Journal of Mathematical Research & Exposition
Nov., 2008, Vol. 28, No. 4, pp. 999–1006
DOI:10.3770/j.issn:1000-341X.2008.04.034
Http://jmre.dlut.edu.cn

Degrees of Fuzzy Compactness in I-Fuzzy Topological Spaces

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Abstract In this paper, the concept of degree of fuzzy compactness in I-fuzzy topological spaces is introduced by means of inequation. Its properties are discussed.

Keywords I-fuzzy topology; value set of fuzzy compactness; degrees of fuzzy compactness.

Document code A
MR(2000) Subject Classification 54A40; 54D30; 03E72
Chinese Library Classification O189.1

1. Introduction and preliminaries

Since Chang[1] introduced fuzzy theory into topology, many topological notions were introduced and discussed in fuzzy setting. As is well known, the concept of compactness is one of the most important concepts in L-topology, on which a lot of work has been done [2–9]. Lowen introduced fuzzy compactness[3,4] in I-topological spaces in 1976, Wang characterized it in terms of nets[7], subsequently he generalized it to L-topology[10], Kubiák also generalized fuzzy compactness to L-topological spaces by means of closed L-sets and the way-below relation[11]. Recently, Shi introduced a new definition of fuzzy compactness in L-topological spaces by using an inequality[9], when L is a completely distributive de Morgan algebra, it is equivalent to Wang's definition[10], he also presented its 25 characterizations by means of neighborhoods, remote neighborhoods and greatest minimal family. On the other hand, Zhou generalized the N-compactness[8] to I-fuzzy topological spaces[13], Yue introduced the notion of N-compactness in the general framework of Fuzzifying topological spaces and discussed the relations with Zhou's N-compactness[14]. The aim of this paper is to generalize fuzzy compactness to I-fuzzy topological spaces by the Shi's inequality[12], introduce the concept of degrees of compactness and discuss its properties.

In this paper, X is a nonempty set, $I=[0,1]$ and $I_0=(0,1]$. The family of all fuzzy sets on X is denoted by I^X. Let $\underline{0}$ and $\underline{1}$ denote the constant fuzzy set on X taking the value 0 and 1, respectively. χ_U denotes the characteristic function of U. For a subfamily $\Phi\subseteq I^X$, $2^{(\Phi)}$ denotes the set of all finite subfamily of Φ.

Received date: 2006-08-29; **Accepted date**: 2007-05-23
Foundation item: the Safety Production Science and Technology Plan Program (No. 07-379); Shandong Soft Science Development Foundation (No. 2007RKB241).

Definition 1.1[15−17] *An I-fuzzy topology on a set X is a map $\tau: I^X \to I$ such that*

(1) $\tau(\underline{1}) = \tau(\underline{0}) = 1$;

(2) $\forall U, V \in I^X, \tau(U \wedge V) \geqslant \tau(U) \wedge \tau(V)$;

(3) $\forall U_j \in I^X, j \in J, \tau(\bigvee_{j\in J} U_j) \geqslant \bigwedge_{j\in J} \tau(U_j)$.

The real number $\tau(U)$ will be called the degree of openness of the fuzzy set U; $\tau^*(U) = \tau(U')$ will be called the degree of closedness of U. The pair (X, τ) is called an I-fuzzy topological space.

A map $f: (X, \tau) \to (Y, \delta)$ is called continuous with respect to I-fuzzy topologies τ and δ if $\tau(f^{\leftarrow}(U)) \geqslant \delta(U)$ for all $U \in I^Y$, where $f^{\leftarrow}$ is defined by $f^{\leftarrow}(U)(x) = U(f(x))$.

Definition 1.2[18−20] *(1) Let τ be an I-fuzzy topology on X and $\mathcal{B}: I^X \to I$ be a function with $\mathcal{B} \leqslant \tau$. Then $\mathcal{B}$ is called a base of τ if $\mathcal{B}$ satisfies the following condition:*

$$\forall A \in I^X, \forall x_\lambda \in pt(I^X), \mathcal{Q}_{x_\lambda}(A) \leqslant \bigvee_{x_\lambda \nleqslant B', B \leqslant A} \mathcal{B}(B),$$

where $Q_{x_\lambda}(A) = \bigvee_{x_\lambda \nleqslant B', B\leqslant A} \tau(B)$. A function $\mathcal{B}: I^X \to I$ is a base of τ if and only if

$$\tau(A) = \bigvee_{\bigvee_{\lambda\in\Lambda} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \mathcal{B}(B_\lambda) \quad \text{for all } \ A \in I^X.$$

(2) Let $\phi: I^X \to I$ be a function. Then ϕ is called a subbase of τ iff $\phi^{(\sqcap)}: I^X \to I$ is a base, where

$$\phi^{(\sqcap)}(A) = \bigvee_{\sqcap_{\lambda\in\Lambda} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \phi(B_\lambda)$$

with $(\sqcap)$ standing for "finite intersection".

