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Maximum Likelihood Estimation of Ratios of Means and Standard Deviations from Normal Populations with Different Sample Numbers under Semi-Order Restriction

SHI Hai Fang, LI Shu You, JI Yong Gang
(Department of Mathematics and Physics, Liaoning University of Technology, Liaoning 121001, China)
(E-mail: lnshuyouli@163.com)

Abstract For two normal populations with unknown means μ_i and variances $\sigma_i^2 > 0$, $i = 1, 2$, assume that there is a semi-order restriction between ratios of means and standard deviations and sample numbers of two normal populations are different. A procedure of obtaining the maximum likelihood estimators of μ_i's and σ_i's under the semi-order restrictions is proposed. For $i = 3$ case, some connected results and simulations are given.

Keywords semi-order restriction; maximum likelihood estimator; likelihood function; PAVA algorithm.

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1. Introduction

The study about maximum likelihood estimation of means and variances from normal populations has long been a topic of interest to statisticians.

Let X_{ij}, $j = 1, 2, \ldots, n_i$ be observations normally distributed with unknown means μ_i and known variances σ_i^2, $i = 1, 2, \ldots, k$. Assume that there is simple semi-order restriction between means, namely

$$\mu_1 \le \mu_2 \le \cdots \le \mu_k. \tag{1.1}$$

Barlow et al[2] gave an algorithm of finding the maximum likelihood estimators of μ_i's under the semi-order restriction (1.1). He used the Pool Adjacent Violators Algorithm (PAVA) that Ayer et al. gave in [1]. If means are known, variances are unknown and restricted by the following simple semi-orders

$$\sigma_1^2 \ge \sigma_2^2 \ge \cdots \ge \sigma_k^2 > 0. \tag{1.2}$$

Robertson[3] gave an algorithm of obtaining the maximum likelihood estimators of σ_i^2's under the semi-order restriction (1.2). For normal populations with unknown means μ_i and variances $\sigma_i^2 > 0$, $i = 1, 2, \ldots, k$, assume $\mu = (\mu_1, \mu_2, \ldots, \mu_k)'$ and $\sigma^2 = (\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2)'$ are restricted by (1.1) and (1.2) respectively. Shi Ningzhong[4] discussed some properties of maximum likelihood

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estimators of μ_i's and σ_i^2's under the semi-order restriction (1.1) and (1.2), and gave an algorithm of getting the maximum likelihood estimators of μ_i's and σ_i^2's under restrictions.

For two normal populations with unknown means μ_i and variances $\sigma_i^2 > 0$, $i = 1, 2$, this paper considers the maximum likelihood estimators of μ_i's and σ_i^2's under the following restriction

$$\frac{\mu_1}{\sigma_1} \leq \frac{\mu_2}{\sigma_2} \tag{1.3}$$

when sample numbers are different. For $i = 3$ case, some connected results and simulations are given in Section 2.

This problem will always be met in the product quality testing, stock investment, biomedical testing and so on.

2. Main results

Let X_{ij}, $j = 1, 2, \ldots, n_i$, be observations normally distributed with unknown means μ_i and variances σ_i^2, $i = 1, 2$. The likelihood function is

$$L(X, \mu_1, \mu_2, \sigma_1, \sigma_2) = (2\pi)^{-\frac{n_1+n_2}{2}} \sigma_1^{-n_1} \sigma_2^{-n_2} \exp\Big\{ -\frac{n_1}{2}[\frac{S_1^2}{\sigma_1^2} + (\frac{\bar{X}_1}{\sigma_1} - \lambda_1)^2] - \frac{n_2}{2}[\frac{S_2^2}{\sigma_2^2} + (\frac{\bar{X}_2}{\sigma_2} - \lambda_2)^2] \Big\},$$

where $\bar{X}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij}$, $S_i^2 = \frac{1}{n_i}\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_i)^2$, $\lambda_i = \frac{\mu_i}{\sigma_i}$, $i = 1, 2$.

