{d}y=\frac{1}{\psi_q n^\lambda},$$

by (2.3), we have $\theta_p(n)=\frac{\phi_p}{\lambda n^{\psi_q}}$.

Hence for $0<\psi_q\le 1$, we have

$$\begin{aligned}\omega_\lambda(p,n) &> n^{\phi_p}\int_1^\infty \frac{1}{\max\{y^\lambda,n^\lambda\}}(\frac{1}{y})^{1-\psi_q}\mathrm{d}y\\ &= n^{\phi_p}\Big[\int_0^\infty \frac{1}{\max\{y^\lambda,n^\lambda\}}(\frac{1}{y})^{1-\psi_q}\mathrm{d}y-\int_0^1 \frac{1}{\max\{y^\lambda,n^\lambda\}}(\frac{1}{y})^{1-\psi_q}\mathrm{d}y\Big]\\ &= \frac{\lambda}{\phi_p\psi_q}\Big[1-\theta_p(n)\Big].\end{aligned}$$

Inequality (2.2) is valid. The lemma is proved. □

Lemma 2.2 *If* $0<p<1,\frac{1}{p}+\frac{1}{q}=1,\lambda>0,0<\phi_p\le 1,0<\psi_q\le 1,\phi_p+\psi_q=\lambda$ *and* $0<\varepsilon<p\psi_q$, *then we have*

$$I:=\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{n^{-1+\phi_p-\frac{\varepsilon}{q}}}{\max\{m^\lambda,n^\lambda\}}m^{-1+\psi_q-\frac{\varepsilon}{p}}<\Big[\frac{1}{\psi_q-\frac{\varepsilon}{p}}+\frac{1}{\phi_p+\frac{\varepsilon}{p}}\Big]\sum_{n=1}^\infty\frac{1}{n^{1+\varepsilon}}. \tag{2.4}$$

Proof For $0<p<1$, we obtain

$$\begin{aligned}I &< \sum_{n=1}^\infty\int_0^\infty \frac{n^{-1+\phi_p-\frac{\varepsilon}{q}}}{\max\{y^\lambda,n^\lambda\}}y^{-1+\psi_q-\frac{\varepsilon}{p}}\mathrm{d}y\\ &= \sum_{n=1}^\infty\Big[n^{-1+\phi_p-\frac{\varepsilon}{q}}\Big(\int_0^n\frac{y^{-1+\psi_q-\frac{\varepsilon}{p}}}{n^\lambda}\mathrm{d}y+\int_n^\infty\frac{y^{-1+\psi_q-\frac{\varepsilon}{p}}}{y^\lambda}\mathrm{d}y\Big)\Big]\\ &= \Big[\frac{1}{\psi_q-\frac{\varepsilon}{p}}+\frac{1}{\phi_p+\frac{\varepsilon}{p}}\Big]\sum_{n=1}^\infty\frac{1}{n^{1+\varepsilon}}.\end{aligned}$$

Hence, (2.4) is valid. The lemma is proved. □

3. Main results and applications

Theorem 3.1 *If* $0<p<1,\frac{1}{p}+\frac{1}{q}=1,\lambda>0,0<\phi_p\le 1,0<\psi_q\le 1,\phi_p+\psi_q=\lambda$ *and* $a_n,b_n\ge 0$ *satisfy* $0<\sum_{n=1}^\infty n^{p(1-\psi_q)-1}a_n^p<\infty$ *and* $0<\sum_{n=1}^\infty n^{q(1-\phi_p)-1}b_n^q<\infty$, *then we have*

$$\sum_{n=1}^\infty\sum_{m=1}^\infty\frac{a_mb_n}{\max\{m^\lambda,n^\lambda\}}>\frac{\lambda}{\phi_p\psi_q}\Big\{\sum_{n=1}^\infty[1-\theta_p(n)]n^{p(1-\psi_q)-1}a_n^p\Big\}^{\frac{1}{p}}\Big\{\sum_{n=1}^\infty n^{q(1-\phi_p)-1}b_n^q\Big\}^{\frac{1}{q}}, \tag{3.1}$$

where $0<\theta_p(n)=\frac{\phi_p}{\lambda n^{\psi_q}}<1$ *and the constant factor* $\frac{\lambda}{\phi_p\psi_q}$ *is the best possible. In particular,*