(3) Let $\{(X_j, \tau_j)\}_{j\in J}$ be a collection of I-fuzzy topological spaces and $P_j: \prod_{j\in J} X_j \to X_j$ be the projection. Then the I-fuzzy topology whose subbase is defined by

$$\forall A \in I^{\prod_{j\in J} X_j}, \quad \phi(A) = \bigvee_{j\in J} \bigvee_{P_j^{\leftarrow}(U) = A} \tau_j(U)$$

is called the product I-fuzzy topology of $\{\tau_j\}_{j\in J}$, denoted by $\prod_{j\in J} \tau_j$, and $(\prod_{j\in J} X_j, \prod_{j\in J} \tau_j)$ is called the product space of $\{(X_j, \tau_j)\}_{j\in J}$.

Lemma 1.3[21] *Let (X, ξ) be a fuzzifying topological space and let $\omega(\xi): I^X \to I$ be defined by $\omega(\xi)(A) = \bigwedge_{r\in I} \xi(\sigma_r(A))$ for $A \in I^X$, where $\sigma_r(A) = \{x | A(x) > r, x \in X\}$. Then $\omega(\xi)$ is an I-fuzzy topology on X, it is also called generated I-fuzzy topology by fuzzifying topology ξ.*

2. Definitions and properties of degrees of compactness

Definition 2.1 *Let $\tau: L^X \to L$ be a map. $\forall \alpha \in I_0$, let $\tau_\alpha = \{U \in I^X, \tau(U) \geqslant \alpha\}$. $\forall \mu \in I^X$, let*

$$\mathcal{S}_\tau(\mu) = \{\alpha | \bigwedge_{x\in X} \Big(\mu'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\Big) \leqslant \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X} \Big(\mu'(x) \vee \bigvee_{A\in\mathcal{V}} A(x)\Big), \forall \mathcal{U} \subseteq \tau_\alpha\}$$

and $DC_\tau(\mu) = \bigvee_{\alpha\in\mathcal{S}_\tau(\mu)} \alpha' = (\bigwedge_{\alpha\in\mathcal{S}_\tau(\mu)} \alpha)'$. If (X, τ) is I-fuzzy topological space, then $\mathcal{S}_\tau(\mu)$ is called value set of compactness of μ with respect to I-fuzzy topology τ, and $DC_\tau(\mu)$ is called degree of compactness of μ with respect to τ.

Let $(X,\mathcal{T})$ be a fuzzy topological space. Then $\mathcal{T}$ can be regarded as map $\tau : I^X \to I$ (such that $\tau(A)=1$ when $A\in\mathcal{T}$ and $\tau(A)=0$ others). If regarding $(X,\mathcal{T})$ as a special I-fuzzy topological space (X,τ), we can easily prove the following theorem.

Theorem 2.2 *Let $(X,\mathcal{T})$ be an I-topological space. $\forall\mu\in I^X$, $DC_\tau(\mu)=1$ if and only if μ is fuzzy compact in $(X,\mathcal{T})$.*

Theorem 2.3 *Let (X,τ) be an I-fuzzy topological space. For any $\mu,\lambda\in I^X$, then $DC_\tau(\mu\vee\lambda)\geqslant DC_\tau(\mu)\wedge DC_\tau(\lambda)$.*

Proof Let $\alpha\in\mathcal{S}_\tau(\mu)$ and $\beta\in\mathcal{S}_\tau(\lambda)$. For any $\mathcal{U}\subseteq\tau_{\alpha\vee\beta}$, since

$$\begin{aligned}&\bigwedge_{x\in X}\Big((\mu\vee\lambda)'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)\Big)\\ &\quad=\bigwedge_{x\in X}\Big(\mu'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)\Big)\wedge\bigwedge_{x\in X}\Big(\lambda'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)\Big)\\ &\quad\leqslant\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\Big(\mu'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)\Big)\wedge\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\Big(\lambda'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)\Big)\\ &\quad\leqslant\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\Big((\mu'(x)\wedge\lambda'(x))\vee\bigvee_{A\in\mathcal{U}}A(x)\Big)=\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\Big((\mu\vee\lambda)'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)\Big),\end{aligned}$$

$\alpha\vee\beta\in\mathcal{S}_\tau(\mu\vee\lambda)$. Hence $\bigwedge_{\alpha\in\mathcal{S}_\tau(\mu),\beta\in\mathcal{S}_\tau(\lambda)}(\alpha\vee\beta)\geqslant\bigwedge_{\gamma\in\mathcal{S}_\tau(\mu\vee\lambda)}\gamma$, i.e.,

$$\begin{aligned}\Big(\bigwedge_{\gamma\in\mathcal{S}_\tau(\mu\vee\lambda)}\gamma\Big)'&\geqslant\Big(\bigwedge_{\alpha\in\mathcal{S}_\tau(\mu),\beta\in\mathcal{S}_\tau(\lambda)}(\alpha\vee\beta)\Big)'\\ &=\Big((\bigwedge_{\alpha\in\mathcal{S}_\tau(\mu)}\alpha)\vee(\bigwedge_{\beta\in\mathcal{S}_\tau(\lambda)}\beta)\Big)'=\Big(\bigwedge_{\alpha\in\mathcal{S}_\tau(\mu)}\alpha\Big)'\wedge\Big(\bigwedge_{\beta\in\mathcal{S}_\tau(\lambda)}\beta\Big)'.\end{aligned}$$

