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Projection Scheme for Zero Points of Maximal Monotone Operators in Banach Spaces

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Abstract A new projection scheme with errors for zero points of maximal monotone operators is introduced and is proved to be strongly convergent to zero points of maximal monotone operators in Banach space by using the techniques of Lyapunov functional and generalized projection operator, etc.

Keywords Lyapunov functional; generalized projection operator; maximal monotone operator.

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1. Introduction and preliminaries

Constructing iterative schemes to approximate zero points of maximal monotone operators is a very active topic in applied mathematics. However, most of the existing iterative schemes are restricted in the frame of Hilbert spaces.

Actually, many important problems related to practical problems are generally defined in Banach spaces. For example, the maximal monotone operator related to elliptic boundary value problem has Sobolev space $W^{1,p}(\Omega)$ as its natural domain of definition[1]. Based on these reasons, we began our study and obtained some results that the proximal point schemes strongly or weakly converged to zero points of maximal monotone operators in Banach space[2−5]. Motivated by the ideas of Yanes and Xu[6] in Hilbert space, we will construct a new projection iterative scheme with errors in Banach space and use some techniques such as Lyapunov functional and generalized projection operator to prove that the iterative sequence converges strongly to zero point of maximal monotone operator.

Let E be a real Banach space and E^* its dual space. The normalized duality mapping $J \subset E \times E^*$ is defined by:

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, x \in E$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between E and E^*. We use " $\to$" and " $\rightharpoonup$" to represent strong or weak convergence in E or E^*, respectively. A multi-valued operator $A \subset$

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$E \times E^*$ is said to be monotone: if for $\forall x_i \in D(A), y_i \in Ax_i, i = 1, 2$, we have $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$. Monotone operator A is said to be maximal monotone: if $R(J + rA) = E^*$, for $\forall r > 0$. For a monotone operator A, we denote by $A^{-1}0 = \{x \in E : 0 \in Ax\}$ the kernel of A.

Lemma 1.1[7,8] *If E is a real reflexive and smooth Banach space, then $J : E \to E^*$ is a single-valued mapping and $JE = E^*$; if E is a real smooth and uniformly convex Banach space, then $J^{-1} : E^* \to E$ is also a duality mapping and is uniformly continuous on each bounded subset of E^*.*

Lemma 1.2[8] *Let E be a real smooth and uniformly convex Banach space, $A \subset E \times E^*$ be a maximal monotone operator, then $A^{-1}0$ is a closed and convex subset of E. Moreover, the graph of A, $G(A)$, is demi-closed in the sense that: $\forall \{x_n\} \subset D(A)$, $x_n \rightharpoonup x$, $(n \to \infty)$, $\forall y_n \in Ax_n$, $y_n \to y$, $(n \to \infty) \Rightarrow x \in D(A)$ and $y \in Ax$.*

Definition 1.1 *Let E be a real smooth and uniformly convex Banach space, $A \subset E \times E^*$ be a maximal monotone operator. Then $\forall r > 0$, define the operator $Q_r^A : E \to E$ by $Q_r^A x = (J + rA)^{-1} Jx$.*

Definition 1.2 *Let E be a real smooth Banach space. Then Lyapunov functional $\varphi : E \times E \to R^+$ is defined as follows:*

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \forall x, y \in E.$$

Lemma 1.3[3] *Let E be a real reflexive, strictly convex and smooth Banach space, C be a nonempty closed and convex subset of E. Then for $\forall x \in E$, there exists a unique $x_0 \in C$, such that $\varphi(x_0, x) = \inf\{\varphi(z, x) : z \in C\}$. In this case, for $\forall x \in E$, define $Q_C : E \to C$ by $Q_C x = x_0$, which is called the generalized projection operator from E onto C.*

Lemma 1.4[2] *Let E be a real reflexive, strictly convex and smooth Banach space, C be a nonempty closed and convex subset of E. Then $\forall x \in E$, $\forall y \in C$, it follows that*

$$\varphi(y, Q_C x) + \varphi(Q_C x, x) \leq \varphi(y, x).$$

Lemma 1.5[3] *Let E be a real smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\varphi(x_n, y_n) \to 0$, as $n \to \infty$, then $x_n - y_n \to 0$, as $n \to \infty$.*

