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A Class of Standard Bases of Polynomial Algebras and Its Applications

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Abstract Let $H=\mathcal{U}_q(\mathrm{sl}(2))$ or $\mathcal{U}(\mathrm{sl}(2))$. By means of the standard basis of polynomial algebras, the Clebsch-Gordan formula and quantum Clebsch-Gordan formula are proved by a unified method, and the explicit formula of the decomposition of $V(1)^{\otimes n}$ into the direct sum of simple modules is given in this paper.

Keywords standard basis; quantum group; Clebsch-Gordan formula; Grothendieck ring.

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1. Introduction

Quantum group or quantum enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ is a one-parameter deformation of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a semisimple finite dimensional Lie algebra $\mathfrak{g}$, introduced by Drinfeld[1,2], Jimbo[3] and Kulish-Reshetikhin[4] in their study of the quantum Yang-Baxter equation. The simplest and most important example is the Drinfeld-Jimbo quantum group $\mathcal{U}_q(\mathrm{sl}(2))$, which will be investigated in this paper.

Let k be an algebraically closed field with characteristic 0, and $q\in k^*=k\backslash 0$ be not a root of unitary. We write $H=\mathcal{U}_q(\mathrm{sl}(2))$, which is an associative algebra generated by variables E,F,K,K^{-1} with relations $KK^{-1}=K^{-1}K=1$, $KEK^{-1}=q^2E$, $KFK^{-1}=q^{-2}F$, $[E,F]=\frac{K-K^{-1}}{q-q^{-1}}$. The following relations endow H with a Hopf algebra structure.

$$\Delta(K)=K\otimes K, S(K)=K^{-1}, \varepsilon(K)=\varepsilon(K^{-1})=1;$$

$$\Delta(E)=E\otimes K+1\otimes E, S(E)=-EK^{-1}, \varepsilon(E)=0;$$

$$\Delta(F)=F\otimes 1+K^{-1}\otimes F, S(F)=-KF, \varepsilon(F)=0.$$

Let V be a left H-module and $\lambda\in k^*$. An element $v\neq 0$ of V is a highest weight vector of weight λ if $Ev=0$ and if $Kv=\lambda v$. V is a highest weight module of highest weight λ if it is generated by a highest weight vector of weight λ. Now it is well known that: (1) Any simple

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finite dimensional H-module is generated by a highest weight vector; (2) Two finite dimensional H-modules generated by highest weight vectors of the same weight are isomorphic; (3) Any finite dimensional H-module is semisimple. Let $V(n)$ denote the unique $(n+1)$-dimensional simple H-module generated by a highest weight vector of weight q^n. It is known that the tensor product of two finite dimensional H-modules can be decomposed into the direct sum of simple modules. By the distributivity of the tensor product with respect to direct sums, it is enough to decompose $V(n)\otimes V(m)$ into the sum of simple modules in order to decompose the tensor product of any two finite dimensional modules into the sum of simple modules. Thanks to quantum Clebsch-Gordan formula, $V(n)\otimes V(m)$ is isomorphic to $\oplus_{i=0}^{m}V(n+m-2i)$ as left H-module, where n and m are nonnegative integers with $n\geqslant m$. For the enveloping algebra $\mathcal{U}(\mathrm{sl}(2))$ of sl(2), we have the similar results. The main aim of the present paper is to determine all coefficients in the decomposition of $V(1)^{\otimes n}$.

We organize this paper as follows. In Section 2, we prove the main theorem concerning the standard basis of an associative algebra. This is stated in Theorem 2.1 which generalizes the property of Grothendieck algebra of $\mathcal{U}_q(\mathrm{sl}(2))$. The notion of Grothendieck algebra is recalled in Section 4. It is interesting to us that Theorem 2.1 provides the foundation of the whole paper. The role of Section 3 is a preparation for determining the coefficients in the decomposition of $V(1)^{\otimes n}$. Two important combinatorial formulae are obtained in this section. Main results appear in Section 4. Together with the results in previous sections and the property of Grothendieck algebra of $\mathcal{U}_q(\mathrm{sl}(2))$, we obtain the unified proof of quantum Clebsch-Gordan formula and Clebsch-Gordan formula, and the explicit formula of the decomposition of $V(1)^{\otimes n}$ into the direct sum of simple modules. In addition, we prove the commutativity of tensor product of two H-modules without using the braided condition of H.