Therefore, $DC_\tau(\mu\vee\lambda)\geqslant DC_\tau(\mu)\wedge DC_\tau(\lambda)$. □

Theorem 2.4 *Let (X,τ) be an I-fuzzy topological space and $\mu,\lambda\in I^X$. If $\tau^*(\lambda)\geqslant\alpha$ for any $\alpha\in\mathcal{S}_\tau(\mu)$, then $DC_\tau(\mu\wedge\lambda)\geqslant DC_\tau(\mu)$.*

Proof Let $\alpha\in\mathcal{S}_\tau(\mu),\mathcal{U}\subseteq\tau_\alpha$ and $\mathcal{V}=\{\lambda'\}\bigcup\mathcal{U}$. Since $\tau(\lambda')=\tau^*(\lambda)\geqslant\alpha$, $\lambda'\in\tau_\alpha$. Hence

$$\begin{aligned}&\bigwedge_{x\in X}\Big((\mu\wedge\lambda)'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)\Big)=\bigwedge_{x\in X}\Big((\mu'(x)\vee\bigvee_{B\in\mathcal{V}}B(x)\Big)\\ &\quad\leqslant\bigvee_{\mathcal{W}\in 2^{(\mathcal{V})}}\bigwedge_{x\in X}\Big(\mu'(x)\vee\bigvee_{C\in\mathcal{W}}C(x)\Big)\\ &\quad=\bigvee_{\phi\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\Big(\mu'(x)\vee\lambda'(x)\vee\bigvee_{D\in\phi}D(x)\Big)\vee\bigvee_{\phi\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\Big(\mu'(x)\vee\bigvee_{D\in\phi}D(x)\Big)\\ &\quad=\bigvee_{\phi\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\Big(\mu'(x)\vee\lambda'(x)\vee\bigvee_{D\in\phi}D(x)\Big)=\bigvee_{\phi\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\Big((\mu\wedge\lambda)'(x)\vee\bigvee_{D\in\phi}D(x)\Big).\end{aligned}$$

This is to say $\alpha\in\mathcal{S}_\tau(\mu\wedge\lambda)$. Hence $\mathcal{S}_\tau(\mu)\subseteq\mathcal{S}_\tau(\mu\wedge\lambda)$. Therefore, $DC_\tau(\mu\wedge\lambda)\geqslant DC_\tau(\mu)$. □

Theorem 2.5 *Let $\tau,\mathcal{B}: L^X\to L$ be two maps and satisfy $\mathcal{B}\leqslant\tau$. Then $\forall\mu\in I^X$, $DC_\tau(\mu)\leqslant$*

$DC_{\mathcal{B}}(\mu)$.

Proof $\forall \mu \in I^X$, suppose $\alpha \in \mathcal{S}_\tau(\mu)$ and $\mathcal{U} \subseteq \mathcal{B}_\alpha$. Then $\mathcal{U} \subseteq \tau_\alpha$ since $\mathcal{B} \leqslant \tau$. Thus $\alpha \in \mathcal{S}_{\mathcal{B}}(\mu)$. This is to say that $\mathcal{S}_\tau(\mu) \subseteq \mathcal{S}_{\mathcal{B}}(\mu)$. Therefore, $DC_\tau(\mu) \leqslant DC_{\mathcal{B}}(\mu)$. □

Corollary 2.6 *Let $(X, \tau), (X, \mathcal{B})$ be two I-fuzzy topological spaces and satisfy $\mathcal{B} \leqslant \tau$. Then $\forall \mu \in I^X$, $DC_\tau(\mu) \leqslant DC_{\mathcal{B}}(\mu)$.*

Theorem 2.7 *If $f : (X, \tau) \to (Y, \delta)$ is continuous with respect to I-fuzzy topologies τ and δ, then $DC_\tau(\mu) \leqslant DC_\delta(f^{\to}(\mu))$.*