Theorem 1 *If $\frac{\bar{X}_1}{S_1} \leq \frac{\bar{X}_2}{S_2}$, then the maximum likelihood estimators of μ_i's and σ_i^2's under the restriction (1.3) are $\hat{\mu}_i = \bar{X}_i$ and $\hat{\sigma}_i^2 = S_i^2$, respectively, $i = 1, 2$.*

Proof The log-likelihood function is

$$\ln L = -\frac{n_1+n_2}{2}\ln(2\pi) - n_1 \ln\sigma_1 - n_2 \ln\sigma_2 - \frac{n_1}{2}[\frac{S_1^2}{\sigma_1^2} + (\frac{\bar{X}_1}{\sigma_1} - \frac{\mu_1}{\sigma_1})^2] - \frac{n_2}{2}[\frac{S_2^2}{\sigma_2^2} + (\frac{\bar{X}_2}{\sigma_2} - \frac{\mu_2}{\sigma_2})^2].$$

Let $\frac{\partial \ln L}{\partial \sigma_1} = 0$. We get

$$\frac{S_1^2}{\sigma_1^2} + \frac{(\bar{X}_1 - \mu_1)^2}{\sigma_1^2} = 1, \tag{2.1}$$

similarly,

$$\frac{S_2^2}{\sigma_2^2} + \frac{(\bar{X}_2 - \mu_2)^2}{\sigma_2^2} = 1. \tag{2.2}$$

We also get

$$\frac{\partial \ln L}{\partial \mu_1} = n_1(\frac{\bar{X}_1}{\sigma_1} - \frac{\mu_1}{\sigma_1})(-\frac{1}{\sigma_1}).$$

Let $\frac{\partial \ln L}{\partial \mu_1} = 0$. We obtain

$$\hat{\mu}_1 = \bar{X}_1, \tag{2.3}$$

similarly,

$$\hat{\mu}_2 = \bar{X}_2. \tag{2.4}$$

Substituting (2.3), (2.4) into (2.1) and (2.2), respectively, we have

$$\hat{\sigma}_1^2 = S_1^2, \quad \hat{\sigma}_2^2 = S_2^2.$$

Theorem 2 *If $\frac{\bar{X}_1}{S_1} > \frac{\bar{X}_2}{S_2}$, then the maximum likelihood estimators of μ_i's and σ_i^2's under the restriction (1.3) are*

(a) If $\bar{X}_1\bar{X}_2 > 0$, then

$$\hat{\mu}_1 = \frac{\bar{X}_1}{c_1} + \frac{\bar{X}_2}{c_2}\frac{a_1c_1}{a_2c_2}\frac{\sqrt{m_2-\sqrt{\Delta}}}{\sqrt{m_1-\sqrt{\Delta}}}, \quad \hat{\mu}_2 = \frac{\bar{X}_2}{c_2} + \frac{\bar{X}_1}{c_1}\frac{a_2c_2}{a_1c_1}\frac{\sqrt{m_1-\sqrt{\Delta}}}{\sqrt{m_2-\sqrt{\Delta}}},$$

$$\hat{\sigma}_1^2 = \frac{m_2-\sqrt{\Delta}}{2a_2^2c_2^2}, \qquad \hat{\sigma}_2^2 = \frac{m_1-\sqrt{\Delta}}{2a_1^2c_1^2}.$$

