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Tightness and Fan Tightness on Multifunction Spaces

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Abstract In this paper, we discuss tightness and fan tightness of multifunction spaces with pointwise convergence topology or compact-open topology, and generalize some results on continuous single-valued function spaces to continuous multifunction spaces.

Keywords multifunction; pointwise convergence topology; compact-open topology; tightness; fan tightness.

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1. Introduction

Multifunction theory has extensive applications in cybernetics, functional analysis and quantitative economics, etc. The basic question in multifunction theory is the relationship between topological properties of multifunction spaces and those of base spaces, that is to say, to find topological property P and Q such that multifunction space $\mathbf{C}(X,\mathbb{R})$ has property Q if and only if base space X has property P, where $\mathbb{R}$ denotes real line and $\mathbf{C}(X,\mathbb{R})$ denotes continuous multifunction family from X to $\mathbb{R}$ with some topological structure. There have been descriptions about tightness and fan tightness in continuous single-valued function spaces([1,2]), but these two characteristics have not been solved in continuous multifunction spaces. The reason is that there are no homogeneity and extendable property of continuous multifunctions. But these two characteristics are the key to investigate tightness and fan tightness. In this paper, we do not use homogeneity and extendable property of continuous multifunctions and give the characteristics of continuous multifunction family $\mathbf{C}_k(X,\mathbb{R})$, and obtain two dual theorems about space X and $\mathbf{C}_k(X,\mathbb{R})$.

In this paper, topological spaces X and Y are completely regular T_1, $\mathbf{M}(X,Y)$ is the multifunction family from space X to Y, $K(X)$ denotes the family of all the non-empty compact subsets of X, $\mathbb{R}$ denotes real line, $\aleph_0$ is countable cardinal, and λ is any infinite cardinal. In the paper the terminologies and symbols are referred to [3].

Suppose $f\in\mathbf{M}(X,Y)$. For $A\subset X$, let $f(A)=\bigcup_{x\in A}f(x)$ and for $B\subset Y$, let

$$f^+(B)=\{x\in X: f(x)\subset B\}$$

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and

$$f^{-}(B)=\{x\in X: f(x)\cap B\neq\emptyset\}.$$

Let $f\in \mathbf{M}(X,Y)$, $x_0\in X$. f is said to be upper continuous at x_0, if $f(x_0)\subset U$ for any open set U of Y, there exists an open neighbourhood V of x_0 such that when $x\in V$, we have $f(x)\subset U$. f is said to be lower continuous at x_0, if $f(x_0)\cap U\neq\emptyset$ for any open set U of Y, there exists an open neighbourhood V of x_0 such that when $x\in V$, we have $f(x)\cap U\neq\emptyset$. f is said to be continuous at x_0, if it is both upper continuous and lower continuous at x_0. If f is continuous at every point of X, it is said to be continuous. Obviously, f is continuous if and only if for any open set U of Y, $f^{+}(U)$ and $f^{-}(U)$ are all open in X.

For $K\subset X$, $U,V\subset Y$, let

$$W^{+}[K,U]=\{f\in \mathbf{M}(X,Y): f(x)\subset U, x\in K\};$$

$$W^{-}[K,V]=\{f\in \mathbf{M}(X,Y): f(x)\cap V\neq\emptyset, x\in K\}.$$

The compact-open topology $\mathcal{T}_k$ on $\mathbf{M}(X,Y)$ is the one having all sets of the forms $W^{+}[K,U]$ and $W^{-}[K,V]$ as a subbase, where K is a compact set of X, and U and V are open in Y. The pointwise convengence topology $\mathcal{T}_p$ on $\mathbf{M}(X,Y)$ is the one having all sets of the forms $W^{+}[\{x\},U]$ and $W^{-}[\{x\},V]$ as a subbase, where $x\in X$, and U and V are open in Y.