Proof Suppose $\alpha \in \mathcal{S}_\tau(\mu)$ and $\mathcal{U} \subseteq \delta_\alpha$. Let $\mathcal{V} = \{f^{\leftarrow}(A) | A \in \mathcal{U}\}$. For any $B \in \mathcal{V}$, there exists an $A \in \mathcal{U}$ such that $B = f^{\leftarrow}(A)$. $\tau(B) = \tau(f^{\leftarrow}(A)) \geqslant \delta(A) \geqslant \alpha$ since f is continuous with respect to τ and δ, i.e., $\mathcal{V} \subseteq \tau_\alpha$. Since

$$\begin{aligned}
&\bigwedge_{y\in Y} \Big(f^{\to}(\mu)'(y) \vee \bigvee_{A\in\mathcal{U}} A(y)\Big) = \bigwedge_{y\in Y} \Big(\bigwedge_{f(x)=y} \mu'(x) \vee \bigvee_{A\in\mathcal{U}} A(f(x))\Big) \\
&\quad = \bigwedge_{y\in Y} \Big(\bigwedge_{f(x)=y} \Big(\mu'(x) \vee \bigvee_{A\in\mathcal{U}} f^{\leftarrow}(A)(x)\Big)\Big) = \bigwedge_{x\in X} \Big(\mu'(x) \vee \bigvee_{B\in\mathcal{V}} B(x)\Big) \\
&\quad \leqslant \bigvee_{\mathcal{W}\in 2^{(\mathcal{V})}} \bigwedge_{x\in X} \Big(\mu'(x) \vee \bigvee_{C\in\mathcal{W}} C(x)\Big) \\
&\quad = \bigvee_{\phi\in 2^{(\mathcal{U})}} \bigwedge_{y\in Y} \Big(\bigwedge_{f(x)=y} \Big(\mu'(x) \vee \bigvee_{D\in\phi} f^{\leftarrow}(D)(x)\Big)\Big) = \bigvee_{\phi\in 2^{(\mathcal{U})}} \bigwedge_{y\in Y} \Big(f^{\to}(\mu)'(y) \vee \bigvee_{D\in\phi} D(y)\Big),
\end{aligned}$$

$\alpha \in \mathcal{S}_\delta(f^{\to}(\mu))$. Hence $\mathcal{S}_\tau(\mu) \subseteq \mathcal{S}_\delta(f^{\to}(\mu))$. Therefore, $DC_\tau(\mu) \leq DC_\delta(f^{\to}(\mu))$. □

Definition 2.8 *Let (X, ξ) be a fuzzifying topological space and $\xi_\alpha = \{U \in P(X), \xi(U) \geqslant \alpha\}$, $\alpha \in I_0$. For any $G \in P(X)$, a set*

$$\mathcal{S}_\xi(G) = \{\alpha | \text{every cover } \mathcal{U} \subseteq \xi_\alpha \text{ of } G, \text{ there exists a finite subfamily } \mathcal{V} \text{ of } \mathcal{U} \text{ is a cover of } G\}$$

is called value set of compactness of G with respect to ξ. The degree of compactness of G with respect to ξ is defined by $DC_\xi(G) = \bigvee_{\alpha\in\mathcal{S}_\xi(G)} \alpha' = (\bigwedge_{\alpha\in\mathcal{S}_\xi(G)} \alpha)'$.

Theorem 2.9 *Let (X, ξ) be a fuzzifying topological space and $\omega(\xi)$ be generated I-fuzzy topology by ξ. Then $DC_\xi(X) = DC_{\omega(\xi)}(\underline{1})$.*

Proof Let $\alpha \in \mathcal{S}_{\omega(\xi)}(\underline{1})$. For any cover $\mathcal{U} \subseteq \xi_\alpha$ of X, let $\mathcal{V} = \{\chi_U | U \in \mathcal{U}\}$. Then $\mathcal{V} \subseteq \omega(\xi)_\alpha$ since $\omega(\xi)(V) = \bigwedge_{r\in I} \xi(\sigma_r(V)) = \bigwedge_{r\in I} \xi(U) = \xi(U) \geqslant \alpha$ for any $V = \chi_U \in \mathcal{V}$. Thus $1 \leqslant \bigwedge_{x\in X} \bigvee_{A\in\mathcal{V}} A(x) \leqslant \bigvee_{\mathcal{W}\in 2^{(\mathcal{V})}} \Big(\bigwedge_{x\in X} \bigvee_{B\in\mathcal{W}} B(x)\Big)$ since $\mathcal{U}$ is a cover of X. We know that there exists a finite subfamily $\mathcal{W}$ of $\mathcal{V}$ such that $\bigwedge_{x\in X} \bigvee_{B\in\mathcal{W}} B(x) \geqslant \frac{1}{2}$. Let $\mathcal{Q} = \{U \in \mathcal{U} | B = \chi_U \in \mathcal{W}\}$. Then $\mathcal{Q} \subseteq \mathcal{U}$ is finite subcover of X. Thus $\alpha \in \mathcal{S}_\xi(X)$. This is to say $\mathcal{S}_{\omega(\xi)}(\underline{1}) \subseteq \mathcal{S}_\xi(X)$. Therefore, $DC_{\omega(\xi)}(\underline{1}) \leqslant DC_\xi(X)$.

On the other hand, let $\alpha \in \mathcal{S}_\xi(X)$. For any $\mathcal{U} \subseteq \omega(\xi)_\alpha$, let $\bigwedge_{x\in X} \bigvee_{A\in\mathcal{U}} A(x) = \beta$.