(a) for $\phi_p=\psi_q=\frac{\lambda}{2}$, *we have*

$$\sum_{n=1}^\infty\sum_{m=1}^\infty\frac{a_mb_n}{\max\{m^\lambda,n^\lambda\}}>\frac{4}{\lambda}\Big\{\sum_{n=1}^\infty[1-\frac{1}{2n^{\frac{\lambda}{2}}}]n^{p(1-\frac{\lambda}{2})-1}a_n^p\Big\}^{\frac{1}{p}}\Big\{\sum_{n=1}^\infty n^{q(1-\frac{\lambda}{2})-1}b_n^q\Big\}^{\frac{1}{q}}; \tag{3.2}$$

(b) for $\lambda=1$, *we have*

$$\sum_{n=1}^\infty\sum_{m=1}^\infty\frac{a_mb_n}{\max\{m,n\}}>\frac{1}{\phi_p\psi_q}\Big\{\sum_{n=1}^\infty[1-\frac{\phi_p}{n^{\psi_q}}]n^{p\phi_p-1}a_n^p\Big\}^{\frac{1}{p}}\Big\{\sum_{n=1}^\infty n^{q\psi_q-1}b_n^q\Big\}^{\frac{1}{q}}. \tag{3.3}$$

Proof By the reverse Hölder's inequality[9], since $0<p<1$ and $q<0$, we have

$$H(a_m,b_n):=\sum_{n=1}^\infty\sum_{m=1}^\infty\frac{a_mb_n}{\max\{m^\lambda,n^\lambda\}}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{\max\{m^\lambda, n^\lambda\}}\Big[\frac{m^{(1-\psi_q)/q}}{n^{(1-\phi_p)/p}}a_m\Big]\Big[\frac{n^{(1-\phi_p)/p}}{m^{(1-\psi_q)/q}}b_n\Big] \\
&\geq \Big\{\sum_{m=1}^{\infty}\Big[\sum_{n=1}^{\infty}\frac{m^{\psi_q}}{\max\{m^\lambda, n^\lambda\}}\cdot\frac{1}{n^{1-\phi_p}}\Big]m^{p(1-\psi_q)-1}a_m^p\Big\}^{\frac{1}{p}} \times \\
&\quad \Big\{\sum_{n=1}^{\infty}\Big[\sum_{m=1}^{\infty}\frac{n^{\phi_p}}{\max\{m^\lambda, n^\lambda\}}\cdot\frac{1}{m^{1-\psi_q}}\Big]n^{q(1-\phi_p)-1}b_n^q\Big\}^{\frac{1}{q}}. \qquad (3.4)
\end{aligned}
$$

Since $\lambda > 0$, and $1-\phi_p \geq 0$, $1-\psi_q \geq 0$, in view of (2.1) , we rewrite (3.4) as:

$$
H(a_m, b_n) > \Big\{\sum_{n=1}^{\infty}\omega_\lambda(p,n)n^{p(1-\psi_q)-1}a_n^p\Big\}^{\frac{1}{p}}\Big\{\sum_{n=1}^{\infty}\omega_\lambda(q,n)n^{q(1-\phi_p)-1}b_n^q\Big\}^{\frac{1}{q}},
$$

and then by (2.2), we have (3.1).

For $0 < \varepsilon < p\psi_q$, set a_n' and b_n' as: $a_n' = n^{-1+\psi_q-\frac{\varepsilon}{p}}$, $b_n' = n^{-1+\phi_p-\frac{\varepsilon}{q}}$, $n \in N$. Since $\phi_p > 0$, we have

$$
\begin{aligned}
&\Big\{\sum_{n=1}^{\infty}[1-\theta_p(n)]n^{p(1-\psi_q)-1}a_n'^p\Big\}^{\frac{1}{p}}\Big\{\sum_{n=1}^{\infty}n^{q(1-\phi_p)-1}b_n'^q\Big\}^{\frac{1}{q}} \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\Big\{1-\frac{\sum_{n=1}^{\infty}\frac{\phi_p}{n^{\psi_q+1+\varepsilon}}}{\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}}\Big\}^{\frac{1}{p}} \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}(1-o(1))^{\frac{1}{p}}(\varepsilon \to 0^+). \qquad (3.5)
\end{aligned}
$$