2. Tightness of multifunction spaces

The tightness of a space X is defined as $t(X)=\sup\{t(X,x): x\in X\}$, where the tightness at x of X is defined as $t(X,x)=\aleph_0+\min\{\lambda$: for $Y\subset X$, if $x\in\overline{Y}$, there exists $Z\subset Y$, such that $|Z|\leq\lambda$ and $x\in\overline{Z}\}$.

A family $\mathcal{U}$ of subsets of X is called α cover[1] of X, if for each K of α, there exists $U\in\mathcal{U}$, such that $K\subset U$. If $\mathcal{U}$ consists of open sets of X, and α consists of non-empty compact sets of X, then $\mathcal{U}$ is called open k cover of X.

The k-Lindelöf number of X is defined as $kL(X)=\aleph_0+\min\{\lambda$: each open k cover of X has an open k subcover $\mathcal{U}$ such that $|\mathcal{U}|\leq\lambda\}$.

A family $\mathcal{U}$ of open subsets of X is called open ω cover of X, if for each finite subset F of X, there exists $U\in\mathcal{U}$ such that $F\subset U$.

The p-Lindelöf number of X is defined as $pL(X)=\aleph_0+\min\{\lambda$: each open ω cover of X has an open ω subcover $\mathcal{U}$ such that $|\mathcal{U}|\leq\lambda\}$.

$C(X,\mathbb{R})$ denotes the family of all the single-valued continuous functions from X to $\mathbb{R}$ and $\mathbf{C}_k(X,\mathbb{R})$ denotes the family of all the point-compact continuous multifunctions from X to $\mathbb{R}$. We shall use the symbol $C(X)$ and $\mathbf{C}_k(X)$ instead of $C(X,\mathbb{R})$ and $\mathbf{C}_k(X,\mathbb{R})$.

Theorem 2.1 *For every space X, the following conditions are equivalent:*

(1) $(C(X),\mathcal{T}_p)$ is a closed subspace of $(\mathbf{C}_k(X),\mathcal{T}_p)$;

(2) $(C(X),\mathcal{T}_k)$ is a closed subspace of $(\mathbf{C}_k(X),\mathcal{T}_k)$.

Definition 2.2 *For any $A,B\subset\mathbb{R}$, put $\rho(x,A)=\inf\{|x-y|: y\in A\}$, $\rho(A,B)=\sup\{\rho(x,B):$*

$x \in A\}$ and $d(A,B) = \sup\{\rho(A,B), \rho(B,A)\}$. For $x \in \mathbb{R}$, put $d(x,A) = d(\{x\},A)$.

Theorem 2.3 *For every space X, we have $t(\mathbf{C}_k(X), \mathcal{T}_k) = kL(X)$.*

Proof Suppose that $t(\mathbf{C}_k(X), \mathcal{T}_k) = \lambda$, and let $\mathcal{U}$ be an open k cover of X. Then for each compact set $A \subset X$, there exists $U_A \in \mathcal{U}$ such that $A \subset U_A$. Let $F = \{f_A : A \in K(X), f_A \in \mathbf{C}_k(X), f_A(A) = \{0\}, f_A(X - U_A) \subset \{1\}\}$. Then $F \neq \emptyset$. We can take a single-valued function $f_A \in C(X)$ such that $f_A(A) = \{0\}, f_A(X - U_A) \subset \{1\}$. Let V be an open neighbourhood of 0 in $\mathbb{R}$. Take $f_A \in F$. Then $f_A \in W^+[A, V]$. Because $f_0 \equiv \{0\} \in \overline{F}$, there exists a subset F' of F such that $|F'| \leq \lambda$ and $f_0 \in \overline{F'}$. Let $\mathcal{V}= \{U_A : f_A \in F'\}$. Then $\mathcal{V}$ is a k subcover of $\mathcal{U}$. In fact, suppose $A \in K(X)$. Then $W^+[A, (-1,1)]$ is a neighbourhood of f_0. So there exists $B \in K(X)$ such that $f_B \in F' \cap W^+[A, (-1,1)]$. For each $x \in A$, we have $\max\{f_B(x)\} < 1$, and for each $x \in X - U_B$, we have $f_B(x) = 1$, so $A \subset U_B$. Hence $\mathcal{V}$ is a k subcover of $\mathcal{U}$ and $|\mathcal{V}| \leq \lambda$. Finally, we have $kL(X) \leq t(\mathbf{C}_k(X), \mathcal{T}_k)$.

