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On a Version of Rosenthal's Inequality for Locally Square Integrable Martingales

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Abstract In this paper, we study the constants in a version of Rosenthal's inequality for locally square integrable martingales. We prove that the order of growth rates of the constants is the same as in the case of discrete time martingales.

Keywords locally square integrable martingale; Garsia's lemma; L_p inequality; Rosenthal's inequality; order of growth rate.

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1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables and $S_n = \sum_{i=1}^n X_i$. Rosenthal[1] established the following inequality:

$$\begin{aligned}
&c_p^{-1}\{[(\sum_{i=1}^n EX_i^2)^{p/2}]^{1/p} + (\sum_{i=1}^n E|X_i|^p)^{1/p}\} \leq (E|S_n|^p)^{1/p} \\
&\leq C_p\{[(\sum_{i=1}^n EX_i^2)^{p/2}]^{1/p} + (\sum_{i=1}^n E|X_i|^p)^{1/p}\}, \quad p \geq 2, \qquad (1)
\end{aligned}$$

where c_p and C_p are positive constants depending only on p. This inequality is a generalization of Khintchine's inequality and is known as Rosenthal's inequality. Rosenthal's inequality is now a fundamental inequality with wide applications in probability and statistics. Recently, a lot of attention has been given to the best constants or growth rates of the constants appearing in various Rosenthal's inequalities. In 1973, Burkholder[2] extended Rosenthal's inequality to discrete time martingales and gave the following result.

Let $\{f_n, \mathcal{F}_n, n \geq 0\}$ be a discrete time martingale, $f_0 = 0$. Then for $p \geq 2$, the following Rosenthal's inequality holds:

$$\begin{aligned}
&d_p^{-1}\{(E(\sum_{i=1}^n E[d_i^2/\mathcal{F}_{i-1}])^{p/2})^{1/p} + (E[\sup_{1\leq i\leq n}|d_i|]^p)^{1/p}\} \leq (E[\sup_{1\leq i\leq n}|f_i|]^p)^{1/p} \\
&\leq D_p\{(E(\sum_{i=1}^n E[d_i^2/\mathcal{F}_{i-1}])^{p/2})^{1/p} + (E[\sup_{1\leq i\leq n}|d_i|]^p)^{1/p}\}, \qquad (2)
\end{aligned}$$

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where $\{d_n = f_n - f_{n-1}, n \geq 1\}$ is the martingale difference sequence with respect to filtration $\{\mathcal{F}_n, n \geq 0\}$, d_p and D_p are positive constants depending only on p.

As for the the growth rates of constants, Rosenthal's proof yielded only exponential of p estimate for the growth rate of C_p (for the best constant) as $p \to \infty$. Burkholder did not give the growth rate of D_p either, but from the proof one can show that $D_p \leq Kp(\ln p)^{\varepsilon}$, where K is an absolute constant and ε is any positive number. In 1985, Johnson, Schechtman and Zinn[3] showed that the actual growth rate of C_p is $p/\ln p$. This is somewhat unexpected since the growth rate in Khintchine's inequality is $\sqrt{p}$ (for the best constant). Talagrand extended this result to the case of independent Banach space valued random variables. In 1990, Hitczenko[4] proved that for discrete-time martingales d_p grows like $\sqrt{p}$ and D_p grows like $p/\ln p$ as $p \to \infty$, and the growth rate of the constants is best possible. Rosenthal's inequality for locally square integrable martingales was first established by Dzhaparidze and Valkeila in 1990. We shall work within the framework of general martingale theory and use the standard notions of general theory of stochastic processes, thus we consider martingales with cadalag paths. This class of course includes discrete-time martingales, but is much larger. We introduce some notations and conventions concerning martingales needed for the proofs of our results.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $(\mathcal{F}_t)_{t\geq 0}$ be a filtration satisfying usual conditions on $(\Omega, \mathcal{F}, P)$. Denote by $M = \{M_t, t \geq 0\}$ a locally square integrable martingales based on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. $[M]$ is the quadratic variation of M and $\langle M \rangle$ is the predictable quadratic variation of M. We write

$$M_t^* = \sup_{s\leq t} |M_s|, \quad (\triangle M)_t^* = \sup_{s\leq t} |\triangle M_s|.$$

$$\triangle M_\tau = M_\tau I(\tau < \infty) - M_{\tau^-} I(\tau < \infty),$$

$$M^\tau = (M_{t\wedge\tau})_{t\geq 0}, \quad M^{\tau^-} = M^\tau - \triangle M_\tau I([[\tau, \infty))).$$