If the constant factor $\frac{\lambda}{\phi_p\psi_q}$ in (3.1) is not the best possible, then there exists a positive number K (with $K > \frac{\lambda}{\phi_p\psi_q}$), such that (3.1) is still valid if we replace $\frac{\lambda}{\phi_p\psi_q}$ by K. In particular, by (3.5) and (2.4), we have

$$
\begin{aligned}
&K\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\{1-o(1)\}^{\frac{1}{p}} = K\Big\{\sum_{n=1}^{\infty}[1-\theta_p(n)]n^{p(1-\psi_q)-1}a_n'^p\Big\}^{\frac{1}{p}}\Big\{\sum_{n=1}^{\infty}n^{q(1-\phi_p)-1}b_n'^q\Big\}^{\frac{1}{q}} \\
&< I < \Big[\frac{1}{\psi_q-\frac{\varepsilon}{p}}+\frac{1}{\phi_p+\frac{\varepsilon}{p}}\Big]\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}.
\end{aligned}
$$

Hence

$$
K\Big\{1-o(1)\Big\}^{\frac{1}{p}} < \Big[\frac{1}{\psi_q-\frac{\varepsilon}{p}}+\frac{1}{\phi_p+\frac{\varepsilon}{p}}\Big],
$$

and then $K \leq \frac{\lambda}{\phi_p\psi_q}$ $(\varepsilon \to 0^+)$. By this contradiction we can conclude that the constant $\frac{\lambda}{\phi_p\psi_q}$ in (3.1) is the best possible. Thus the theorem is proved. □

Theorem 3.2 *If $0 < p < 1, \frac{1}{p}+\frac{1}{q} = 1, \lambda > 0, 0 < \phi_p \leq 1, 0 < \psi_q \leq 1, \phi_p+\psi_q = \lambda$ and $a_n \geq 0$ satisfies $0 < \sum_{n=1}^{\infty}n^{p(1-\psi_q)-1}a_n^p < \infty$, then we have*

$$
\sum_{n=1}^{\infty}n^{p\phi_p-1}\Big[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda, n^\lambda\}}\Big]^p > \Big(\frac{\lambda}{\phi_p\psi_q}\Big)^p\sum_{n=1}^{\infty}\Big[1-\theta_p(n)\Big]n^{p(1-\psi_q)-1}a_n^p, \qquad (3.6)
$$

where $0 < \theta_p(n) = \frac{\phi_p}{\lambda n^{\psi_q}} < 1$, and the constant factor $\Big(\frac{\lambda}{\phi_p\psi_q}\Big)^p$ is the best possible. Inequality (3.6) is equivalent to (3.1). In particular,

(a) *for* $\phi_p = \psi_q = \frac{\lambda}{2}$, *we have*

$$\sum_{n=1}^{\infty} n^{\frac{p\lambda}{2}-1}\Big[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\Big]^p > \Big(\frac{4}{\lambda}\Big)^p \sum_{n=1}^{\infty}\Big[1-\frac{1}{2n^{\frac{\lambda}{2}}}\Big]n^{p(1-\frac{\lambda}{2})-1}a_n^p; \tag{3.7}$$

(b) *for* $\lambda = 1$, *we have*

$$\sum_{n=1}^{\infty} n^{p\phi_p-1}\Big[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m,n\}}\Big]^p > \Big(\frac{1}{\phi_p\psi_q}\Big)^p \sum_{n=1}^{\infty}\Big[1-\frac{\phi_p}{n^{\psi_q}}\Big]n^{p\phi_p-1}a_n^p. \tag{3.8}$$