(b) If $\bar{X}_1\bar{X}_2 < 0$, then

$$\hat{\mu}_1 = \frac{\bar{X}_1}{c_1} + \frac{\bar{X}_2}{c_2}\frac{a_1c_1}{a_2c_2}\frac{\sqrt{m_2+\sqrt{\Delta}}}{\sqrt{m_1+\sqrt{\Delta}}}, \quad \hat{\mu}_2 = \frac{\bar{X}_2}{c_2} + \frac{\bar{X}_1}{c_1}\frac{a_2c_2}{a_1c_1}\frac{\sqrt{m_1+\sqrt{\Delta}}}{\sqrt{m_2+\sqrt{\Delta}}},$$

$$\hat{\sigma}_1^2 = \frac{m_2+\sqrt{\Delta}}{2a_2^2c_2^2}, \qquad \hat{\sigma}_2^2 = \frac{m_1+\sqrt{\Delta}}{2a_1^2c_1^2}$$

respectively, where $c_1 = \frac{n_1+n_2}{n_1}$, $c_2 = \frac{n_1+n_2}{n_2}$, $a_1^2 = c_2S_1^2 + \bar{X}_1^2$, $a_2^2 = c_1S_2^2 + \bar{X}_2^2$,

$$m_1 = 2a_1^2a_2^2c_1 + c_2\bar{X}_1^2\bar{X}_2^2 - c_1\bar{X}_1^2\bar{X}_2^2, \; m_2 = 2a_1^2a_2^2c_2 + c_1\bar{X}_1^2\bar{X}_2^2 - c_2\bar{X}_1^2\bar{X}_2^2,$$

$$\Delta = 4a_1^2a_2^2c_1c_2\bar{X}_1^2\bar{X}_2^2 + (c_1-c_2)^2\bar{X}_1^4\bar{X}_2^4.$$

Proof From the Pool Adjacent Violators Algorithm, as $\frac{\bar{X}_1}{S_1} > \frac{\bar{X}_2}{S_2}$, the maximum likelihood estimator of μ_i's and σ_i^2's under the restriction (1.3) is the unique solution $\lambda = \lambda_1 = \lambda_2 = \frac{n_1}{n_1+n_2}\frac{\mu_1}{\sigma_1} + \frac{n_2}{n_1+n_2}\frac{\mu_2}{\sigma_2}$ such that the likelihood function reaches the maximal value. The likelihood function is

$$L(X,\lambda,\sigma_1,\sigma_2) = (2\pi)^{\frac{n_1+n_2}{2}}\sigma_1^{-n_1}\sigma_2^{-n_2}\exp\Big\{-\frac{n_1}{2}[\frac{S_1^2}{\sigma_1^2} + (\frac{\bar{X}_1}{\sigma_1}-\lambda)^2] - \frac{n_2}{2}[\frac{S_2^2}{\sigma_2^2} + (\frac{\bar{X}_2}{\sigma_2}-\lambda)^2]\Big\}.$$

Let $\frac{\partial \ln L}{\partial \sigma_1} = 0$. We obtain

$$\frac{S_1^2}{\sigma_1^2} + (\frac{\bar{X}_1}{\sigma_1}-\lambda)(\frac{\bar{X}_1}{\sigma_1} - \frac{n_1}{n_1+n_2}\frac{\mu_1}{\sigma_1}) - \frac{n_2}{n_1+n_2}\frac{\mu_1}{\sigma_1}(\frac{\bar{X}_2}{\sigma_2}-\lambda) = 1 \tag{2.5}$$

similarly,

$$\frac{S_2^2}{\sigma_2^2} + (\frac{\bar{X}_2}{\sigma_2}-\lambda)(\frac{\bar{X}_2}{\sigma_2} - \frac{n_2}{n_1+n_2}\frac{\mu_2}{\sigma_2}) - \frac{n_1}{n_1+n_2}\frac{\mu_2}{\sigma_2}(\frac{\bar{X}_1}{\sigma_1}-\lambda) = 1. \tag{2.6}$$

Let $\frac{\partial \ln L}{\partial \mu_1} = 0$. We obtain

$$\lambda = \frac{n_1}{n_1+n_2}\frac{\bar{X}_1}{\sigma_1} + \frac{n_2}{n_1+n_2}\frac{\bar{X}_2}{\sigma_2}. \tag{2.7}$$