$\mu = \mu((0,t] \times B) = \sum_{s\leq t} I\{\triangle M_s \in B\}$ $(B \in \mathcal{B}(R), t \geq 0)$ is the jump measure of M, and ν is the compensating random measure of μ. Denote

$$|x|^p * \mu_t = \int_{[0,\ t]\times B} |x|^p \mathrm{d}\mu, \quad t \geq 0,$$

$$|x|^p * \nu_t = \int_{[0,\ t]\times B} |x|^p \mathrm{d}\nu, \quad t \geq 0.$$

In 1990, Dzhaparidze and Valkeila[5] established the following different versions of Rosenthal's inequality for locally square integrable martingales

$$\begin{aligned} &A_p^{-1}\{E[\langle M\rangle_T^{p/2} + (\triangle M)_T^{*p}]\}^{1/p} \leq [E(M_T^*)^p]^{1/p} \\ &\quad \leq B_p\{E[\langle M\rangle_T^{p/2} + (\triangle M)_T^{*p}]\}^{1/p}, \quad p \geq 2. \end{aligned} \tag{3}$$

$$\begin{aligned} &A_p^{-1}\{E[\langle M\rangle_T^{p/2} + |x|^p * \nu_T]\}^{1/p} \leq [E(M_T^*)^p]^{1/p} \\ &\quad \leq B_p\{E[\langle M\rangle_T^{p/2} + |x|^p * \nu_T]\}^{1/p}\}, \quad p \geq 2. \end{aligned} \tag{4}$$

Where A_p and B_p are positive constants depending only on p. Dzhaparidze and Valkeila could not give the growth rates of A_p and B_p, they only pointed out that $A_p \to \infty$, $B_p \to \infty$, when

$p \to \infty$. Wood studied the Rosenthal's inequality for point process martingales and proved that A_p and B_p appearing in (4) are the same constants as in the discrete-time case for point process martingales and marked point process martingales. In 2003, Ren and Tian[6] showed that for general locally square integrable martingales the constants A_p and B_p appearing in (3) have the same orders as in (2) , i.e., $A_p = O(\sqrt{p})$, $B_p = O(p/\ln p)$.

In this paper, we prove that the orders of A_p and B_p appearing in (4) for general locally square integrable martingales are the same as that for the discrete-time case too.

2. Main result

We need the following Garsia's lemma for the proof of our theorem.

Lemma (Garsia's lemma)[7] *Let $A = \{A_t, t \geq 0\}$ be an adapted increasing process, Φ a moderate increasing and convex function on R_+, $\Phi(0) = 0$, ξ and η be nonnegative integrable random variables, and $E[\Phi(\xi)] < \infty$, $\xi \geq A_\infty$, a.s., $\xi \in \mathcal{F}_\infty$, if one of the two conditions is satisfied:*

(a) For any stoping time T

$$E(\xi/\mathcal{F}_T) - A_{T^-} I(T > 0) \leq E(\eta/\mathcal{F}_T) \quad \text{a.s.}.$$

(b) A is predictable, $A_0 = 0$, and for any predictable time T

$$E(\xi/\mathcal{F}_T) - A_T \leq E(\eta/\mathcal{F}_T) \quad \text{a.s.}.$$

Then $E[\Phi(\xi)] \leq \rho^{\rho+1} E[\Phi(\eta)]$. If taking $\Phi(t) = t^p$ $(1 < p < \infty)$, we have better inequality $E(\xi^p) \leq p^p E(\eta^p)$.

Theorem 1 *Let $M = \{M_t, t \geq 0\}$ be a locally square integrable martingale with $M_0 = 0$. Then for $p \geq 2$, the following Rosenthal's inequality holds:*

$$A_p^{-1}\{E[\langle M\rangle_T^{p/2} + |x|^p * \nu_T]\}^{1/p} \leq [E(M_T^*)^p]^{1/p} \leq B_p\{E[\langle M\rangle_T^{p/2} + |x|^p * \nu_T]\}^{1/p}\} \tag{5}$$

for any stopping time T, and $A_p = O(\sqrt{p})$, $B_p = O(p/\ln p)$.