On the other hand, suppose that $kL(X) = \lambda$, $F \subset (\mathbf{C}_k(X), \mathcal{T}_k)$ and $f \in \overline{F}$. For each $A \in K(X), n \in N, x \in A$, let $U^{(n)}_{f(x)} = \{y \in R : d(y, f(x)) < \frac{1}{2n}\}$. Then $U^{(n)}_{f(x)}$ is an open set and $f(x) \subset U^{(n)}_{f(x)}$, so $\{U^{(n)}_{f(x)} : x \in A\}$ is an open cover of $f(A)$, hence there exists a subcover $\{U^{(n)}_{f(x_i)} : x_i \in A, 1 \leq i \leq k\}$. Because f is continuous, there exists a closed neighbourhood $U^{(n)}_{x_i}$ of x such that when $x' \in U^{(n)}_{x_i}$, we always have $f(x') \subset U^{(n)}_{f(x_i)}$. Thus $f \in W^+[A, \bigcup_{i=1}^k U^{(n)}_{f(x_i)}]$. Take $g^{(n)}_A \in F \cap (W^+[A, \bigcup_{i=1}^k U^{(n)}_{f(x_i)}])$ and $U^{(n)}_A = \{x \in X : d(g^{(n)}_A(x), f(x)) < \frac{1}{n}\}$. Then $A \subset U^{(n)}_A$. This is because if $x \in A$, then there exists some $U^{(n)}_{f(x_i)}$ such that $x \in U^{(n)}_{f(x_i)}$. From $g^{(n)}_A \in W^+[A, \bigcup_{i=1}^k U^{(n)}_{f(x_i)}]$, we have $g^{(n)}_A(x) \subset \bigcup_{i=1}^k U^{(n)}_{f(x_i)}$, and for each $y \in g^{(n)}_A(x)$, there exists $U^{(n)}_{f(x_i)}$ such that when $y \in U^{(n)}_{f(x_i)}$, we have $d(y, f(x_i)) < \frac{1}{2n}$. But $d(f(x_i), f(x)) < \frac{1}{2n}$, so $d(g^{(n)}_A(x), f(x)) < \frac{1}{n}$, and $A \subset U^{(n)}_A$. Obviously, $U^{(n)}_A$ is an open set, so $\mathcal{V}_n = \{U^{(n)}_A : A \in K(X)\}$ is an open k cover of X, hence there exists an open k subcover $\mathcal{W}_n$ whose cardinal does not exceed λ. Let $F' = \{g^{(n)}_A : U^{(n)}_A \in \mathcal{W}_n\}$. Then $F' \subset F$ and $|F'| \leq \lambda$. For each open base neighbourhood $W[B, \mathcal{V}]$ of f, $B \in K(X)$, $\mathcal{V}$ is a finite open set in $\mathbb{R}$. Let $V \in \mathcal{V}$. Then for $x \in B$, we have $f(x) \cap V \neq \emptyset$. Take $y_{(x,V)} \in f(x) \cap V$ and $\delta_{(x,V)} > 