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Drazin Invertibility of Operators AB and BA

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Abstract In this note an alternative proof of the equivalence of Drazin invertibility of operators AB and BA is given. As an application, we will prove that $\sigma_D(AB)=\sigma_D(BA)$ and $\sigma_D(A)=\sigma_D(\widetilde{A})$, where $\sigma_D(M)$ and $\widetilde{M}$ denote the Drazin spectrum and the Aluthge transform of an operator $M\in\mathcal{B}(\mathcal{H})$, respectively.

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1. Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space with the inner product $\langle .\,,\,.\rangle$ and let $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For $A\in B(\mathcal{H})$, the range, the null space, the adjoint, spectrum and its isolated points of spectrum of A are denoted by $R(A)$, $N(A)$, $A^*\sigma(A)$ and iso$\sigma(A)$, respectively. As is well known, that AB is invertible does not imply that BA is invertible for A and $B\in B(\mathcal{H})$.

Example 1.1 Let S be the unilateral shift operator on $\mathcal{H}$. Then $S^*S=I$ is invertible, but SS^* is not invertible.

Similarly, that AB is Moore-Penrose invertible does not imply that BA is Moore-Penrose invertible.

Example 1.2 Let

$$A=\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix} \text{ and } B=\begin{pmatrix} 0 & 0\\ C & 0\end{pmatrix},$$

with respect to the space decomposition $\mathcal{H}=\mathcal{M}\oplus\mathcal{M}^\perp$, where the range $R(C)$ of operator $C(\in\mathcal{B}(\mathcal{M},\mathcal{M}^\perp))$ is not closed. Then

$$AB=\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix} \text{ and } BA=\begin{pmatrix} 0 & 0\\ C & 0\end{pmatrix}.$$

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This shows that $R(AB) = \{0\}$ is closed and $R(BA) = R(C)$ is not closed. Recall that an operator T is Moore-Penrose invertible if and only if the range $R(T)$ of T is closed. So AB is Moore-Penrose invertible, but BA is not.

Two examples above encourage us to consider whether AB is Drazin invertible implies BA is Drazin invertible. In this note, we shall give a proof of the equivalence between Drazin invertibility of AB and BA by Riesz functional calculus which is different from the proof given by Dajic and Koliha in [1]. Recall that an operator $A \in B(\mathcal{H})$ is called Drazin invertible if there exists an operator $A^D \in B(\mathcal{H})$ such that

$$AA^D = A^D A,\ A^D A A^D = A^D,\ A^{k+1} A^D = A^k,$$

for some nonnegative integer k. In this case, the least positive integer k satisfying the operator equation $A^{k+1}A^D = A^k$ is called the Drazin index of A and denote it by $\mathrm{ind}(A) = k$. For $A \in B(\mathcal{H})$, if A is Drazin invertible, then the Drazin inverse A^D of A is unique[2]. The ascent (descent) of $A \in B(\mathcal{H})$ is defined by the smallest integer n such that $N(A^n) = N(A^{n+1})(R(A^n) = R(A^{n+1}))$, that is, $\mathrm{asc}(A) = \min\{n : N(A^n) = N(A^{n+1})\}(\mathrm{des}(A) = \min\{n : R(A^n) = R(A^{n+1})\}$. If such n does not exist, then $\mathrm{asc}(A) = \infty$ $(\mathrm{des}(A) = \infty)$. It is well known that $\mathrm{des}(A) = \mathrm{asc}(A)$ if $\mathrm{asc}(A)$ and $\mathrm{des}(A)$ are finite[3] and A is Drazin invertible if and only if the ascent and the descent of A are both finite, equivalently, 0 is a finite order pole of the resolvent operator $R_\lambda(A) = (\lambda I - A)^{-1}$. In this case, $\mathrm{asc}(A) = \mathrm{des}(A) = \mathrm{ind}(A)$ and 0 is not the accumulated point of $\sigma(A)$[4]. Clearly, an invertible operator A is Drazin invertible with $A^D = A^{-1}$ and $\mathrm{ind}(A) = 0$. A nilpotent operator B is Drazin invertible and $B^D = 0$.