Lemma 1.6[3] *Let E be a real reflexive, strictly convex and smooth Banach space, $A \subset E \times E^*$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$. Then $\forall x \in E$, $y \in A^{-1}0$ and $r > 0$, we have $\varphi(y, Q_r^A x) + \varphi(Q_r^A x, x) \leq \varphi(y, x)$.*

Lemma 1.7[3] *Let E be a real smooth Banach space, C be a nonempty closed and convex subset of E, $x \in E$, $x_0 \in C$. Then $\varphi(x_0, x) = \inf\{\varphi(z, x) : z \in C\}$ if and only if $\langle z - x_0, Jx_0 - Jx \rangle \geq 0, \forall z \in C$.*

2. Main results

In this section, unless otherwise stated, we always assume that E is a real smooth and uniformly convex Banach and $A \subset E \times E^*$ is a maximal monotone operator such that $A^{-1}0 \neq \emptyset$, and suppose both J and J^{-1} are weakly sequentially continuous. The projection scheme is introduced by the following:

$$\begin{cases} x_0 \in E, r_0 > 0, \\ y_n = Q_{r_n}^A x_n, n \geq 0, \\ Ju_n = \beta_n Jy_n + (1-\beta_n)Je_n, n \geq 0 \\ Jz_n = \alpha_n Jx_n + (1-\alpha_n)Ju_n, n \geq 0 \\ H_n = \{v \in E : \varphi(v, z_n) \leq (\alpha_n + \beta_n - \alpha_n\beta_n)\varphi(v, x_n) + (1-\alpha_n)(1-\beta_n)\varphi(v, e_n)\}, n \geq 0 \\ W_n = \{z \in E : \langle z - x_n, Jx_0 - Jx_n \rangle \leq 0\}, n \geq 0 \\ x_{n+1} = Q_{H_n \bigcap W_n}(x_0), n \geq 0 \end{cases} \tag{2.1}$$

where $\{r_n\} \subset (0, +\infty)$, $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$, and $\{e_n\}$ is the error sequence.

Lemma 2.1 *The sequence $\{x_n\}$ generated by scheme* (2.1) *is meaningful.*

Proof It is very easy to check that W_n is a closed and convex subset of E. Since

$$\begin{aligned} &\varphi(v, z_n) \leq (\alpha_n + \beta_n - \alpha_n\beta_n)\varphi(v, x_n) + (1-\alpha_n)(1-\beta_n)\varphi(v, e_n) \\ &\Leftrightarrow \|z_n\|^2 - (\alpha_n + \beta_n - \alpha_n\beta_n)\|x_n\|^2 - (1-\alpha_n)(1-\beta_n)\|e_n\|^2 \\ &\leq 2\langle v, Jz_n - (\alpha_n + \beta_n - \alpha_n\beta_n)Jx_n - (1-\alpha_n)(1-\beta_n)Je_n \rangle, \end{aligned}$$

H_n is also a closed and convex subset of E.

Let $p \in A^{-1}0$. From Definition 1.1, we know that there exists $y_0 \in E$ such that $y_0 = Q_{r_0}^A(x_0)$. Lemma 1.6 implies that $\varphi(p, y_0) \leq \varphi(p, x_0)$. Therefore

$$\begin{aligned} \varphi(p, z_0) &\leq \alpha_0\varphi(p, x_0) + (1-\alpha_0)\varphi(p, u_0) \\ &\leq (\alpha_0 + \beta_0 - \alpha_0\beta_0)\varphi(p, x_0) + (1-\alpha_0)(1-\beta_0)\varphi(p, e_0). \end{aligned}$$

Thus $p \in H_0$. Since $W_0 = E$, $p \in H_0 \bigcap W_0$. Therefore, $x_1 = Q_{H_0 \bigcap W_0}(x_0)$ is well-defined.