0$ such that $(y_{(x,V)} - \delta_{(x,V)}, y_{(x,V)} + \delta_{(x,V)}) \subset V$. Then there exists an open neighbourhood $U_{(x,V)}$ of x such that $U_{(x,V)} \subset f^-((y_{(x,V)} - \frac{\delta_{(x,V)}}{2}, y_{(x,V)} + \frac{\delta_{(x,V)}}{2}))$. Because $\{U_{(x,V)} : x \in B\}$ covers B, there exists a finite subcover $\{U_{(x_i,V)} : 1 \leq i \leq k_V\}$. Take Lebesgue number δ of $\mathcal{V}$ about $f(B)$. Then the open cover $\{(y - \frac{\delta}{2}, y + \frac{\delta}{2}) : y \in f(B)\}$ of $f(B)$ refines $\mathcal{V}$. So there exists a finite subcover $\{(y_i - \frac{\delta}{2}, y_i + \frac{\delta}{2}) : y_i \in f(B), 1 \leq i \leq k\}$. Take $n \in N$, such that $\frac{1}{n} < \min\{\frac{\delta_{(x_i,V)}}{2}, \frac{\delta}{2} : 1 \leq i \leq k_V, V \in \mathcal{V}\}$. Then for $x \in B$, $V \in \mathcal{V}$, there exists $U_{(x_i,V)}$ such that $x \in U_{(x_i,V)}$. Take $U^{(n)}_A$ and $g^{(n)}_A \in F'$ such that $B \subset U^{(n)}_A$ and $d(g^{(n)}_A(x), f(x)) < \frac{1}{n}$. Then $f(x) \cap (y_{(x_i,V)} - \frac{\delta_{(x_i,V)}}{2}, y_{(x_i,V)} + \frac{\delta_{(x_i,V)}}{2}) \neq \emptyset$. Thus $d(g^{(n)}_A(x), y_{(x_i,V)}) < \delta_{(x_i,V)}$, that is, $g^{(n)}_A(x) \cap V \neq \emptyset$. For each $x \in B$, we have $f(x) \subset \bigcup_{i=1}^k (y_i - \frac{\delta}{2}, y_i + \frac{\delta}{2})$, and for each $y \in g^{(n)}_A(x)$, we have $d(y, f(x)) < \frac{1}{n} < \frac{\delta}{2}$, then for each $y' \in f(x)$, there exists some $(y_i - \frac{\delta}{2}, y_i + \frac{\delta}{2})$ such that $y' \in (y_i - \frac{\delta}{2}, y_i + \frac{\delta}{2})$, so $y \in (y_i - \delta, y_i + \delta)$ and $g^{(n)}_A(x) \subset \bigcup \mathcal{V}$. Finally, we have $g^{(n)}_A \subset W[B, \mathcal{V}] \cap F'$. Thus $f \in \overline{F'}$, $t((\mathbf{C}_k(X), \mathcal{T}_k), f) \leq \lambda$ and $t(\mathbf{C}_k(X), \mathcal{T}_k) \leq \lambda$.