Proof Setting

$$b_n = n^{p\phi_p-1}\Big[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\Big]^{p-1},$$

by (3.1), we have

$$\begin{aligned}\Big[\sum_{n=1}^{\infty} n^{q(1-\phi_p)-1}b_n^q\Big]^p &= \Big\{\sum_{n=1}^{\infty} n^{p\phi_p-1}\Big[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\Big]^p\Big\}^p \\ &= \Big[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}}\Big]^p \\ &\geq \Big(\frac{\lambda}{\phi_p\psi_q}\Big)^p\Big\{\sum_{n=1}^{\infty}[1-\theta_p(n)]n^{p(1-\psi_q)-1}a_n^p\Big\}\Big\{\sum_{n=1}^{\infty} n^{q(1-\phi_p)-1}b_n^q\Big\}^{p-1};\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} n^{q(1-\phi_p)-1}b_n^q &= \sum_{n=1}^{\infty} n^{p\phi_p-1}\Big[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\Big]^p \\ &\geq \Big(\frac{\lambda}{\phi_p\psi_q}\Big)^p\sum_{n=1}^{\infty}[1-\theta_p(n)]n^{p(1-\psi_q)-1}a_n^p > 0.\end{aligned} \tag{3.10}$$

If $\sum_{n=1}^{\infty} n^{q(1-\phi_p)-1}b_n^q = \infty$, since

$$\sum_{n=1}^{\infty}[1-\theta_n(p)]n^{p(1-\psi_q)-1}a_n^p < \sum_{n=1}^{\infty} n^{p(1-\psi_q)-1}a_n^p < \infty,$$

(3.10) takes the form of strict inequality, so does (3.9). If $0 < \sum_{n=1}^{\infty} n^{q(1-\phi_p)-1}b_n^q < \infty$, by using (3.1), (3.9) takes the form of strict inequality, so does (3.10). Hence we have (3.6).

On the other hand, if (3.6) is valid, by the reverse Hölder's inequality, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}} &= \sum_{n=1}^{\infty}\Big[\sum_{m=1}^{\infty}\frac{n^{\frac{1}{q}-1+\phi_p}a_m}{\max\{m^\lambda,n^\lambda\}}\Big]\Big[n^{1-\phi_p-\frac{1}{q}}b_n\Big] \\ &\geq \Big\{\sum_{n=1}^{\infty} n^{p\phi_p-1}\Big[\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^\lambda,n^\lambda\}}\Big]^p\Big\}^{\frac{1}{p}}\Big\{\sum_{n=1}^{\infty} n^{q(1-\phi_p)-1}b_n^q\Big\}^{\frac{1}{q}}.\end{aligned} \tag{3.11}$$

By (3.6), we have (3.1). Hence, inequalities (3.1) and (3.6) are equivalent. If the constant factor in (3.6) is not the best possible, we can conclude that the constant factor in (3.1) is not the best possible by (3.11). The theorem is proved. □

Theorem 3.3 *If* $0 < p < 1, \frac{1}{p}+\frac{1}{q} = 1, \lambda > 0, 0 < \phi_p \leq 1, 0 < \psi_q \leq 1, \phi_p + \psi_q = \lambda$ *and* $b_n \geq 0$

satisfies $0<\sum_{n=1}^{\infty}n^{q(1-\phi_p)-1}b_n^q<\infty$, *then we have*

$$\sum_{m=1}^{\infty}[1-\theta_p(m)]^{1-q}m^{q\psi_q-1}\Big[\sum_{n=1}^{\infty}\frac{b_n}{\max\{m^\lambda,n^\lambda\}}\Big]^q<\Big(\frac{\lambda}{\phi_p\psi_q}\Big)^q\sum_{n=1}^{\infty}n^{q(1-\phi_p)-1}b_n^q, \tag{3.12}$$

where $0<\theta_p(m)=\frac{\phi_p}{\lambda m^{\psi_q}}<1$ *and the constant factor* $\Big(\frac{\lambda}{\phi_p\psi_q}\Big)^q$ *is the best possible. Inequality (3.12) is equivalent to (3.1). In particular,*

(a) for $\phi_p=\psi_q=\frac{\lambda}{2}$, *we have*

$$\sum_{m=1}^{\infty}[1-\frac{1}{2m^{\frac{\lambda}{2}}}]^{1-q}m^{\frac{q\lambda}{2}-1}\Big[\sum_{n=1}^{\infty}\frac{b_n}{\max\{m^\lambda,n^\lambda\}}\Big]^q<\Big(\frac{4}{\lambda}\Big)^q\sum_{n=1}^{\infty}n^{q(1-\frac{\lambda}{2})-1}b_n^q; \tag{3.13}$$