In this paper, tensor product will be over k. We refer the reader to [5,6] for basic results about Hopf algebras, and to [7, 8] for basic results about $\mathcal{U}_q(\mathrm{sl}(2))$ and $\mathcal{U}(\mathrm{sl}(2))$.

2. Standard bases of polynomial algebras

In this section, we prove some basic results. The proofs are very elementary, but the results are exciting.

Theorem 2.1 *Let A be any associative algebra, $\{e_0, e_1, \ldots, e_n, \ldots\}$ be a set of linearly independent elements in A satisfying*

$$e_0e_0 = e_0,\ e_ne_1 = e_1e_n = e_{n+1} + e_{n-1},$$

where $n\in\mathbb{N}$, $e_{-1}=0$. Then we have

(1) If $n\geqslant m$, then

$$e_me_n = e_ne_m = \sum_{i=0}^{m} e_{n+m-2i}\,.$$

(2) As algebras $k[e_1]\cong k[x]$, where $k[x]$ is the polynomial algebra over k in one variable x and $k[e_1]$ is the subalgebra of A generated by e_1.

Proof (1) We use induction on both m and n. Firstly, we show that $e_0e_n = e_ne_0 = e_n$ for all $n \in \mathbb{N}$. It is easy to see that the result holds when $n = 0, 1$. Now suppose $n > 1$ and that $e_te_0 = e_t$ for all $t \leqslant n-1$. On the one hand, $e_{n-1}e_1 = e_n + e_{n-2}$. On the other hand, $e_{n-1}e_1 = (e_{n-1}e_1)e_0 = (e_n + e_{n-2})e_0 = e_ne_0 + e_{n-2}$, so $e_ne_0 = e_n$. In a similar way, we have $e_0e_n = e_n$. Thus $e_0e_n = e_ne_0 = e_n$ for all $n \in \mathbb{N}$.

Secondly, suppose $m > 0$ and that the following holds for all $t \leqslant m-1$,

$$e_te_n = e_ne_t = \sum_{i=0}^{t} e_{n+t-2i}.$$

When $n \geqslant m$, we consider $(e_ne_{m-1})e_1$. On the one hand, we have

$$\begin{aligned}(e_ne_{m-1})e_1 &= e_n(e_{m-1}e_1) = e_n(e_m + e_{m-2}) = e_ne_m + e_ne_{m-2} \\ &= e_ne_m + \sum_{i=0}^{m-2} e_{n+m-2i-2} = e_ne_m + e_{n+m-2} + \cdots + e_{n-(m-2)}.\end{aligned}$$

On the other hand, we get

$$(e_ne_{m-1})e_1 = \Big(\sum_{i=0}^{m-1} e_{n+m-1-2i}\Big)e_1 = e_{n+m} + 2\sum_{i=1}^{m-1} e_{n+m-2i} + e_{n-m}.$$

Comparing the above two identities, we have

$$e_ne_m = \sum_{i=0}^{m} e_{n+m-2i}.$$

Similarly, from the equation $e_1(e_{m-1}e_n) = (e_1e_{m-1})e_n$, one obtains

$$e_me_n = \sum_{i=0}^{m} e_{n+m-2i}.$$

This shows Part (1).

(2) From Part (1), one can show by induction that $e_0, e_1, e_1^2, \ldots, e_1^n, \ldots$ can be written as linear combinations of $e_0, e_1, \ldots, e_n, \ldots$. On the other hand, using induction on n, it is easy to see that $e_0, e_1, \ldots, e_n, \ldots$ can also be written as linear combinations of $e_0, e_1, e_1^2, \ldots, e_1^n, \ldots$. Thus $e_0, e_1, e_1^2, \ldots, e_1^n, \ldots$ are linearly independent. It follows that subalgebra $k[e_1]$ generated by e_1 is isomorphic to the polynomial algebra $k[x]$. □

Definition 2.2 *Let $\{e_0, e_1, \ldots, e_n, \ldots\}$ be a basis of an associative algebra A. It is called a standard basis of A if it satisfies $e_ie_j = \sum_{s\leqslant i+j} a_se_s$ with $a_s \in \mathbb{N}$.*

By Definition 2.2, $\{1, x, x^2, \ldots, x^n, \ldots\}$ is clearly a standard basis for $k[x]$. In addition, if we take a basis $\{e_0, e_1, \ldots, e_n, \ldots\}$ of A as stated in Theorem 2.1, then $\{e_0, e_1, \ldots, e_n, \ldots\}$ is the standard basis of A. The following theorem shows that there is a standard basis in $k[x]$ satisfying Theorem 2.1(1).