Corollary 2.4 *The tightness of space $(\mathbf{C}_k(X), \mathcal{T}_k)$ is countable if and only if each open k cover of X has a countable k subcover.*

If we replace the topology $\mathcal{T}_k$ of Theorem 2.3 with $\mathcal{T}_p$, and replace k cover with ω cover, then we have following results.

Theorem 2.5 *For every space X, we have $t(\mathbf{C}_k(X), \mathcal{T}_p) = pL(X)$.*

Corollary 2.6 *The tightness of space $(\mathbf{C}_k(X), \mathcal{T}_p)$ is countable if and only if each open ω cover of X has a countable ω subcover.*

Lemma 2.7[1] *For every space X, we have $\sup\{L(X^n) : n \in N\} = pL(X)$.*

Corollary 2.8 *For every space X, we have $t(\mathbf{C}_k(X), \mathcal{T}_p) = \sup\{L(X^n) : n \in N\}$.*

Corollary 2.9 *The tightness of space $(\mathbf{C}_k(X), \mathcal{T}_p)$ is countable if and only if for each $n \in N$, the Cartesian product space X^n is Lindelöf space.*

3. Fan tightness of multifunction spaces

Definition 3.1 *Let $\mathcal{Y}$ be a family of topological spaces and $\prod_{Y_\alpha \in \mathcal{Y}} Y_\alpha$ be its Tychonoff product topology. For each $Y_\alpha \in \mathcal{Y}$, let $p_{Y_\alpha} : \prod_{Y_\alpha \in \mathcal{Y}} Y_\alpha \longrightarrow Y_\alpha$ be the projection mapping. If X is a topological space, we define a topological product multifunction $T : \mathbf{M}(X, \prod_{Y_\alpha \in \mathcal{Y}} Y_\alpha) \longrightarrow \prod_{Y_\alpha \in \mathcal{Y}} \mathbf{M}(X, Y_\alpha)$ as follows: for each $f \in \mathbf{M}(X, \prod_{Y_\alpha \in \mathcal{Y}} Y_\alpha)$ and $Y_\alpha \in \mathcal{Y}$, we have*

$$p_{\mathbf{M}(X,Y_\alpha)}(T(f)) = p_{Y_\alpha}(f).$$

Theorem 3.2 *Let $\mathcal{Y}$ be a family of topological spaces and X be a topological space. Then topological product multifunction $T : \mathbf{C}_k(X, \prod_{Y_\alpha \in \mathcal{Y}} Y_\alpha) \longrightarrow \prod_{Y_\alpha \in \mathcal{Y}} \mathbf{C}_k(X, Y_\alpha)$ is a homeomorphism.*

Proof We define a mapping $T' : \prod_{Y_\alpha \in \mathcal{Y}} \mathbf{C}_k(X, Y_\alpha) \longrightarrow \mathbf{C}_k(X, \prod_{Y_\alpha \in \mathcal{Y}} Y_\alpha)$ as follows: $p_{Y_\alpha} \circ T'(g) = p_{\mathbf{C}_k(X,Y_\alpha)}(g)$, where $g \in \prod_{Y_\alpha \in \mathcal{Y}} \mathbf{C}_k(X, Y_\alpha)$, $Y_\alpha \in \mathcal{Y}$. Let $f \in \mathbf{C}_k(X, \prod_{Y_\alpha \in \mathcal{Y}} Y_\alpha)$. Then for each $Y_\alpha \in \mathcal{Y}$, we have $p_{Y_\alpha} \circ T'(T(f)) = p_{\mathbf{C}_k(X,Y_\alpha)}(T(f)) = p_{Y_\alpha}(f)$, so $T' \circ T(f) = f$. On the other hand, let $g \in \prod_{Y_\alpha \in \mathcal{Y}} \mathbf{C}_k(X, Y_\alpha)$. Then for each $Y_\alpha \in \mathcal{Y}$, we have $p_{\mathbf{C}_k(X,Y_\alpha)} \circ T(T'(g)) = p_{Y_\alpha}(T'(g)) = p_{\mathbf{C}_k(X,Y_\alpha)}(g)$, and $T \circ T'(g) = g$. Combining the two conclusions, we have $T(\mathbf{C}_k(X, \prod_{Y_\alpha \in \mathcal{Y}} Y_\alpha)) = \prod_{Y_\alpha \in \mathcal{Y}} \mathbf{C}_k(X, Y_\alpha)$. So T is a bijection. Next we prove that T is a continuous mapping.

For each $Y_\alpha \in \mathcal{Y}$, compact set $A \subset X$, and open set $U_\alpha \subset Y_\alpha$, we have

$$T^{-1}(p^{-1}_{\mathbf{C}_k(X,Y_\alpha)}(W^+[A, U_\alpha])) = W^+[A, p^{-1}_{\mathbf{C}_k(X,Y_\alpha)}(U)].$$

Similarly, $T^{-1}(p^{-1}_{\mathbf{C}_k(X,Y_\alpha)}(W^-[A, U_\alpha])) = W^-[A, p^{-1}_{\mathbf{C}_k(X,Y_\alpha)}(U_\alpha)]$. So T is continuous.

The fan tightness of a space X is defined by $ft(X) = \sup\{ft(X, x) : x \in X\}$, where the fan tightness at x of X is defined by $ft(X, x) = \aleph_0 + \min\{\lambda :$ for subset sequence $\{A_n\}$ of X and $x \in \bigcap_{n \in N} \overline{A_n}$, there exists a subset B_n of A_n such that $|B_n| \le \lambda$ and $x \in \overline{\bigcup_{n \in N} B_n}\}$.