Substituting (2.7) into (2.5) and (2.6), respectively, we get

$$c_2\sigma_1^2\sigma_2 - a_1^2\sigma_2 + \sigma_1\bar{X}_1\bar{X}_2 = 0, \tag{2.8}$$

$$c_1\sigma_2^2\sigma_1 - a_2^2\sigma_1 + \sigma_2\bar{X}_1\bar{X}_2 = 0, \tag{2.9}$$

where $c_1 = \frac{n_1+n_2}{n_1}$, $c_2 = \frac{n_1+n_2}{n_2}$, $a_1^2 = c_2 S_1^2 + \bar{X}_1^2$, $a_2^2 = c_1 S_2^2 + \bar{X}_2^2$.

Calculate above equations, we obtain

(a) If $\bar{X}_1\bar{X}_2 > 0$, then

$$\hat{\sigma}_1^2 = \frac{m_2 - \sqrt{\Delta}}{2a_2^2c_2^2}, \quad \hat{\sigma}_2^2 = \frac{m_1 - \sqrt{\Delta}}{2a_1^2c_1^2}$$

and

$$\hat{\lambda} = \frac{\bar{X}_1}{c_1\hat{\sigma}_1} + \frac{\bar{X}_2}{c_2\hat{\sigma}_2} = \frac{\bar{X}_1}{c_1}\frac{\sqrt{2}a_2c_2}{\sqrt{m_2-\sqrt{\Delta}}} + \frac{\bar{X}_2}{c_2}\frac{\sqrt{2}a_1c_1}{\sqrt{m_1-\sqrt{\Delta}}}.$$

$$\hat{\mu}_1 = \hat{\sigma}_1\hat{\lambda} = \frac{\bar{X}_1}{c_1} + \frac{\bar{X}_2}{c_2}\frac{a_1c_1}{a_2c_2}\frac{\sqrt{m_2-\sqrt{\Delta}}}{\sqrt{m_1-\sqrt{\Delta}}}, \quad \hat{\mu}_2 = \hat{\sigma}_2\hat{\lambda} = \frac{\bar{X}_2}{c_2} + \frac{\bar{X}_1}{c_1}\frac{a_2c_2}{a_1c_1}\frac{\sqrt{m_1-\sqrt{\Delta}}}{\sqrt{m_2-\sqrt{\Delta}}}$$

(b) If $\bar{X}_1\bar{X}_2 < 0$, then

$$\hat{\sigma}_1^2 = \frac{m_2 + \sqrt{\Delta}}{2a_2^2c_2^2}, \quad \hat{\sigma}_2^2 = \frac{m_1 + \sqrt{\Delta}}{2a_1^2c_1^2},$$

$$\hat{\mu}_1 = \hat{\sigma}_1\hat{\lambda} = \frac{\bar{X}_1}{c_1} + \frac{\bar{X}_2}{c_2}\frac{a_1c_1}{a_2c_2}\frac{\sqrt{m_2+\sqrt{\Delta}}}{\sqrt{m_1+\sqrt{\Delta}}}, \ \hat{\mu}_2 = \hat{\sigma}_2\hat{\lambda} = \frac{\bar{X}_2}{c_2} + \frac{\bar{X}_1}{c_1}\frac{a_2c_2}{a_1c_1}\frac{\sqrt{m_1+\sqrt{\Delta}}}{\sqrt{m_2+\sqrt{\Delta}}},$$

where $\Delta = 4a_1^2a_2^2c_1c_2\bar{X}_1^2\bar{X}_2^2 + (c_1-c_2)^2\bar{X}_1^4\bar{X}_2^4$,

$$m_1 = 2a_1^2a_2^2c_1 + c_2\bar{X}_1^2\bar{X}_2^2 - c_1\bar{X}_1^2\bar{X}_2^2, \ m_2 = 2a_1^2a_2^2c_2 + c_1\bar{X}_1^2\bar{X}_2^2 - c_2\bar{X}_1^2\bar{X}_2^2.$$

For three normal populations with unknown means μ_i and variances $\sigma_i^2 > 0$, $i = 1, 2, 3$, by applying the results of Theorems 1 and 2, the maximum likelihood estimators of μ_i's and σ_i's under the following restriction

$$\frac{\mu_1}{\sigma_1} \le \frac{\mu_2}{\sigma_2} \le \frac{\mu_3}{\sigma_3} \tag{2.10}$$

are given as follows.