Theorem 2.3 *Let $k[x]$ be a polynomial algebra over k. Then there is a standard basis*

$\{e_0, e_1, \ldots, e_n, \ldots\}$ *satisfying*

$$e_m e_n = e_n e_m = \sum_{i=0}^{m} e_{n+m-2i}$$

for all $n \geqslant m \geqslant 0$.

Proof By Theorem 2.1, it suffices to find a basis $\{e_0, e_1, \ldots, e_n, \ldots\}$ satisfying

$$e_0 e_0 = 1,\ e_1 e_n = e_n e_1 = e_{n+1} + e_{n-1},$$

for all $n \in \mathbb{N}$. Using induction on n, we take $e_0 = 1, e_1 = x, e_2 = x^2 - 1 = e_1^2 - e_0$. Obviously, they satisfy $e_1 e_n = e_n e_1 = e_{n+1} + e_{n-1}$, for all $n \leqslant 1$. Suppose that we have obtained e_t which is a monic polynomial of e_1 of degree t and satisfies $e_1 e_{t-1} = e_{t-1} e_1 = e_t + e_{t-2}$ for all $t \leqslant n-1$. Then we take $e_n = e_{n-1} e_1 - e_{n-2}$. By induction, e_n is a monic polynomial of e_1 of degree n. Following the construction and Theorem 2.1(2), $\{e_0, e_1, \ldots, e_n, \ldots\}$ is a basis for $k[x]$. This completes the proof. $\square$

Proposition 2.4 *Let* $k[x]$ *be a polynomial algebra over* k. *Then* $\left\{ \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n-i}{i} x^{n-2i} \,|\, n \in \mathbb{N} \right\}$ *is exactly the standard basis constructed in Theorem 2.3, where* $\lfloor \frac{n}{2} \rfloor$ *is the largest integer which does not exceed* $\frac{n}{2}$.

Proof Let $\{e_n\}$ be the standard basis in $k[x]$ as constructed in the proof of Theorem 2.3. Then it is easy to see that

$$e_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i T_n^i x^{n-2i} \quad \text{for some } T_n^i \in \mathbb{N}.$$

When $n = 2k$, $k \in \mathbb{N}$, we have

$$\begin{aligned}
e_n e_1 - e_{n-1} &= e_{2k} e_1 - e_{2k-1} \\
&= \sum_{i=0}^{\lfloor \frac{2k}{2} \rfloor} (-1)^i T_{2k}^i x^{2k-2i} \cdot x - \sum_{i=0}^{\lfloor \frac{2k-1}{2} \rfloor} (-1)^i T_{2k-1}^i x^{2k-2i-1} \\
&= \sum_{i=0}^{k} (-1)^i T_{2k}^i x^{2k-2i+1} + \sum_{i=0}^{k-1} (-1)^{i+1} T_{2k-1}^i x^{2k-2i-1} \\
&= \sum_{i=0}^{k} (-1)^i T_{2k}^i x^{2k-2i+1} + \left(0 + \sum_{i=0}^{k-1} (-1)^{i+1} T_{2k-1}^i x^{2k-2i-1}\right) \\
&= \sum_{i=0}^{k} (-1)^i T_{2k}^i x^{2k-2i+1} + \sum_{i=0}^{k} (-1)^i T_{2k-1}^{i-1} x^{2k-2i+1} \\
&= \sum_{i=0}^{k} (-1)^i \left(T_{2k}^i + T_{2k-1}^{i-1}\right) x^{2k-2i+1} \\
&= \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^i \left(T_n^i + T_{n-1}^{i-1}\right) x^{n+1-2i},
\end{aligned}$$

where $T_{2k-1}^{-1} = 0$.

When $n=2k-1$, $k\in\mathbb{N}$, a similar argument shows that

$$e_ne_1-e_{n-1}=\sum_{i=0}^{\lfloor\frac{n+1}{2}\rfloor}(-1)^i\left(T_n^i+T_{n-1}^{i-1}\right)x^{n+1-2i},$$

where $T_{2k-2}^{-1}=0$ and $T_{2k-1}^{k}=0$.