Suppose $p \in H_{n-1} \bigcap W_{n-1}$ and x_n is well-defined, for $n \geq 1$. From Definition 1.1, we know that there exists $y_n \in E$ such that $y_n = Q_{r_n}^A(x_n)$. Then Lemma 1.6 implies that $\varphi(p, y_n) \leq \varphi(p, x_n)$. Therefore

$$\begin{aligned} \varphi(p, z_n) &\leq \alpha_n\varphi(p, x_n) + (1-\alpha_n)[\beta_n\varphi(p, y_n) + (1-\beta_n)\varphi(p, e_n)] \\ &\leq (\alpha_n + \beta_n - \alpha_n\beta_n)\varphi(p, x_n) + (1-\alpha_n)(1-\beta_n)\varphi(p, e_n). \end{aligned}$$

Thus $p \in H_n$. Moreover, Lemma 1.7 implies that

$$\langle p - x_n, Jx_0 - Jx_n \rangle = \langle p - Q_{H_{n-1} \bigcap W_{n-1}}(x_0), Jx_0 - JQ_{H_{n-1} \bigcap W_{n-1}}(x_0) \rangle \leq 0.$$

Thus $p \in W_n$, and then $p \in H_n \bigcap W_n$. Therefore, $x_{n+1} = Q_{H_n \bigcap W_n}(x_0)$ is well-defined.

By using the method of mathematical induction, the sequence $\{x_n\}$ defined by (2.1) is meaningful. This completes the proof. □

Remark 2.1 From the proof of Lemma 2.1, we can see that $A^{-1}0 \subset H_n \bigcap W_n$, for $\forall n \geq 0$.

Theorem 2.1 *Suppose $\{x_n\}$ is generated by iterative scheme (2.1), $\liminf_{n\to\infty} r_n > 0$, $\liminf_{n\to\infty} \alpha_n > 0$, $\lim_{n\to\infty} \beta_n = 1$ and there exists a positive constant M such that $\|e_n\| \leq M$, then $x_n \to Q_{A^{-1}0}(x_n)$, as $n \to \infty$.*

Proof Our proof is split into three steps.

Step 1. $\{x_n\}$ is bounded.

In fact: $\forall p \in A^{-1}0 \subset H_n \bigcap W_n$, it follows from Lemma 1.4 that

$$\varphi(p, Q_{W_n}x_0) + \varphi(Q_{W_n}x_0, x_0) \leq \varphi(p, x_0).$$

In view of the definition of W_n, Lemmas 1.3 and 1.3, we know that $x_n = Q_{W_n}x_0$. Then $\varphi(p, x_n) + \varphi(x_n, x_0) \leq \varphi(p, x_0)$. Therefore, $\{x_n\}$ is bounded.

Step 2. $\omega(x_n) \subset A^{-1}0$, where $\omega(x_n)$ is the set consisting of all the weak limit points of $\{x_n\}$.

In fact, from Step 1, we know that $\omega(x_n) \neq \emptyset$. Then for $\forall w \in \omega(x_n)$, there exists $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup w$, as $i \to \infty$.

Since $\varphi(x_{n+1}, x_n) + \varphi(x_n, x_0) \leq \varphi(x_{n+1}, x_0)$, $\lim_{n\to\infty} \varphi(x_n, x_0)$ exists. Therefore

$$\varphi(x_{n+1}, x_n) \to 0, n \to \infty.$$

Since $x_{n+1} \in H_n$, we have

$$\varphi(x_{n+1}, z_n) \leq (\alpha_n + \beta_n - \alpha_n\beta_n)\varphi(x_{n+1}, x_n) + (1 - \alpha_n)(1 - \beta_n)\varphi(x_{n+1}, e_n).$$

From the assumptions, we know that $\varphi(x_{n+1}, z_n) \to 0$, as $n \to \infty$. Then $z_{n_i} \rightharpoonup w$, as $i \to \infty$. Since both J and J^{-1} are weakly sequentially continuous, we have $y_{n_i} \rightharpoonup w, i \to \infty$. In view of the definition of y_{n_i}, there exists $v_{n_i} \in Ay_{n_i}$ such that $Jy_{n_i} + r_{n_i}v_{n_i} = Jx_{n_i}$. Therefore $v_{n_i} \to 0$, as $i \to \infty$. Then Lemma 1.2 implies that $w \in A^{-1}0$.

Step 3. $x_n \to Q_{A^{-1}0}x_0$, as $n \to \infty$.