Theorem 3.3 *For a space X, the following conditions are equivalent:*

(1) $ft(\mathbf{C}_k(X), \mathcal{T}_k) = \aleph_0$;

(2) $ft(\mathbf{C}_k^\omega(X), \mathcal{T}_k) = \aleph_0$;

(3) For each open k over sequence $\{\mathcal{U}_n\}$ of X, there exists finite subsets $\mathcal{U}'_n$ of $\mathcal{U}_n$ such that $\bigcup_{n\in N}\mathcal{U}'_n$ is a k over of X.

Proof (1) $\Rightarrow$ (3). Suppose that $\{\mathcal{U}_n\}$ is an open k cover sequence of X, for each $n \in N$, put $A_n = \{f \in (\mathbf{C}_k(X), \mathcal{T}_k)$: there exists $U \in \mathcal{U}_n$ such that $f(X-U) \subset \{0\}\}$. Next we will prove that A_n is a dense subset of $(\mathbf{C}_k(X), \mathcal{T}_k)$.

Let $W[K, V_1, V_2, \ldots, V_k] = \{f \in (\mathbf{C}_k(X), \mathcal{T}_k) : f(x) \cap V_i \neq \emptyset, x \in K, 1 \le i \le k, f(K) \subset \bigcup_{i=1}^k V_i\}$ be a base open set of $(\mathbf{C}_k(X), \mathcal{T}_k)$, where K is compact in X, $V_1, V_2, \ldots, V_k$ are open in $\mathbb{R}$. Take $U \in \mathcal{U}_n$ such that $K \subset U$ and let $f \in W[K, V_1, V_2, \ldots, V_k]$. Then $f(x) \cap V_i \neq \emptyset, x \in K, 1 \le i \le k, f(K) \subset \bigcup_{i=1}^k V_i$. For $x \in K$, by the complete regularity of X, there exists $f_x^{(i)} \in C(X)$ such that $f_x^{(i)}(x) \in f(x) \cap V_i$, $f_x^{(i)}(X-U) = \{0\}$. Thus there exists an open neighbourhood $U_x^{(i)}$ of x such that for each $x' \in U_x^{(i)}$, we have $f_x^{(i)}(x') \in V_i$. Since $\{U_x^{(i)} : x \in K\}$ covers K, there exists a finite subcover $\mathcal{V}_i = \{U_{x_j}^{(i)} : x_j \in K, 1 \le j \le j_i\}$. We define $g \in \mathbf{M}(X, \mathbb{R})$ by $g(x) = \{f_{x_j}^{(i)}(x) : 1 \le i \le k, 1 \le j \le j_i\}$. Obviously, g is point-compact and $g \in W[K, V_1, V_2, \ldots, V_k]$. Because for any $x \in K, 1 \le i \le k$, there exists $U_{x_j}^{(i)} \in \mathcal{V}_i$ such that $x \in U_{x_j}^{(i)}$, $f_{x_j}^{(i)}(U_{x_j}) \subset V_i$. Thus we have $f_{x_j}^{(i)}(x) \in V_i$, $g(x) \cap V_i \neq \emptyset$ and $g(x) \subset \bigcup_{i=1}^k V_i$. Therefore $g \in W[K, V_1, V_2, \ldots, V_k]$, and finally g is continuous. For any $x \in X$ and open set V of $\mathbb{R}$, if $g(x) \cap V \neq \emptyset$, then there exists some $f_{x_j}^{(i)}$ such that $f_{x_j}^{(i)}(x) \in V$. Because each $f_{x_j}^{(i)}$ is single-valued and continuous, there exists an open neighbourhood U_x of x such that for any $x' \in U_x$, we always have $f_{x_j}^{(i)}(x') \in V$. So $U_x \subset g^-(V)$ and $g^-(V)$ is open. For any $x \in X$, if $g(x) \subset V$, then for each $f_{x_j}^{(i)}$, we have $f_{x_j}^{(i)}(x) \in V$. So there exists an open neighbourhood $U_{x_j}^{(i)}$ of x such that for any $x' \in U_{x_j}^{(i)}$, we have $f_{x_j}^{(i)}(x') \in V$. Put $U_x = \bigcap_{1\le j\le j_i, 1\le i\le k} U_{x_j}^{(i)}$. Then $U_x \subset g^+(V)$, and $g^+(V)$ is open. Thus g is continuous. Obviously, we have $g(X-U) \subset \{0\}$. Therefore, $g \in A_n$, so $g \in A_n \cap W[K, V_1, V_2, \ldots, V_k]$.