(b) for $\lambda=1$

$$\sum_{m=1}^{\infty}[1-\frac{\phi_p}{m^{\psi_q}}]^{1-q}m^{q\psi_q-1}\Big[\sum_{n=1}^{\infty}\frac{b_n}{\max\{m,n\}}\Big]^q<\Big(\frac{1}{\phi_p\psi_q}\Big)^q\sum_{n=1}^{\infty}n^{q\psi_q-1}b_n^q. \tag{3.14}$$

Proof Setting

$$a_m=[1-\theta_p(m)]^{1-q}m^{q\psi_q-1}\Big[\sum_{n=1}^{\infty}\frac{b_n}{\max\{m^\lambda,n^\lambda\}}\Big]^{q-1},$$

then by (3.1), we have

$$\begin{aligned}
&\Big[\sum_{m=1}^{\infty}[1-\theta_p(m)]m^{p(1-\psi_q)-1}a_m^p\Big]^q=\Big\{\sum_{m=1}^{\infty}[1-\theta_p(m)]^{1-q}m^{q\psi_q-1}\Big[\sum_{n=1}^{\infty}\frac{b_n}{\max\{m^\lambda,n^\lambda\}}\Big]^q\Big\}^q\\
&=\Big[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{\max\{m^\lambda,n^\lambda\}}\Big]^q\\
&\le\Big(\frac{\lambda}{\phi_p\psi_q}\Big)^q\Big\{\sum_{m=1}^{\infty}[1-\theta_p(m)]m^{p(1-\psi_q)-1}a_m^p\Big\}^{q-1}\sum_{n=1}^{\infty}n^{q(1-\phi_p)-1}b_n^q
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
0&<\sum_{m=1}^{\infty}[1-\theta_p(m)]m^{p(1-\psi_q)-1}a_m^p=\sum_{m=1}^{\infty}[1-\theta_p(m)]^{1-q}m^{q\psi_q-1}\Big[\sum_{n=1}^{\infty}\frac{b_n}{\max\{m^\lambda,n^\lambda\}}\Big]^q\\
&\le\Big(\frac{\lambda}{\phi_p\psi_q}\Big)^q\sum_{n=1}^{\infty}n^{q(1-\phi_p)-1}b_n^q<\infty.
\end{aligned} \tag{3.16}$$

By using (3.1), (3.15) and (3.16) take the form of strict inequality, and we have (3.12).

On the other hand, if (3.12) is valid, by the reverse Hölder's inequality, we have

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{\max\{m^\lambda,n^\lambda\}}\\
&=\sum_{m=1}^{\infty}\Big\{[1-\theta_p(m)]^{\frac{1}{p}}m^{1-\psi_q-\frac{1}{p}}a_m\Big\}\Big\{\sum_{n=1}^{\infty}\frac{[1-\theta_p(m)]^{-\frac{1}{p}}m^{-1+\psi_q+\frac{1}{p}}b_n}{\max\{m^\lambda,n^\lambda\}}\Big\}\\
&\ge\Big\{\sum_{m=1}^{\infty}[1-\theta_p(m)]m^{p(1-\psi_q)-1}a_m^p\Big\}^{\frac{1}{p}}\times\\
&\quad\Big\{\sum_{m=1}^{\infty}[1-\theta_p(m)]^{1-q}m^{q\psi_q-1}[\sum_{n=1}^{\infty}\frac{b_n^q}{\max\{m^\lambda,n^\lambda\}}]^q\Big\}^{\frac{1}{q}}.
\end{aligned} \tag{3.17}$$

By (3.12), we have (3.1). Hence, inequalities (3.1) and (3.12) are equivalent. If the constant factor in (3.12) is not the best possible, we can conclude that the constant factor in (3.1) is not the best possible by (3.17). The theorem is proved. $\square$

Remark Inequalities (3.1), (3.6) and (3.12) are equivalent.

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