(1) If $\frac{\bar{X}_1}{S_1} \le \frac{\bar{X}_2}{S_2} \le \frac{\bar{X}_3}{S_3}$, then $\hat{\mu}_i = \bar{X}_i$ and $\hat{\sigma}_i = S_i$, $i = 1, 2, 3$.

(2) If $\frac{\bar{X}_1}{S_1} > \frac{\bar{X}_2}{S_2}$, then

(a) If $\bar{X}_1\bar{X}_2 > 0$ and $\frac{\bar{X}_1}{c_1}\frac{\sqrt{2}a_2c_2}{\sqrt{m_2-\sqrt{\Delta}}} + \frac{\bar{X}_2}{c_2}\frac{\sqrt{2}a_1c_1}{\sqrt{m_1-\sqrt{\Delta}}} \le \frac{\bar{X}_3}{S_3}$, then

$$\hat{\mu}_1 = \frac{\bar{X}_1}{c_1} + \frac{\bar{X}_2}{c_2}\frac{a_1c_1}{a_2c_2}\frac{\sqrt{m_2-\sqrt{\Delta}}}{\sqrt{m_1-\sqrt{\Delta}}}, \ \hat{\mu}_2 = \frac{\bar{X}_2}{c_2} + \frac{\bar{X}_1}{c_1}\frac{a_2c_2}{a_1c_1}\frac{\sqrt{m_1-\sqrt{\Delta}}}{\sqrt{m_2-\sqrt{\Delta}}}, \ \hat{\mu}_3 = \bar{X}_3,$$

$$\hat{\sigma}_1^2 = \frac{m_2-\sqrt{\Delta}}{2a_2^2c_2^2}, \ \hat{\sigma}_2^2 = \frac{m_1-\sqrt{\Delta}}{2a_1^2c_1^2}, \ \hat{\sigma}_3^2 = S_3^2.$$

(b) If $\bar{X}_1\bar{X}_2 < 0$ and $\frac{\bar{X}_1}{c_1}\frac{\sqrt{2}a_2c_2}{\sqrt{m_2+\sqrt{\Delta}}} + \frac{\bar{X}_2}{c_2}\frac{\sqrt{2}a_1c_1}{\sqrt{m_1+\sqrt{\Delta}}} \le \frac{\bar{X}_3}{S_3}$, then

$$\hat{\mu}_1 = \frac{\bar{X}_1}{c_1} + \frac{\bar{X}_2}{c_2}\frac{a_1c_1}{a_2c_2}\frac{\sqrt{m_2+\sqrt{\Delta}}}{\sqrt{m_1+\sqrt{\Delta}}}, \ \hat{\mu}_2 = \frac{\bar{X}_2}{c_2} + \frac{\bar{X}_1}{c_1}\frac{a_2c_2}{a_1c_1}\frac{\sqrt{m_1+\sqrt{\Delta}}}{\sqrt{m_2+\sqrt{\Delta}}}, \ \hat{\mu}_3 = \bar{X}_3,$$

$$\hat{\sigma}_1^2 = \frac{m_2+\sqrt{\Delta}}{2a_2^2c_2^2}, \ \hat{\sigma}_2^2 = \frac{m_1+\sqrt{\Delta}}{2a_1^2c_1^2}, \ \hat{\sigma}_3^2 = S_3^2.$$