Now we take $f_1 \equiv \{1\}$. Since each A_n is a dense subset of $(\mathbf{C}_k(X), \mathcal{T}_k)$, $f_1 \in \overline{A_n}$. So $f_1 \in \bigcap_{n\in N}\overline{A_n}$. From $ft(\mathbf{C}_k(X), \mathcal{T}_k) = \aleph_0$, there exists a finite subset B_n of A_n such that $f_1 \in \overline{\bigcup_{n\in N} B_n}$. Let $B_n = \{f_{(n,i)} : 1 \le i \le k_n\}$. Then there exists $U_{(n,i)} \in \mathcal{U}_n$ such that $f_{(n,i)}(X - U_{(n,i)}) \subset \{0\}$. Put $\mathcal{U}'_n = \{U_{(n,i)} : 1 \le i \le k_n\}$. Then $\bigcup_{n\in N}\mathcal{U}'_n$ is a k cover of X. In fact, for each compact subset K, from $f_1 \in W^+[K, (0,2)]$, there exists $n \in N, i \le k_n$ such that $f_{(n,i)} \in W^+[K, (0,2)]$. Hence $K \subset U_{(n,i)}$.

(3) $\Rightarrow$ (2). By Theorem 3.2, $(\mathbf{C}_k^\omega(X, \mathbb{R}), \mathcal{T}_k)$ is homeomorphic to $(\mathbf{C}_k(X, \mathbb{R}^\omega), \mathcal{T}_k)$. Next we prove that $ft(\mathbf{C}_k(X, \mathbb{R}^\omega), \mathcal{T}_k) = \aleph_0$. Put $f \in \bigcap_{n\in N}\overline{A_n}$, where $A_n \subset (\mathbf{C}_k(X, \mathbb{R}^\omega), \mathcal{T}_k)$. Let $\mathcal{U}$ be a countable base of $\mathbb{R}$ and $\mathcal{U}^\sharp$ be a finite subfamily of $\mathcal{U}$. Then $\{\mathcal{U}^\sharp : \mathcal{U}^\sharp \subset \mathcal{U}\}$ is also countable. The base of hyper-space $K(\mathbb{R})$ of compact sets of $\mathbb{R}$ is of all the sets of $\langle\mathcal{U}^\sharp\rangle = \{K \in K(R) : K \subset \bigcup\mathcal{U}^\sharp, K \cap U \neq \emptyset, U \in \mathcal{U}^\sharp\}$. Thus the base of compact subset hyper-space $(K(\mathbb{R}))^\omega$ of product space $\mathbb{R}^\omega$ is of all the sets of $\langle\Pi_{n\in N}\langle\mathcal{U}_{(n,1)}^\sharp\rangle, \Pi_{n\in N}\langle\mathcal{U}_{(n,2)}^\sharp\rangle, \ldots, \Pi_{n\in N}\langle\mathcal{U}_{(n,k)}^\sharp\rangle\rangle$, where $k \in N, \mathcal{U}_{(n,i)}^\sharp (1 \le i \le k)$ are $\langle\mathbb{R}\rangle$ except finite n. So the base of $(K(\mathbb{R}))^\omega$ is countable