Let $w^* = Q_{A^{-1}0}x_0$. Since $x_{n+1} = Q_{H_n \bigcap W_n}(x_0)$ and $w^* \in A^{-1}0 \subset H_n \bigcap W_n$, we have $\varphi(x_{n+1}, x_0) \leq \varphi(w^*, x_0)$. Therefore:

$$\begin{aligned}\varphi(x_n, w^*) &= \varphi(x_n, x_0) + \varphi(x_0, w^*) - 2\langle x_n - x_0, Jw^* - Jx_0\rangle \\ &\leq \varphi(w^*, x_0) + \varphi(x_0, w^*) - 2\langle x_n - x_0, Jw^* - Jx_0\rangle.\end{aligned}$$

For $\forall\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup p$, as $i \to \infty$, we have

$$\begin{aligned}&\limsup_{i\to\infty} \varphi(x_{n_i}, w^*) \leq \varphi(w^*, x_0) + \varphi(x_0, w^*) - 2\langle p - x_0, Jw^* - Jx_0\rangle \\ &= 2\langle w^* - p, Jw^* - Jx_0\rangle \leq 0.\end{aligned}$$

Therefore, $\varphi(x_{n_i}, w^*) \to 0$, as $i \to \infty$. Then $x_{n_i} \to w^*$, as $i \to \infty$.

By now, we have proved that $\{x_n\}$ is weakly convergent to w^*. Since each weakly convergent subsequence of $\{x_n\}$ converges strongly to w^*, it follows $x_n \to w^* = Q_{A^{-1}0}x_0$, as $n \to \infty$. This completes the proof. □

Remark 2.2 Compared with the proof of convergence of proximal point schemes in [2–5], the proof here is simpler.

Remark 2.3 If $E = H$ is reduced to Hilbert space, then iterative scheme (2.1) is reduced to the following:

$$\begin{cases} x_0 \in H, r_0 > 0, \\ y_n = J_{r_n}^A x_n, n \geq 0, \\ z_n = \alpha_n x_n + (1-\alpha_n)\beta_n y_n + (1-\alpha_n)(1-\beta_n)e_n, n \geq 0 \\ H_n = \{v \in H : \|z_n\|^2 \leq 2\langle v, z_n\rangle + (\alpha_n + \beta_n - \alpha_n\beta_n)(\|x_n\|^2 - 2\langle v, x_n\rangle) + \\ \qquad (1-\alpha_n)(1-\beta_n)(\|e_n\|^2 - 2\langle v, e_n\rangle)\}, n \geq 0 \\ W_n = \{z \in H : \langle z - x_n, x_0 - x_n\rangle \leq 0\}, n \geq 0 \\ x_{n+1} = P_{H_n \bigcap W_n}(x_0), n \geq 0 \end{cases}$$

where $J_r^A x = (I + rA)^{-1}x$.

Remark 2.4 Modify iterative scheme (2.1) slightly, we can get the following iterative scheme:

$$\begin{cases} x_0 \in E, r_0 > 0, \\ y_n = Q_{r_n}^A x_n, n \geq 0, \\ Ju_n = \beta_n Jy_n + (1-\beta_n)Je_n, n \geq 0 \\ Jz_n = \alpha_n Jx_0 + (1-\alpha_n)Ju_n, n \geq 0 \\ H_n = \{v \in E : \varphi(v, z_n) \leq \alpha_n\varphi(v, x_0) + (1-\alpha_n)\beta_n\varphi(v, x_n) + \\ \qquad (1-\alpha_n)(1-\beta_n)\varphi(v, e_n)\}, n \geq 0 \\ W_n = \{z \in E : \langle z - x_n, Jx_0 - Jx_n\rangle \leq 0\}, n \geq 0 \\ x_{n+1} = Q_{H_n \bigcap W_n}(x_0), n \geq 0. \end{cases} \tag{2.2}$$

Similarly to the proof of Theorem 2.1, we obtain the following result:

Theorem 2.2 *Suppose* $\{x_n\}$ *is generated by iterative scheme (2.2),* $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 1$ *and there exists a positive constant* M *such that* $\|e_n\| \leq M$*. Then* $x_n \to Q_{A^{-1}0}(x_n)$*, as* $n \to \infty$.

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