If $\beta = 0$, then $\bigwedge_{x\in X} \bigvee_{A\in\mathcal{U}} A(x) = 0 \leqslant \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X} \bigvee_{B\in\mathcal{V}} B(x)$;

Else $\forall 0<\gamma<\beta$, we know that $\mathcal{W}_\gamma=\{\sigma_\gamma(A)|A\in\mathcal{U}\}$ is a cover of X and $\mathcal{W}_\gamma\subseteq\xi_\alpha$ since $\xi(C)=\xi(\sigma_\gamma(A))\geqslant\bigwedge_{r\in I}\xi(\sigma_r(A))=\omega(\xi)(A)\geqslant\alpha$ for any $C=\sigma_\gamma(A)\in\mathcal{W}_\gamma$, where $A\in\mathcal{U}$. Since $\alpha\in\mathcal{S}_\xi(X)$, there exists a finite subfamily $\mathcal{Q}_\gamma$ of $\mathcal{W}_\gamma$ such that $\mathcal{Q}_\gamma$ is cover of X. Let $\mathcal{P}_\gamma=\{A\in\mathcal{U}|\sigma_\gamma(A)\in\mathcal{Q}_\gamma\}$. Then $\mathcal{P}_\gamma$ is a finite subfamily of $\mathcal{U}$, i.e., $\mathcal{P}_\gamma\in 2^{(\mathcal{U})}$. Hence

$$\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\bigvee_{B\in\mathcal{V}}B(x)\geqslant\bigvee_{0<\gamma<\beta}\bigwedge_{x\in X}\bigvee_{A\in\mathcal{P}_\gamma}A(x)\geqslant\bigvee_{0<\gamma<\beta}\gamma=\beta=\bigwedge_{x\in X}\bigvee_{A\in\mathcal{U}}A(x).$$

Thus $\alpha\in\mathcal{S}_{\omega(\xi)}(\underline{1})$. This means that $\mathcal{S}_\xi(X)\subseteq\mathcal{S}_{\omega(\xi)}(\underline{1})$. Therefore, $DC_\xi(X)\leqslant DC_{\omega(\xi)}(\underline{1})$. So $DC_\xi(X)=DC_{\omega(\xi)}(\underline{1})$. □

Theorem 2.10 *Let (X,τ) be an I-fuzzy topological space and ϕ be a subbase of τ. Then $DC_\tau(\mu)=DC_\phi(\mu)$ for any $\mu\in I^X$.*

Proof For any $\mu\in I^X$, $DC_\tau(\mu)\leqslant DC_\phi(\mu)$ is obvious by $\phi\leqslant\tau$ and Corollary 2.6. We need only prove $DC_\phi(\mu)\leqslant DC_\tau(\mu)$.

We say that $\mathcal{A}\subseteq I^X$ has the finite property for μ if

$$\bigwedge_{x\in X}\Big(\mu'(x)\vee\bigvee_{A\in\mathcal{A}}A(x)\Big)\nleqslant\bigvee_{\mathcal{V}\in 2^{(\mathcal{A})}}\bigwedge_{x\in X}\Big(\mu'(x)\vee\bigvee_{A\in\mathcal{V}}A(x)\Big).$$

Suppose $\alpha\in\mathcal{S}_\phi(\mu)$, now we prove that for any $\beta\in I$ and $\beta>\alpha$, $\beta\in\mathcal{S}_\tau(\mu)$.

Let $\mathcal{U}\subseteq\tau_\beta$. If $\mathcal{U}$ has the finite property for μ, let

$$\Gamma=\{\mathcal{P}|\mathcal{U}\subseteq\mathcal{P}\subseteq\tau_\beta \text{ and } \mathcal{P} \text{ has the finite property for } \mu\}.$$

Then $(\Gamma,\subseteq)$ is nonempty partially ordered set. Now we prove that each chain in $(\Gamma,\subseteq)$ has an upper bound.

For any chain Λ in Γ, let $\mathcal{Q}=\bigcup\limits_{\mathcal{P}\in\Lambda}\mathcal{P}$. Since $\forall\mathcal{P}\in\Lambda$, $\mathcal{P}$ has the finite property for μ,

$$\begin{aligned}\bigwedge_{x\in X}\Big(\mu'(x)\vee\bigvee_{A\in\mathcal{Q}}A(x)\Big)&=\bigwedge_{x\in X}\Big(\mu'(x)\vee\bigvee_{\mathcal{P}\in\Lambda}\bigvee_{A\in\mathcal{P}}A(x)\Big)\\&=\bigwedge_{x\in X}\bigvee_{\mathcal{P}\in\Lambda}\Big(\mu'(x)\vee\bigvee_{A\in\mathcal{P}}A(x)\Big)\\&\geqslant\bigvee_{\mathcal{P}\in\Lambda}\bigwedge_{x\in X}\Big(\mu'(x)\vee\bigvee_{A\in\mathcal{P}}A(x)\Big)\\&>\bigvee_{\mathcal{V}\in 2^{(\mathcal{Q})}}\bigwedge_{x\in X}\Big(\mu'(x)\vee\bigvee_{A\in\mathcal{V}}A(x)\Big).\end{aligned}$$

Thus $\mathcal{Q}$ has the finite property for μ, i.e., $\mathcal{Q}\in\Gamma$. Therefore, $\mathcal{Q}$ is upper bound of Λ. Considering the arbitrariness of Λ, we conclude that each chain in $(\Gamma,\subseteq)$ has an upper bound.