Using the fact that $e_{n+1}=e_ne_1-e_{n-1}$, we have

$$\sum_{i=0}^{\lfloor\frac{n+1}{2}\rfloor}(-1)^iT_{n+1}^ix^{n+1-2i}=\sum_{i=0}^{\lfloor\frac{n+1}{2}\rfloor}(-1)^i\left(T_n^i+T_{n-1}^{i-1}\right)x^{n+1-2i}.$$

for all $n\in\mathbb{N}$. Comparing the coefficients, we can obtain $T_{n+1}^i=T_n^i+T_{n-1}^{i-1}$, where $0\leqslant i\leqslant\lfloor\frac{n+1}{2}\rfloor$, $T_{n-1}^{-1}=0$, and $T_n^{\lfloor\frac{n+1}{2}\rfloor}=0$ when n is odd.

We claim that

$$T_n^i=\frac{1}{i!}\prod_{s=i}^{2i-1}(n-s)=\frac{(n-i)!}{i!(n-2i)!}=\binom{n-i}{i},$$

where $0\leqslant i\leqslant\lfloor\frac{n}{2}\rfloor$.

We will prove the above assertion by induction on n. It is obvious that the assertion is true for $n=0,1$. We suppose $m\geqslant 1$ and that the assertion is true for all $i,n\in\mathbb{N}$ with $0\leqslant n\leqslant m$ and $0\leqslant i\leqslant\lfloor\frac{n}{2}\rfloor$. Now we consider all $i\in\mathbb{N}$ with $0\leqslant i\leqslant\lfloor\frac{m+1}{2}\rfloor$. If $i=0$, then $T_{m+1}^0=T_m^0+T_{m-1}^{-1}=T_m^0=\binom{m}{0}=\binom{m+1}{0}$. Now suppose $1\leqslant i\leqslant\lfloor\frac{m+1}{2}\rfloor$. If $i>\lfloor\frac{m}{2}\rfloor$, then m is odd and $i=\lfloor\frac{m+1}{2}\rfloor=\frac{m+1}{2}$, and consequently $T_{m+1}^i=T_m^i+T_{m-1}^{i-1}=T_{m-1}^{i-1}=\binom{m-i}{i-1}=1=\binom{m+1-i}{i}$. If $1\leqslant i\leqslant\lfloor\frac{m}{2}\rfloor$, then

$$T_{m+1}^i=T_m^i+T_{m-1}^{i-1}=\binom{m-i}{i}+\binom{m-i}{i-1}=\binom{m+1-i}{i}.$$

This completes the proof. □

3. Two combinatorial formulae

Following Theorem 2.1, we obtain two important combinatorial formulae which will be used in Section 4. Taking $\{e_n|n\in\mathbb{N}\}$ as in Theorem 2.1, one can get the following two equations for all $n\in\mathbb{N}$:

$$e_1^{2n}=\sum_{i=0}^{n}A_n^ie_{2n-2i},\quad\text{for all } n\geqslant 1,\tag{3.1}$$

$$e_1^{2n+1}=\sum_{i=0}^{n}B_n^ie_{2n+1-2i},\quad\text{for all } n\geqslant 0,\tag{3.2}$$

where $A_n^i,B_n^i\in\mathbb{N}$,

Proposition 3.1 *Let A_n^i and B_n^i be given as the above, where $i,n\in\mathbb{N}$ with $0\leqslant i\leqslant n$. Put $A_0^0=1$ and $A_n^i=B_n^i=0$ when $i<0$ or $i>n$. Then we have*

(1) The following relations are satisfied

$$A_n^i=B_{n-1}^{i-1}+B_{n-1}^i,\quad 0\leqslant i\leqslant n,$$

$$A_n^i = A_{n-1}^{i-2} + 2A_{n-1}^{i-1} + A_{n-1}^i, \quad 0 \leqslant i \leqslant n-1,$$

$$A_n^n = A_{n-1}^{n-2} + A_{n-1}^{n-1}.$$

(2) The following relations are satisfied

$$B_n^i = A_n^{i-1} + A_n^i, \quad 0 \leqslant i \leqslant n,$$

$$B_n^i = B_{n-1}^{i-2} + 2B_{n-1}^{i-1} + B_{n-1}^i, \quad 0 \leqslant i \leqslant n.$$