(3) If $\frac{\bar{X}_2}{S_2} > \frac{\bar{X}_3}{S_3}$, then

(a) If $\bar{X}_2\bar{X}_3 > 0$ and $\frac{\bar{X}_1}{S_1} \le \frac{\bar{X}_2}{c_2}\frac{\sqrt{2}a_3c_3}{\sqrt{m_4-\sqrt{\Delta'}}} + \frac{\bar{X}_3}{c_3}\frac{\sqrt{2}a_2c_2}{\sqrt{m_3-\sqrt{\Delta'}}}$, then

$$\hat{\mu}_1 = \bar{X}_1,\ \hat{\mu}_2 = \frac{\bar{X}_2}{c_2} + \frac{\bar{X}_3}{c_3}\frac{a_2c_2}{a_3c_3}\frac{\sqrt{m_4-\sqrt{\Delta'}}}{\sqrt{m_3-\sqrt{\Delta'}}},\ \hat{\mu}_3 = \frac{\bar{X}_3}{c_3} + \frac{\bar{X}_2}{c_2}\frac{a_3c_3}{a_2c_2}\frac{\sqrt{m_3-\sqrt{\Delta'}}}{\sqrt{m_4-\sqrt{\Delta'}}},$$

$$\hat{\sigma}_1^2 = S_1^2,\ \hat{\sigma}_2^2 = \frac{m_4-\sqrt{\Delta'}}{2a_3^2c_3^2},\ \hat{\sigma}_3^2 = \frac{m_3-\sqrt{\Delta'}}{2a_2^2c_2^2}.$$

(b) If $\bar{X}_2\bar{X}_3 < 0$ and $\frac{\bar{X}_1}{S_1} \le \frac{\bar{X}_2}{c_2}\frac{\sqrt{2}a_3c_3}{\sqrt{m_4+\sqrt{\Delta'}}} + \frac{\bar{X}_3}{c_3}\frac{\sqrt{2}a_2c_2}{\sqrt{m_3+\sqrt{\Delta'}}}$, then

$$\hat{\mu}_1 = \bar{X}_1,\ \hat{\mu}_2 = \frac{\bar{X}_2}{c_2} + \frac{\bar{X}_3}{c_3}\frac{a_2c_2}{a_3c_3}\frac{\sqrt{m_4+\sqrt{\Delta'}}}{\sqrt{m_3+\sqrt{\Delta'}}},\ \hat{\mu}_3 = \frac{\bar{X}_3}{c_3} + \frac{\bar{X}_2}{c_2}\frac{a_3c_3}{a_2c_2}\frac{\sqrt{m_3+\sqrt{\Delta'}}}{\sqrt{m_4+\sqrt{\Delta'}}},$$

$$\hat{\sigma}_1^2 = S_1^2,\ \hat{\sigma}_2^2 = \frac{m_4+\sqrt{\Delta'}}{2a_3^2c_3^2},\ \hat{\sigma}_3^2 = \frac{m_3+\sqrt{\Delta'}}{2a_2^2c_2^2},$$

where $\Delta' = 4a_2^2a_3^2c_2c_3\bar{X}_2^2\bar{X}_3^2 + (c_2-c_3)^2\bar{X}_2^4\bar{X}_3^4$,

$$m_3 = 2a_2^2a_3^2c_2 + c_3\bar{X}_2^2\bar{X}_3^2 - c_2\bar{X}_2^2\bar{X}_3^2,\ m_4 = 2a_2^2a_3^2c_3 + c_2\bar{X}_2^2\bar{X}_3^2 - c_3\bar{X}_2^2\bar{X}_3^2.$$

(4) (a) If $\frac{\bar{X}_1}{S_1} > \frac{\bar{X}_2}{S_2}$ and $\frac{\bar{X}_1}{c_1}\frac{\sqrt{2}a_2c_2}{\sqrt{m_2-\sqrt{\Delta}}} + \frac{\bar{X}_2}{c_2}\frac{\sqrt{2}a_1c_1}{\sqrt{m_1-\sqrt{\Delta}}} > \frac{\bar{X}_3}{S_3}$(If $\bar{X}_1\bar{X}_2 > 0$) or $\frac{\bar{X}_1}{c_1}\frac{\sqrt{2}a_2c_2}{\sqrt{m_2+\sqrt{\Delta}}} + \frac{\bar{X}_2}{c_2}\frac{\sqrt{2}a_1c_1}{\sqrt{m_1+\sqrt{\Delta}}} > \frac{\bar{X}_3}{S_3}$ (If $\bar{X}_1\bar{X}_2 < 0$).