Hence by Zorn's Lemma, Γ has a maximal element Ω. Now we prove that Ω satisfies the following conditions:

Ω_1) $\Omega\subseteq\tau_\beta$;

Ω_2) $\forall B\in\tau_\beta$, if $C\in\Omega$ and $C\geqslant B$, then $B\in\Omega$;

Ω_3) If $B,C\in\tau_\beta$, $B\wedge C\in\Omega$, then $B\in\Omega$ or $C\in\Omega$.

We only verify Ω_3). If $B \notin \Omega$ and $C \notin \Omega$, then neither $\{B\} \cup \Omega$, nor $\{C\} \cup \Omega$ has the finite property. We obtain that

$$\begin{aligned}
&\bigwedge_{x\in X}\Big(\mu'(x) \vee (B\wedge C)(x) \vee \bigvee_{A\in\Omega} A(x)\Big)\\
&\quad\leqslant \bigwedge_{x\in X}\Big(\mu'(x) \vee B(x) \vee \bigvee_{A\in\Omega} A(x)\Big) \wedge \bigwedge_{x\in X}\Big(\mu'(x) \vee C(x) \vee \bigvee_{A\in\Omega} A(x)\Big)\\
&\quad\leqslant \bigvee_{\mathcal{V}\in 2^{(\Omega)}}\bigwedge_{x\in X}\Big(\mu'(x) \vee B(x) \vee \bigvee_{A\in\mathcal{V}} A(x)\Big) \wedge \bigvee_{\mathcal{V}\in 2^{(\Omega)}}\bigwedge_{x\in X}\Big(\mu'(x) \vee C(x) \vee \bigvee_{A\in\mathcal{V}} A(x)\Big)\\
&\quad= \bigvee_{\mathcal{V}\in 2^{(\Omega)}}\bigwedge_{x\in X}\Big(\mu'(x) \vee (B\wedge C)(x) \vee \bigvee_{A\in\mathcal{V}} A(x)\Big).
\end{aligned}$$

This implies that $B \wedge C \notin \Omega$, which contradicts $B \wedge C \in \Omega$. Ω_3) is proved.

$\forall D \in \Omega$, $\tau(D) = \bigvee_{\bigvee\limits_{\lambda\in\Lambda} V_\lambda = D} \bigwedge_{\lambda\in\Lambda} \bigvee_{(\sqcap)_{\beta\in\Lambda_\lambda} W_{\lambda\beta}=V_\lambda} \bigwedge_{\beta\in\Lambda_\lambda} \phi(W^D_{\lambda\beta}) \geqslant \beta > \alpha$ by Ω_1) and Definition 1.2. Then there exist $\{V_\lambda\}_{\lambda\in\Lambda}$ such that

D_1) $\bigvee_{\lambda\in\Lambda} V_\lambda = D$;

D_2) For each $\lambda \in \Lambda$, there exists $\{W^D_{\lambda\beta}\}_{\beta\in\Lambda_\lambda}$ satisfying $(\sqcap)_{\beta\in\Lambda_\lambda} W^D_{\lambda\beta} = V_\lambda$;

D_3) For each $\beta \in \Lambda_\lambda$, $\phi(W^D_{\lambda\beta}) > \alpha$.

On the other hand, $D = \bigvee_{\lambda\in\Lambda}(\sqcap)_{\beta\in\Lambda_\lambda} W^D_{\lambda\beta} \in \Omega$ by Ω_2) and Ω_3). We obviously know $\forall D \in \Omega$ satisfies the following conditions:

Ω_{D1}) For each $\lambda \in \Lambda$, $(\sqcap)_{\beta\in\Lambda_\lambda} W^D_{\lambda\beta} \in \Omega$;

Ω_{D2}) There exists $\beta^D_\lambda \in \Lambda_\lambda$ such that $W^D_{\lambda\beta^D_\lambda} \in \Omega$.