and we denote it by $\{V_n : n \in N\}$, where V_n is some element in the base. Put $\mathcal{V}_n = \{\{x : g(x) \in V_n\} : g \in A_n\}$. Then $\mathcal{V}_n$ is an open k cover of X. Because for any compact $A \subset X$, any base neighbourhood $W[A, V_n] = \{g \in \mathbf{C}_k(X, \mathbb{R}^\omega) : g(x) \in V_n, x \in A\}$ of f, there exists $g \in W[A, V_n] \cap A_n$ such that $A \subset \{x : g(x) \in V_n\} \in \mathcal{V}_n$. Put $N_1 = \{n \in N : X \in \mathcal{V}_n\}$. If N_1 is infinite, then for any neighbourhood $W[A, \langle U_1, U_2, \ldots, U_k\rangle]$, because of the compactness of $f(A)$, there exists a base open set $V_m (m \in N_1)$ of $(K(\mathbb{R}))^\omega$ such that for each $x \in A$, we have $f(x) \in V_m \subset \langle U_1, U_2, \ldots, U_k\rangle$, and there exists $g_m \in A_m$ such that $g_m(X) \in V_m$. Thus $g_m \in W[A, \langle U_1, U_2, \ldots, U_k\rangle]$, therefore $\{g_m : m \in N_1\}$ converge to f. If N_1 is finite, then there exists $n_0 \in N$ such that when $m \geq n_0$, for $g_m \in A_m$, we have $X \neq \{x : g_m(x) \in V_m\}$. But $\{\mathcal{V}_m : m \geq n_0\}$ is an open k cover sequence of X. Then there exists a finite subset $\mathcal{V}'_m$ of $\mathcal{V}_m$ such that $\bigcup_{m \geq n_0} \mathcal{V}'_m$ is an open k cover of X. Let $\mathcal{V}'_m = \{U_{(m,j)} : j \leq i_m\}$. Then there exists $f_{(m,j)} \in A_m$ such that $U_{(m,j)} = \{x : f_{(m,j)}(x) \in V_m\}$. Then $f \in \overline{\{f_{(m,j)} : m \geq n_0, j \leq i_m\}}$. Because for any base neighbourhood $W[A, \langle U_1, U_2, \ldots, U_k\rangle]$ of f, we put $N_2 = \{(m,j) \in N \times N : m \geq n_0, j \leq i_m, A \subset U_{(m,j)}\}$. If N_2 is finite, then for $(m,j) \in N_2$, $U_{(m,j)} \neq X$. We can take $x_{(m,j)} \in X - U_{(m,j)}$ and $A' = A \bigcup \{x_{(m,j)} : (m,j) \in N_2\}$. Then there exists no element in $\bigcup_{m \geq n_0} \mathcal{V}'_m$ containing A', a contradiction. So N_2 is infinite, and there exists $m \geq n_0, j \leq i_m$ such that $A \subset U_{(m,j)} = \{x : f_{(m,j)}(x) \in V_m\}$ and $V_m \subset \langle U_1, U_2, \ldots, U_k\rangle$. Thus for each $x \in A$, we have $f_{(m,j)}(x) \in \langle U_1, U_2, \ldots, U_k\rangle$, so $f_{(m,j)} \in W[A, \langle U_1, U_2, \ldots, U_k\rangle]$, and $f \in \overline{\{f_{(m,j)} : m \geq n_0, j \leq i_m\}}$.

(2) $\Rightarrow$ (1). Because $(\mathbf{C}_k(X), \mathcal{T}_k)$ is a closed subspace of $(\mathbf{C}_k^\omega(X), \mathcal{T}_k)$, the conclusion is obvious.

By Theorem 2.1, $(C(X), \mathcal{T}_p)$ and $(C(X), \mathcal{T}_k)$ are subspaces of $(\mathbf{C}_k(X), \mathcal{T}_p)$ and $(\mathbf{C}_k(X), \mathcal{T}_k)$, respectively. So the results in this paper are right to continuous single-valued function spaces $(C(X), \mathcal{T}_p)$ and $(C(X), \mathcal{T}_k)$.

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