Similar to the difinition of the spectrum of an operator in $\mathcal{B}(\mathcal{H})$, we shall define the Drazin spectrum $\sigma_D(A)$ of an operator $A \in \mathcal{B}(\mathcal{H})$ by

$$\sigma_D(A) = \{\lambda : A - \lambda I \text{ is not Drazin invertible}\}.$$

For the spetrum of AB and BA, we have $\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$, but in general case, $\sigma(AB) \neq \sigma(BA)$. However, in this note we show that $\sigma_D(AB) = \sigma_D(BA)$. As an application, we will prove that an operator A is Drazin invertible if and only if its Aluthge transform $\widetilde{A}$ is Drazin invertible. Let $A = V \mid A \mid$ be a polar decomposition of A, where $\mid A \mid = (A^*A)^{\frac{1}{2}}$ and V is a partial isometry with initial space $\overline{R(A^*)}$ and the final space $\overline{R(A)}$. Then the Aluthge transform $\widetilde{A}$ of A is defined by $\widetilde{A} = \mid A \mid^{\frac{1}{2}} V \mid A \mid^{\frac{1}{2}}$[5].

2. Main results and proofs

We begin with some lemmas.

Lemma 2.1[6] *Let A and $B \in B(\mathcal{H})$. Then $\sigma(AB)\setminus\{0\} = \sigma(BA)\setminus\{0\}$.*

Lemma 2.2[2,3] *Let $A \in B(\mathcal{H})$. Then following statements are equivalent:*

(1) *A is Drazin invertible with $\mathrm{ind}(A) = k$;*

(2) *If $0 \in \sigma(A)$, then 0 is an isolated point of $\sigma(A)$, $A|_{E(\{0\})}$ is nilpotent and $(A|_{E(\{0\})\mathcal{H}})^k = 0$,*

where $E(0)$ is the spectral projection of A according to $\{0\}$ which is defined by

$$E(0) = \frac{1}{2\pi i}\int_{\Gamma}(\lambda I - A)^{-1}\mathrm{d}\lambda,$$

where Γ is a positively oriented curve such that $\{0\} \in \mathrm{ins}\Gamma$ and $\sigma(A)\backslash\{0\} \in \mathrm{out}\Gamma$ and $T|_M$ is the restriction of $T \in B(\mathcal{H})$ onto a subspace $M \subset H$.

The main result in this note is

Theorem 2.3 *Let $A, B \in B(\mathcal{H})$. Then the following statements hold:*

(1) AB is Drazin invertible if and only if BA is Drazin invertible.

(2) If AB is Drazin invertible, then $|\mathrm{ind}(AB) - \mathrm{ind}(BA)| \leq 1$.

Proof By the symmetry, it suffices to prove that AB is Drazin invertible implies that BA is Drazin invertible.

For convenience, we divide the proof into two steps:

Step 1. Suppose $0 \notin \sigma(AB)$, i.e., AB is invertible, then AB is Drazin invertible and $\mathrm{ind}(AB) = 0$. In this case, if $0 \notin \sigma(BA)$, i.e., BA is invertible, then BA is Drazin invertible and $\mathrm{ind}(BA) = 0$; if $0 \in \sigma(BA)$, then 0 is an isolated point of $\sigma(BA)$ by Lemma 2.1 and the assumption that $0 \notin \sigma(AB)$. To prove that BA is Drazin invertible, it suffices to show that $BA|_{P_{BA}(0)\mathcal{H}}$ is nilpotent. Here we shall prove that $P_{BA}(0)BAP_{BA}(0) = 0$. Recall that the Riesz projection $P_{AB}(0)$ of AB according to 0 $A|_{E(\{0\})}$ is nilpotent and $(A|_{E(\{0\})\mathcal{H}})^k = 0$, where $E(0)$ is the spectral projection of A according to $\{0\}$ is defined by

$$P_{AB}(0) = \frac{1}{2\pi i}\int_{\Gamma}(\lambda I - AB)^{-1}\mathrm{d}\lambda,$$

where Γ is a positively oriented curve such that $\{0\} \in \mathrm{ins}\Gamma$ and $\sigma(A)\backslash\{0\} \in \mathrm{out}\Gamma$. Since AB is invertible, $P_{AB}(0) = 0$. Moreover, we have