(b) If $\frac{\bar{X}_2}{S_2} > \frac{\bar{X}_3}{S_3}$ and $\frac{\bar{X}_1}{S_1} > \frac{\bar{X}_2}{c_2}\frac{\sqrt{2}a_3c_3}{\sqrt{m_4-\sqrt{\Delta'}}} + \frac{\bar{X}_3}{c_3}\frac{\sqrt{2}a_2c_2}{\sqrt{m_3-\sqrt{\Delta'}}}$ (If $\bar{X}_2\bar{X}_3 > 0$) or $\frac{\bar{X}_1}{S_1} > \frac{\bar{X}_2}{c_2}\frac{\sqrt{2}a_3c_3}{\sqrt{m_4+\sqrt{\Delta'}}} + \frac{\bar{X}_3}{c_3}\frac{\sqrt{2}a_2c_2}{\sqrt{m_3+\sqrt{\Delta'}}}$ (If $\bar{X}_2\bar{X}_3 < 0$).

(c) If $\frac{\bar{X}_1}{S_1} > \frac{\bar{X}_2}{S_2} > \frac{\bar{X}_3}{S_3}$, we can get the maximum likelihood estimators of μ_i's and σ_i^2's under the restriction (2.10) in the following way.

From the Pool Adjacent Violators Algorithm, as one of (4) holds, the maximum likelihood estimator of ratios of means μ_i's and deviations σ_i^2's under the restriction (2.10) is the unique solution $\lambda^* = \frac{1}{n_1+n_2+n_3}(n_1\frac{\mu_1}{\sigma_1} + n_2\frac{\mu_2}{\sigma_2} + n_3\frac{\mu_3}{\sigma_3})$ such that the likelihood function reaches the maximal value. The likelihood function is

$$L(X,\lambda^*,\sigma_1,\sigma_2,\sigma_3) = (2\pi)^{-\frac{n_1+n_2+n_3}{2}}\sigma_1^{-n_1}\sigma_2^{-n_2}\sigma_3^{-n_3}\exp\Big\{-\frac{n_1}{2}[\frac{S_1^2}{\sigma_1^2} + (\frac{\bar{X}_1}{\sigma_1}-\lambda^*)^2] - \frac{n_2}{2}[\frac{S_2^2}{\sigma_2^2} + (\frac{\bar{X}_2}{\sigma_2}-\lambda^*)^2] - \frac{n_3}{2}[\frac{S_3^2}{\sigma_3^2} + (\frac{\bar{X}_3}{\sigma_3}-\lambda^*)^2]\Big\}.$$

From the equations $\frac{\partial \ln L}{\partial \sigma_i} = 0$, $i = 1,2,3$, and $\frac{\partial \ln L}{\partial \lambda^*} = 0$, we get the following equations

$$\sigma_1^2 + \sigma_1\bar{X}_1\lambda^* - b_1^2 = 0,$$
$$\sigma_2^2 + \sigma_2\bar{X}_2\lambda^* - b_2^2 = 0,$$
$$\sigma_3^2 + \sigma_3\bar{X}_3\lambda^* - b_3^2 = 0,$$
$$\lambda^* = \frac{1}{n_1+n_2+n_3}(n_1\frac{\bar{X}_1}{\sigma_1} + n_2\frac{\bar{X}_2}{\sigma_2} + n_3\frac{\bar{X}_3}{\sigma_3}),$$

where $b_i^2 = \bar{X}_i^2 + S_i^2$, $i = 1,2,3$. We can get the equality

$$\frac{b_1^2-\sigma_1^2}{\sigma_1\bar{X}_1} = \frac{b_2^2-\sigma_2^2}{\sigma_2\bar{X}_2} = \frac{b_3^2-\sigma_3^2}{\sigma_3\bar{X}_3} = \frac{1}{n_1+n_2+n_3}(n_1\frac{\bar{X}_1}{\sigma_1} + n_2\frac{\bar{X}_2}{\sigma_2} + n_3\frac{\bar{X}_3}{\sigma_3}).$$