Let $\mathcal{R} = \{W^D_{\lambda\beta^D_\lambda} | D \in \Omega, W^D_{\lambda\beta^D_\lambda}$ satisfies the conditions $\Omega_{D1}), \Omega_{D2})\}$. Then $\mathcal{R} \subseteq \phi_\alpha$ and $\mathcal{R} \subseteq \Omega$. Thus

$$\begin{aligned}
\bigwedge_{x\in X}\Big(\mu'(x) \vee \bigvee_{D\in\Omega} D(x)\Big) &\leqslant \bigwedge_{x\in X}\Big(\mu'(x) \vee \bigvee_{D\in\Omega}\bigvee_{\lambda\in\Lambda} W^D_{\lambda\beta^D_\lambda}(x)\Big) = \bigwedge_{x\in X}\Big(\mu'(x) \vee \bigvee_{U\in\mathcal{R}} U(x)\Big)\\
&\leqslant \bigvee_{\mathcal{V}\in 2^{(\mathcal{R})}}\bigwedge_{x\in X}\Big(\mu'(x) \vee \bigvee_{V\in\mathcal{V}} V(x)\Big) \leqslant \bigvee_{\mathcal{V}\in 2^{(\Omega)}}\bigwedge_{x\in X}\Big(\mu'(x) \vee \bigvee_{V\in\mathcal{V}} V(x)\Big).
\end{aligned}$$

This is a contradiction. Then $\mathcal{U}$ does not have the finite property. Thus $\beta \in \mathcal{S}_\tau(\mu)$.

Therefore, $DC_\phi(\mu) = \bigvee_{\alpha\in\mathcal{S}_\phi(\mu)} \alpha' = \bigvee\limits_{\alpha\in\mathcal{S}_\phi(\mu)} \Big(\bigwedge_{\beta>\alpha}\beta\Big)' = \bigvee_{\alpha\in\mathcal{S}_\phi(\mu)}\bigvee_{\beta>\alpha}\beta' \leqslant \bigvee_{\gamma\in\mathcal{S}_\tau(\mu)}\gamma' = DC_\tau(\mu)$. The proof is completed. □

Theorem 2.11 *Let (X, τ) be the product I-fuzzy topological space of $\{(X_j, \tau_j)\}_{j\in J}$. Then $DC_\tau(\mu) \geqslant \bigwedge_{j\in J} DC_{\tau_j}(\mu_j)$ for any $\mu = \prod_{j\in J}\mu_j \in I^{\prod_{j\in J} X_j}$, where $\mu_j \in I^{X_j}$ for any $j \in J$.*

Proof Let $\phi : I^X \to I$, $\phi(A) = \bigvee_{j\in J}\bigvee_{P_j^{\leftarrow}(U)=A} \tau_j(U), \forall A \in I^X$ be subbase of product I-fuzzy topological space (X, τ). Then

$$\bigwedge_{j\in J} DC_{\tau_j}(\mu_j) = \Big(\bigvee_{j\in J}\bigwedge_{\alpha\in\mathcal{S}_{\tau_j}(\mu_j)}\alpha\Big)' = \Big(\bigwedge_{f\in\prod\limits_{j\in J}\mathcal{S}_{\tau_j}(\mu_j)}\bigvee_{j\in J} P_j^{\leftarrow}(f)\Big)' = \bigvee_{f\in\prod\limits_{j\in J}\mathcal{S}_{\tau_j}(\mu_j)}\Big(\bigvee_{j\in J} P_j^{\leftarrow}(f)\Big)'.$$

$\forall f \in \prod_{j\in J}\mathcal{S}_{\tau_j}(\mu_j)$, let $\alpha = \bigvee_{j\in J} P_j^{\leftarrow}(f)$. Now we prove that $\forall\beta > \alpha, \beta \in \mathcal{S}_\phi(\mu)$.

Suppose that $\mathcal{U} \subseteq \phi_\beta$. Then $\forall A \in \mathcal{U}$, there exists $j \in J$ and $B \in I^{X_j}$ such that $P_j^{\leftarrow}(B) = A$ and $\tau_j(B) > \alpha$ since $\phi(A) = \bigvee_{j\in J} \bigvee_{P_j^{\leftarrow}(B)=A} \tau_j(B)$ for any $A \in \mathcal{U}$. Let $\mathcal{B}_j = \{B | B \in I^{X_j}, P_j^{\leftarrow}(B) = A, \tau_j(B) > \alpha, A \in U\}$ and $\mathcal{U}_j = \{P_j^{\leftarrow}(B) | B \in I^{X_j}, P_j^{\leftarrow}(B) = A, \tau_j(B) > \alpha, A \in U\}$, $j \in I \subseteq J$. Then $\mathcal{U} = \bigcup_{j\in I} \mathcal{U}_j$ and $\mathcal{B}_j \subseteq (\tau_j)_\alpha, \forall j \in I$. Thus for any $K \in I$,

$$\bigwedge_{x_k\in X_k} \Big(\mu_k'(x_k) \vee \bigvee_{B\in\mathcal{B}_k} B(x_k)\Big) \leqslant \bigvee_{\mathcal{V}_k\in 2^{(\mathcal{B}_k)}} \bigwedge_{x_k\in X_k} \Big(\mu_k'(x_k) \vee \bigvee_{C\in\mathcal{V}_k} C(x_k)\Big).$$