$$\begin{aligned} P_{BA}(0)BAP_{BA}(0) &= (\frac{1}{2\pi i}\int_{\Gamma}(\lambda - BA)^{-1}\mathrm{d}\lambda)BAP_{BA}(0) \\ &= B(\frac{1}{2\pi i}\int_{\Gamma}(\lambda - AB)^{-1}\mathrm{d}\lambda)AP_{BA}(0) \\ &= BP_{AB}(0)AP_{BA}(0) = 0. \end{aligned}$$

So BA is Drazin invertible with $\mathrm{ind}(BA) \leq 1$ by Lemma 2.2.

Step 2. Suppose $0 \in \sigma(AB)$. Since AB is Drazin invertible, we have $0 \in \mathrm{iso}\sigma(AB)$ by Lemma 2.2. In this case, if $0 \notin \sigma(BA)$, i.e., BA is invertible, then BA is Drazin invertible and $\mathrm{ind}(BA) = 0$. From Step 1, we have $\mathrm{ind}(AB) \leq 1$. If $0 \in \sigma(BA)$, then $0 \in \mathrm{iso}\sigma(BA)$ by the assumption $0 \in \mathrm{iso}\sigma(AB)$ and Lemma 2.1. Moreover, suppose $\mathrm{ind}(AB) = k_0$. Then $(AB\,|_{P_{AB}(0)}\,\mathcal{H})^{k_0} = 0$, i.e., $(P_{AB}(0)ABP_{AB}(0))^{k_0} = 0$, so $P_{AB}(0)(AB)^{k_0} = 0$. To prove that BA is Drazin invertible, here we shall prove that $(P_{BA}(0)BAP_{BA}(0))^{k_0+1} = 0$. But

$$\begin{aligned} (P_{BA}(0)BAP_{BA}(0))^{k_0+1} &= P_{BA}(0)(BA)^{k_0+1} \\ &= BP_{AB}(0)(AB)^{k_0}A = 0. \end{aligned}$$

So BA is Drazin invertible with $\mathrm{ind}(BA) \leq k_0 + 1$.

(2) By the proof of (1), it is clear that $|\mathrm{ind}(AB) - \mathrm{ind}(BA)| \leq 1$, if AB is Drazin invertible. The proof is completed. $\square$

Remark 2.4 In fact, if AB is Drazin invertible, then $(AB)^D = \int_\Gamma \lambda^{-1}(\lambda I - AB)^{-1}\mathrm{d}\lambda$ and so $(AB)^D A = A(BA)^D$, where Γ is a positively oriented curve such that $\{0\} \in \mathrm{ins}\Gamma$ and $\sigma(A)\backslash\{0\} \in \mathrm{out}\Gamma$.

Next we give an example to show there exist two operators A and B such that $|\mathrm{ind}(AB) - \mathrm{ind}(BA)| = 1$.

Example 2.5 Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

be 2×2 matrices. Then

$$AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

In this case, $\mathrm{ind}(AB) = 2$, $\mathrm{ind}(BA) = 1$ and so $|\mathrm{ind}(AB) - \mathrm{ind}(BA)| = 1$.

In [7], Buoni and Faires proved that the Drazin invertibility of operators $\lambda I - BA$ and $\lambda I - AB$ is equivalent for any $\lambda \neq 0$ and $\mathrm{ind}(\lambda I - AB) = \mathrm{ind}(\lambda I - BA)(\lambda \neq 0)$. Together with Theorem 2.3 and the definition of Drazin spectrum of an operator, we get the following

Corollary 2.6 *Let A, $B \in \mathcal{B}(\mathcal{H})$. Then $\sigma_D(AB) = \sigma_D(BA)$.*

Recall that an operator A and its Aluthge transform $\widetilde{A}$ share many common properties[8]. Moreover, by Corollary 2.6 and the definition of Aluthge transform of an operator, we also have the following

Corollary 2.7 *Let $A \in B(\mathcal{H})$. Then $\sigma_D(A) = \sigma_D(\widetilde{A})$.*

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