It is very difficult to show concrete solution of the above equation. At first, we can give the value $\hat{\lambda}^* = \frac{1}{n_1+n_2+n_3}(n_1\frac{\bar{X}_1}{S_1} + n_2\frac{\bar{X}_2}{S_2} + n_3\frac{\bar{X}_3}{S_3})$ and obtain $\hat{\sigma}_1$, $\hat{\sigma}_2$ and $\hat{\sigma}_3$, considering the value of the following equality.

$$I = \Big|\frac{b_1^2 - \hat{\sigma}_1^2}{\hat{\sigma}_1\bar{X}_1} - \frac{1}{n_1+n_2+n_3}(n_1\frac{\bar{X}_1}{\hat{\sigma}_1} + n_2\frac{\bar{X}_2}{\hat{\sigma}_2} + n_3\frac{\bar{X}_3}{\hat{\sigma}_3})\Big|$$

$\hat{\sigma}_i > 0$ and $\hat{\mu}_i = \hat{\sigma}_i\hat{\lambda}^*$, $i = 1, 2, 3$, are the maximum likelihood estimators of ratios of means μ_i's and standard deviations σ_i's under the restriction (2.10) as I approaches zero.

The simulations of the iterative algorithm are as follows.

We only give the simulated data with MATLAB[6] when $\frac{\bar{X}_1}{S_1} > \frac{\bar{X}_2}{S_2} > \frac{\bar{X}_3}{S_3}$, as shown in Table 1. Where $X_1 \sim N(2,1)$, $X_2 \sim N(3,4)$, $X_3 \sim N(4,9)$, $c_i = \frac{\hat{\mu}_i}{\hat{\sigma}_i}$, $i = 1, 2, 3$. As for the other two cases in (4), similar method can be used to obtain the parallel results.

(n_1, n_2, n_3)	(1000, 1500, 2000)			(10000, 15000, 20000)			$(100000, 150000, 200000)$		
Accuracy	0.01	0.001	0.0001	0.01	0.001	0.0001	0.01	0.001	0.0001
$\hat{\mu}_1$	1.8095	1.7922	1.7904	1.7927	1.7948	1.8047	1.7905	1.7908	1.7903
$\hat{\mu}_2$	3.0045	2.9773	3.0560	3.0085	3.0109	3.0075	3.0203	3.0191	3.0175
$\hat{\mu}_3$	4.2902	4.2419	4.3402	4.2757	4.2469	4.2408	4.2510	4.2505	4.2511
$\hat{\sigma}_1$	1.1941	1.2058	1.1826	1.1849	1.1917	1.2041	1.1796	1.1938	1.1808
$\hat{\sigma}_2$	1.9826	2.0031	2.0185	1.9885	1.9992	2.0066	1.9898	2.0127	2.0071
$\hat{\sigma}_3$	2.8310	2.8539	2.8667	2.8260	2.8189	2.8295	2.8006	2.8336	2.8277
$c_1 = c_2 = c_3$	1.5154	1.4863	1.5140	1.5130	1.5060	1.4888	1.5179	1.5001	1.5034
Iteration Frequency	3	7	10	3	7	11	3	7	11

Table 1 Simulations of Three Normal Populations

It is obvious to see that iteration results obtained satisfy the semi-order restriction (2.10). Moreover, the convergence speed of the algorithm is very quick.

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