Proof Let $n \geqslant 1$. Then by Eq.(3.2), we have

$$\begin{aligned} e_1^{2n} = e_1^{2n-1}e_1 &= \Big(\sum_{i=0}^{n-1} B_{n-1}^i e_{2n-2i-1}\Big)e_1 \\ &= \sum_{i=0}^{n-1} B_{n-1}^i (e_{2n-2i} + e_{2n-2i-2}) \\ &= \Big(\sum_{i=0}^{n-1} B_{n-1}^i e_{2n-2i} + 0\Big) + \Big(0 + \sum_{i=0}^{n-1} B_{n-1}^i e_{2n-2i-2}\Big) \\ &= \sum_{i=0}^{n} B_{n-1}^i e_{2n-2i} + \sum_{i=0}^{n} B_{n-1}^{i-1} e_{2n-2i} \\ &= \sum_{i=0}^{n} \left(B_{n-1}^{i-1} + B_{n-1}^i\right) e_{2n-2i}. \end{aligned}$$

So $A_n^i = B_{n-1}^{i-1} + B_{n-1}^i$. If $n = 0$, then clearly $B_0^0 = 1 = A_0^{-1} + A_0^0$. Now let $n \geqslant 1$. Then by Eq.(3.1), we have

$$\begin{aligned} e_1^{2n+1} = e_1^{2n}e_1 &= \Big(\sum_{i=0}^{n} A_n^i e_{2n-2i}\Big)e_1 \\ &= \sum_{i=0}^{n} A_n^i (e_{2n-2i+1} + e_{2n-2i-1}) \\ &= \sum_{i=0}^{n} A_n^i e_{2n-2i+1} + \sum_{i=0}^{n} A_n^{i-1} e_{2n-2i+1} \\ &= \sum_{i=0}^{n} \left(A_n^{i-1} + A_n^i\right) e_{2n-2i+1}. \end{aligned}$$

So $B_n^i = A_n^{i-1} + A_n^i$.

Thus we have shown the first equation in Part (1) and the first equation in Part (2). Now the other equations can be obtained by the above two equations. □

Proposition 3.2 *Let A_n^i and B_n^i be given as in Proposition 3.1. Then the following two equations are satisfied*

$$2^{2n} = \sum_{i=0}^{n} (2n-2i+1)A_n^i, \quad 2^{2n} = \sum_{i=0}^{n} (n-i+1)B_n^i.$$

Proof We use induction on n. By Eq.(3.2), it is easy to check that the second equation holds for $n = 0, 1$. Now let $n \geqslant 1$ and suppose

$$2^{2n} = \sum_{i=0}^{n} (n - i + 1)B_n^i.$$

Then from Part (2) of Proposition 3.1, we have

$$\begin{aligned}\sum_{i=0}^{n+1} [(n+1) - i + 1]B_{n+1}^i &= \sum_{i=0}^{n+1} (n - i + 2)(B_n^{i-2} + 2B_n^{i-1} + B_n^i) \\ &= \sum_{i=0}^{n+1} [(n - i) + (2n - 2i + 2) + (n - i + 2)]B_n^i \\ &= \sum_{i=0}^{n} (4n - 4i + 4)B_n^i \\ &= 2^2 \sum_{i=0}^{n} (n - i + 1)B_n^i \\ &= 2^2 \cdot 2^{2n} = 2^{2(n+1)}.\end{aligned}$$

So the second equation holds.

By the second equation and Part (2) of Proposition 3.1, we have

$$\begin{aligned}2^{2n} &= \sum_{i=0}^{n} (n - i + 1)B_n^i \\ &= \sum_{i=0}^{n} (n - i + 1)(A_n^{i-1} + A_n^i) \\ &= \sum_{i=1}^{n} (n - i + 1)A_n^{i-1} + \sum_{i=0}^{n} (n - i + 1)A_n^i \\ &= \sum_{i=0}^{n-1} (n - i)A_n^i + \sum_{i=0}^{n-1} (n - i + 1)A_n^i + A_n^n \\ &= \sum_{i=0}^{n-1} (2n - 2i + 1)A_n^i + A_n^n \\ &= \sum_{i=0}^{n} (2n - 2i + 1)A_n^i.\end{aligned}$$