We know that

$$\begin{aligned}
\gamma &= \bigwedge_{x\in X} \Big(\mu'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\Big) \\
&= \bigwedge_{x\in X} \Big(\bigvee_{j\in J} \mu_j'(P_j(x)) \vee \bigvee_{j\in I} \bigvee_{B\in\mathcal{B}_j} B(P_j(x))\Big) \\
&= \bigwedge_{x\in X} \Big(\bigvee_{j\notin I} \mu_j'(P_j(x)) \vee \bigvee_{j\in I} \mu_j'(P_j(x)) \vee \bigvee_{j\in I} \bigvee_{B\in\mathcal{B}_j} B(P_j(x))\Big) \\
&= \bigwedge_{x\in X} \Big(\bigvee_{j\notin I} \mu_j'(P_j(x)) \vee \bigvee_{j\in I} \Big(\mu_j'(P_j(x)) \vee \bigvee_{B\in\mathcal{B}_j} B(P_j(x))\Big)\Big).
\end{aligned}$$

(1) If $\gamma \leqslant \bigvee_{j\in J} \bigwedge_{x_j\in X_j} \mu_j'(x_j)$, then

$$\gamma \leqslant \bigvee_{j\in J} \bigwedge_{x_j\in X_j} \mu_j'(x_j) \leqslant \bigwedge_{x\in X} \Big(\bigvee_{j\in J} \mu_j'(P_j(x))\Big) = \bigwedge_{x\in X} \mu'(x) \leqslant \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X} \Big(\mu'(x) \vee \bigvee_{A\in\mathcal{V}} A(x)\Big).$$

(2) Else, $\forall j \in J$, there exists $x_j \in X_j$ such that $\mu_j'(x_j) < a < \gamma$.

If $\bigvee_{k\in I} \bigwedge_{x_k\in X_k} \Big(\mu_k'(x_k) \vee \bigvee_{B\in\mathcal{B}_k} B(x_k)\Big) < \gamma$, then $\forall j \in I$, there exists a $y_j \in X_j$ such that $\mu_j'(y_j) \vee \bigvee_{B\in\mathcal{B}_j} B(y_j) < b < \gamma$. Let $z = \{z_j\}_{j\in J}$ such that $z_j = y_j$ when $j \in I$, $z_j = x_j$ otherwise. Then

$$\begin{aligned}
\gamma &= \bigwedge_{x\in X} \Big(\bigvee_{j\notin I} \mu_j'(P_j(x)) \vee \bigvee_{j\in I} \mu_j'(P_j(x)) \vee \bigvee_{j\in I} \bigvee_{B\in\mathcal{B}_j} B(P_j(x))\Big) \\
&\leqslant \bigvee_{j\notin I} \mu_j'(P_j(z)) \vee \bigvee_{j\in I} \mu_j'(P_j(z)) \vee \bigvee_{j\in I} \bigvee_{B\in\mathcal{B}_j} B(P_j(z)) \\
&= \bigvee_{j\notin I} \mu_j'(P_j(z)) \vee \bigvee_{j\in I} \Big(\mu_j'(P_j(z)) \vee \bigvee_{B\in\mathcal{B}_j} B(P_j(z))\Big) \leqslant a \vee b < \gamma.
\end{aligned}$$

This is a contradiction. Thus $\bigvee_{k\in I} \bigwedge_{x_k\in X_k} \Big(\mu_k'(x_k) \vee \bigvee_{B\in\mathcal{B}_k} B(x_k)\Big) \geqslant \gamma$. Ulteriorly, we have that

$$\gamma \leqslant \bigvee_{k\in I} \bigvee_{\mathcal{V}_k\in 2^{(\mathcal{B}_k)}} \bigwedge_{x_k\in X_k} \Big(\mu_k'(x_k) \vee \bigvee_{C_k\in\mathcal{V}_k} C_k(x_k)\Big) \leqslant \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X} \Big(\mu'(x) \vee \bigvee_{C\in\mathcal{V}} C(x)\Big).$$

Hence $\beta \in \mathcal{S}_\phi(\mu)$. Therefore,

$$\bigwedge_{j\in J} DC_{\tau_j}(\mu_j) = \bigvee_{f\in \prod_{j\in J} \mathcal{S}_{\tau_j}(\mu_j)} \Big(\bigvee_{j\in J} P_j^{\leftarrow}(f)\Big)' = \bigvee_{f\in \prod_{j\in J} \mathcal{S}_{\tau_j}(\mu_j)} \alpha'$$

$$= \bigvee_{f \in \prod_{j \in J} \mathcal{S}_{\tau_j}(\mu_j)} \Big(\bigwedge_{\beta > \alpha} \beta\Big)' = \bigvee_{f \in \prod_{j \in J} \mathcal{S}_{\tau_j}(\mu_j)} \bigvee_{\beta > \alpha} \beta'$$

$$\leqslant \bigvee_{\gamma \in \mathcal{S}_\phi(\mu)} \gamma' = DC_\phi(\mu) = DC_\tau(\mu).$$

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