Proof For any stopping time T, define

$$\widetilde{M}_t = (M_{T+t} - M_{T^-} I(T > 0)) I(T < \infty), \quad \mathcal{G}_t = \mathcal{F}_{T+t}, \quad t \geq 0.$$

Then $\widetilde{M} = \{\widetilde{M}_t, \mathcal{G}_t, t \geq 0\}$ is a locally square integrable martingale with $\widetilde{M}_0 = 0$, and we have

$$[\widetilde{M}]_t = ([M]_{T+t} - [M]_{T^-}) I(T > 0), \quad t \geq 0.$$

$$\widetilde{M}^*_\infty \leq M^*_\infty + M^*_{T^-} \leq 2M^*_\infty.$$

By Burkholder-Davis inequality

$$E[\widetilde{M}]_\infty \leq E(\widetilde{M}^*_\infty)^2. \tag{6}$$

We get

$$E([M]_\infty - [M]_{T^-}) \leq E(2M^*_\infty)^2. \tag{7}$$

Since for any set $A \in \mathcal{F}_T$, $\widetilde{M}I_A = \{\widetilde{M}_t I_A, \mathcal{G}_t, t \geq 0\}$ is still a locally square integrable martingale with $\widetilde{M}_0 I_A = 0$, we have

$$E([\widetilde{M}]_\infty/\mathcal{F}_T) \leq E(\widetilde{M}^*_\infty)^2/\mathcal{F}_T) \quad \text{a.s..} \tag{8}$$

So that

$$E([M]_\infty/\mathcal{F}_T) - [M]_{T^-} I(T > 0) \leq E(2M^*_\infty)^2/\mathcal{F}_T) \quad \text{a.s..} \tag{9}$$

Set $A = [M]$, $\xi = [M]_\infty$, $\eta = (2M^*_\infty)^2$. Taking $\Phi(t) = t^{p/2}$ $(2 < p < \infty)$, from Garsia's lemma, we obtain

$$E[M]^{p/2}_\infty \leq (2p)^{p/2} E(M^*_\infty)^p. \tag{10}$$

Since, for $p > 2$

$$E(|x|^p * \nu_\infty) = E(|x|^p * \mu_\infty) = E[\sum_{t<\infty} \mid \Delta M_t \mid^p \ ] \leq E[M]^{p/2}_\infty \leq (2p)^{p/2} E(M^*_\infty)^p,$$

and by the similar approach, we can prove

$$E\langle M\rangle^{p/2}_\infty \leq (2p)^{p/2} E(M^*_\infty)^p. \tag{11}$$

Hence, we obtain

$$E(\langle M\rangle^{p/2}_\infty + |x|^p * \nu_\infty) \leq 2(2p)^{p/2} E(M^*_\infty)^p. \tag{12}$$

For any stopping time T, replace M by stopping martingale M^T, we get

$$E(\langle M\rangle^{p/2}_T + |x|^p * \nu_T) \leq 2(2p)^{p/2} E(M^*_T)^p. \tag{13}$$

Thus the left-hand side inequality of (5) follows with $A_p = (2^{\frac{1}{p}}\sqrt{2p}) \leq 2\sqrt{p}$. Since for discrete-time martingales, $d_p = O(\sqrt{p})$, we have $A_p = O(\sqrt{p})$, and the order of growth rates A_p is best possible.

On the other hand, since $E[(\triangle M)^{*p}_T] \leq E[\sum_{t<\infty} \mid \Delta M_t \mid^p \ ] = E(|x|^p * \mu_\infty) = E(|x|^p * \nu_\infty)$, we have

$$[E(M^*_T)^p]^{1/p} \leq B_p\{E[\langle M\rangle^{p/2}_T + (\triangle M)^{*p}_T]\}^{1/p} \leq B_p\{E[\langle M\rangle^{p/2}_T + |x|^p * \nu_T]\}^{1/p}\}.$$

From [6], $B_p = O(p/\ln p)$.

Thus, we proved that the order of growth rates of the constants A_p and B_p are the same as in the case of discrete-time martingales.

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