So the first identity holds. □

Theorem 3.3 *For all natural numbers i, n with $0 \leqslant i \leqslant n$, we have*

$$A_n^i = \binom{2n}{i} \frac{2n - 2i + 1}{2n - i + 1}. \tag{3.3}$$

Proof For the case of $n = 0$, it is obvious. Now suppose $n \geqslant 1$. From Eq.(3.1) and $A_n^i = A_{n-1}^{i-2} + 2A_{n-1}^{i-1} + A_{n-1}^i$ in Part (1) of Proposition 3.1, we can obtain $A_n^0 = A_{n-1}^0 = \cdots = A_1^0 = A_0^0 = 1$

and

$$A_n^1 = 2\sum_{j=1}^{n-1} A_j^0 + A_1^1 = 2(n-1)+1 = 2n-1.$$

Thus the Eq.(3.3) holds for all $n \in \mathbb{N}$ and $i = 0, 1$. Suppose that Eq.(3.3) holds for all i, n with $1 \leqslant i \leqslant n$. Now let $2 \leqslant i \leqslant n+1$. Then

$$\begin{aligned} A_{n+1}^i &= A_n^{i-2} + 2A_n^{i-1} + A_n^i \\ &= \binom{2n}{i-2}\frac{2n-2i+5}{2n-i+3} + 2\binom{2n}{i-1}\frac{2n-2i+3}{2n-i+2} + \binom{2n}{i}\frac{2n-2i+1}{2n-i+1} \\ &= \frac{(2n+2)!(2n-2i+3)}{(2n-i+2)!i!(2n-i+3)} \\ &= \frac{(2(n+1))![2(n+1)-2i+1]}{[2(n+1)-i]!i![2(n+1)-i+1]} \\ &= \binom{2(n+1)}{i}\frac{2(n+1)-2i+1}{2(n+1)-i+1}. \end{aligned}$$

This completes the proof. □

By relation $B_n^i = A_n^{i-1} + A_n^i$ in Proposition 3.1 and Theorem 3.3, we have the following result.

Corollary 3.4 *For all i, n with $0 \leqslant i \leqslant n$, we have*

$$B_n^i = \binom{2n+1}{i}\frac{2n-2i+2}{2n-i+2}.$$

4. Clebsch-Gordan formula

Let $(H, m, u, \Delta, \epsilon)$ be a bialgebra over a field k. For a finite dimensional left H-module M, let (M) denote the isomorphic class containing M. Let $\mathcal{M}(H)$ denote the set of isomorphic classes of all finite dimensional left H-modules. Let $\mathcal{F}(H) = \{\sum_{i=1}^u n_i(M_i) | u \in \mathbb{N}, n_i \in \mathbb{Z}, (M_i) \in \mathcal{M}(H)\}$ be the free abelian group on symbols (M_i), and $\mathcal{F}_0(H)$ be the subgroup of $\mathcal{F}(H)$ generated by all expressions $(M_2) - (M_1) - (M_3)$, where $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of finite dimensional left H-modules. We call $\mathcal{G}(H) = \mathcal{F}(H)/\mathcal{F}_0(H)$ the Grothendieck group of H. For each finite dimensional left H-module M, we write $[M] = (M) + \mathcal{F}_0(H)$ for the images in $\mathcal{G}(H)$ of (M). Observe that $\mathcal{G}(H)$ is a ring with addition $[M_i] + [M_j] = [M_i \oplus M_j]$ and multiplication $[M_i][M_j] = [M_i \otimes M_j]$. Then we can get an associative algebra $\mathcal{G}(H)^k$ by extending the scalars, which has a basis $\mathcal{X} = \{[X]\,|X$ is a simple finite dimensional left H-module$\}$. We refer the reader to [9] for more information about Grothendieck group.

Let H be a bialgebra. We write $[k] = f_0$, where k is the trivial left H-module, i.e., $h \cdot 1 = \epsilon(h)$ for all $h \in H$. Suppose there is an injective $\varphi : \mathbb{N} \to \mathcal{X}$, $\varphi(n) = [X_n] = f_n$. In particular, $\varphi(0) = [k] = [X_0] = f_0$.

Theorem 4.1 *Let H be a bialgebra. Assume that any finite dimensional left H-module is semisimple. Suppose that $f_0 f_0 = f_0, f_1 f_n = f_n f_1 = f_{n+1} + f_{n-1}$ for all $n \geqslant 0$, where $f_n \in \mathcal{G}(H)$*

are given as before and $f_{-1} = 0$. *Then*

$$X_m \otimes X_n \cong X_n \otimes X_m \cong \oplus_{i=0}^m X_{n+m-2i}$$

for all $n \geqslant m \geqslant 0$.

Proof By Theorem 2.1, $f_m f_n = f_n f_m = \sum_{i=0}^m f_{n+m-2i}$, for all $n \geqslant m$. That is,

$$[X_m][X_n] = [X_n][X_m] = \sum_{i=0}^m [X_{n+m-2i}],$$

i.e.,

$$[X_m \otimes X_n] = [X_n \otimes X_m] = [\oplus_{i=0}^m X_{n+m-2i}].$$

From the fact that every finite dimensional left H-module is semisimple, there exists a positive integer s and simple modules X_j $(j = 1, \ldots, s)$ such that

$$X_m \otimes X_n \cong \oplus_{j=1}^s \alpha_j X_j, \quad \oplus_{i=0}^m X_{n+m-2i} \cong \oplus_{j=1}^s \beta_j X_j,$$

where α_j and β_j are non-negative integers. So we have

$$\sum_{j=1}^s \alpha_j [X_j] = \sum_{j=1}^s \beta_j [X_j].$$

where $[X_j]$ $(j = 1, \ldots, s)$ are elements in $\mathcal{X}$, which implies that $\alpha_j = \beta_j$ $(j = 1, \ldots, s)$. Thus

$$X_m \otimes X_n \cong \oplus_{i=0}^m X_{n+m-2i}.$$

Similarly, we have $X_n \otimes X_m \cong \oplus_{i=0}^m X_{n+m-2i}$. $\square$

In the rest of this section, let $H = \mathcal{U}_q((2))$ or $\mathcal{U}((2))$. From the discussion in the introduction, there is a unique simple module $V(n)$ of dimension $n+1$ for every $n \in \mathbb{N}$.

Proposition 4.2 *There are* H*-module isomorphisms*

$$V(0) \otimes V(0) \cong V(0), \; V(1) \otimes V(n) \cong V(n) \otimes V(1) \cong V(n+1) \oplus V(n-1),$$

where $n \geqslant 0$ *and* $V(-1) = 0$.

Proof We only consider the case $H = \mathcal{U}_q(\mathrm{sl}(2))$. Obviously, $V(0) \otimes V(0) \cong V(0)$ and $V(1) \otimes V(0) \cong V(0) \otimes V(1) \cong V(1) \cong V(0+1) \oplus V(0-1)$, where $V(-1) = 0$. For any $n \geqslant 1$, let $v^{(n)} \in V(n)$ be a highest weight vector of weight q^n, and $v_p^{(n)} = \frac{1}{[p]!} F^p v^{(n)}$, where $p = 0, 1$. We define

$$v_{n+1-2p} = \sum_{i=0}^p (-1)^i \frac{[1-p+i]![n-i]!}{[1-p]![n]!} q^{-i(2-2p+i)} v_i^{(n)} \otimes v_{p-i}^{(1)}.$$

It is easy to see that $v_i^{(n)} \otimes v_{p-i}^{(1)}$ is a weight vector of weight $q^{n-2i+1-2(p-i)} = q^{n+1-2p}$. From the fact that $\triangle(E) = 1 \otimes E + E \otimes K$, we have

$$Ev_{n+1-2p} = \sum_{i=0}^p (-1)^i \frac{[1-p+i]![n-i]!}{[1-p]![n]!} q^{-i(2-2p+i)} v_i^{(n)} \otimes Ev_{p-i}^{(1)} +$$

$$
\begin{aligned}
&\sum_{i=0}^{p}(-1)^i\frac{[1-p+i]![n-i]!}{[1-p]![n]!}q^{-i(2-2p+i)}Ev_i^{(n)}\otimes Kv_{p-i}^{(1)}\\
=&\sum_{i=0}^{p}(-1)^i[2-p+i]\frac{[1-p+i]![n-i]!}{[1-p]![n]!}q^{-i(2-2p+i)}v_i^{(n)}\otimes v_{p-i-1}^{(1)}+\\
&\sum_{i=0}^{p}(-1)^i[n-i+1]\frac{[1-p+i]![n-i]!}{[1-p]![n]!}q^{-i(2-2p+i)+(1-2p+2i)}v_{i-1}^{(n)}\otimes v_{p-i}^{(1)}\\
=&\sum_{i=0}^{p}(-1)^i\Big(\frac{[1-p+i]![n-i+1]!}{[1-p]![n]!}q^{-(i-1)(1-2p+i)}-\frac{[1-p+i]![n-i+1]!}{[1-p]![n]!}\\
&q^{-(i-1)(1-2p+i)}\Big)\times\Big(v_{i-1}^{(n)}\otimes v_{p-i}^{(1)}\Big)\\
&=0.
\end{aligned}
$$

So v_{n+1-2p} is the highest weight vector of weight q^{n+1-2p} in $V(n)\otimes V(1)$. It follows that there is a submodule of $V(n)\otimes V(1)$ which is isomorphic to $V(n+1)\oplus V(n-1)$. In addition, $V(n+1)\oplus V(n-1)$ and $V(n)\otimes V(1)$ have the same dimension. Thus $V(n)\otimes V(1)\cong V(n+1)+V(n-1)$. In a similar way, we have $V(1)\otimes V(n)\cong V(n+1)+V(n-1)$. □

Let $f_0=[k], f_n=[V(n)]$. By Proposition 4.2, we have $f_1f_n=f_nf_1=f_{n+1}+f_{n-1}$. Applying Theorem 4.1, one can get the Clebsch-Gordan formula and quantum Clebsch-Gordan formula with unified method. We state the result as follows.

Corollary 4.3 *There are H-module isomorphisms*

$$V(m)\otimes V(n)\cong V(n)\otimes V(m)\cong\oplus_{i=0}^{m}V(n+m-2i)$$

for all $n,m\geqslant 0$.

It is known that any finite dimensional left H-module is semisimple. By Corollary 4.3, for any finite dimensional H-modules M and N, we have H-module isomorphism

$$M\otimes N\cong N\otimes M.$$

Theorem 4.4 *For any $n\in\mathbb{N}$, we have the following formula*

$$V(1)^{\otimes 2n}=\oplus_{i=0}^{n}A_n^iV(2n-2i),$$
$$V(1)^{\otimes(2n+1)}=\oplus_{i=0}^{n}B_n^iV(2n+1-2i),$$

where A_n^i and B_n^i are the formulae given in Theorem 3.3 and Corollary 3.4, respectively.

Proof Let $f_n=[V(n)]$ for all $n\geqslant 0$. Then $f_0=[k]$. It follows from Proposition 4.2 that $\{f_0,f_1,\ldots,f_n,\ldots\}$ satisfies the conditions stated in Theorem 2.1. From the discussion in Section 3, we have $f_1^{2n}=\sum_{i=0}^{n}A_n^if_{2n-2i}, f_1^{2n+1}=\sum_{i=0}^{n}B_n^if_{2n+1-2i}$. That is,

$$[V(1)^{\otimes 2n}]=[\oplus_{i=0}^{n}A_n^iV(2n-2i)],\quad [V(1)^{\otimes(2n+1)}]=[\oplus_{i=0}^{n}B_n^iV(2n+1-2i)].$$

Then the theorem can be proven by using the same method as in the proof of Theorem 4.1. □

By Proposition 2.4, for all $n\in\mathbb{N}$, $V(n)$ and $V(1)^{\otimes i}(i=0,\ldots,n)$ satisfy the following relations.

Corollary 4.5 *For all $n \in \mathbb{N}$, we have*

$$V(n) \oplus \left(\oplus_{i=1}^{\lfloor \frac{n}{4} \rfloor + a} \binom{n-2i+1}{2i-1} V(1)^{\otimes(n-4i+2)} \right) \cong \oplus_{j=0}^{\lfloor \frac{n}{4} \rfloor} \binom{n-2j}{2j} V(1)^{\otimes(n-4j)},$$

where $V(1)^{\otimes 0} = V(0)$; if $n \equiv 2$ or $3 \pmod 4$, then $a = 1$; otherwise $a = 0$.

Proof By Proposition 2.4, we have

$$[V(n)] = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n-i}{i} [V(1)]^{n-2i},$$

where $[V(1)]^0 = [V(0)]$. Transferring the negative items on the right side to the left side and applying the method in the proof of Theorem 4.1, we obtain the result by considering the cases of $n = 4i, 4i+1, 4i+2, 4i+3$, respectively. $\square$

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