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On Balance Index Set for the General Butterfly Graph

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Abstract In this paper, we introduce the concept of the general butterfly graph $B[m,n;d]$ for integers $m,n \geq 3$, $d \geq 1$, determine its balance index set, and give the necessary and sufficient condition for balanced graph $B[m,n;d]$ to exist.

Keywords friendly labeling; balance labeling; balance index set.

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1. Introduction

Let $G=(V,E)$ be a graph. For a labeling $f: V(G) \to \{0,1\}$, we can induce an edge partial labeling $f^*: E(G) \to \{0,1\}$ as follows:

$$\text{if } f(x)=f(y), \text{ then } f^*(xy)=f(x) \text{ else the edge } xy \text{ has no label for } f^*,$$

where $xy \in E(G)$. Define

$$v_f(i)=|\{x \in V(G): f(x)=i\}| \text{ and } e_f(i)=|\{xy \in E(G): f^*(xy)=i\}|,$$

where $i \in \{0,1\}$. A labeling f of graph G is said to be friendly if $|v_f(0)-v_f(1)| \leq 1$. Furthermore, if $|e_f(0)-e_f(1)| \leq 1$ is also satisfied, then the friendly graph G is said to be balanced. For a given graph G,

$$\mathrm{BI}(G)=\{|e_f(0)-e_f(1)| : f \text{ is friend labeling of } G\}$$

is called the balanced index set of G.

Lee, Liu and Tan[1] introduced the concept of the balanced graph. In [2], several graphs are studied. On Twentieth Mid-West Conference on Combinatorics, Cryptography and Computing(2006), Lee, Wang and Wen gave the balance index sets of $B[3,3;d]$, where integer $d \geq 1$. They proved the following:

The graph $B[3,3;d]$ is balanced if $d=1,2,3,4$.

The balance index set $BI(B[3,3;2k+1])$ is

(1) $\{1,2\}$, if $k=0$,

(2) $\{1,2,3,4\}$, if $k=1$,

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(3) $\{k, k+1, k+2, k+3, k+4\}$, if $k \geq 2$.

The balance index set $\mathrm{BI}(B[3,3;2k])$ is

(1)$\{0,1,2,3,4\}$, if $k=1$,

(2)$\{k-1, k, k+1, k+2, k+3, k+4\}$ if $k \geq 2$.

In this paper, we introduce an extended concept for the butterfly graph $B[3,3;d]$. For integers $n, m \geq 3$ and $d \geq 1$, so-called general butterfly graph $B[m,n;d]$ is a graph that consists of a star $K_{1,d}$ and two cycles C_m and C_n, where the center w of $K_{1,d}$ is also the unique common vertex of C_m and C_n, and there is no other common vertex between $K_{1,d}$, C_m and C_n. Thus, $B[m,n;d]$ is a $(p,p+1)$-graph, where $p=m+n+d-1$ is its vertex number and $p+1$ is its edge number. The two cycles C_m and C_n form the two wings of the $B[m,n;d]$, and the d edges of $K_{1,d}$ form the antennae of the $B[m,n;d]$.

Below, we will determine the balance index sets of the graph $B[m,n;d]$, and give the necessary and sufficient condition for the balanced graph $B[m,n;d]$ to exist.

2. The range of $\mathrm{BI}(B[m,n;d])$

In a $(0,1)$-sequence (or cycle), a section in the form

$$\text{“}1,0,0,\ldots,0,1\text{”} \quad (\text{or “}0,1,1,\ldots,1,0\text{”})$$

is called 0-runs (or 1-runs) with size k, if the number of 0 (or 1) between two ends 1 (or 0) is k. In a $(0,1)$-cycle with size n, if there is runs, then the appearance of 0-runs and 1-runs is alternative, and the numbers of 0-runs and 1-runs are the same. For a given $(0,1)$-labeling f, if a cycle C_n consists of t 0-runs (with size $r_1,\ldots,r_t$) and t 1-runs (with size $s_1,\ldots,s_t$), then such $(0,1)$-cycle is denoted by $C_n(t)$, and $\sum_{i=1}^{t}(r_i+s_i)=n$, where $t \geq 1$. Obviously, the contribution of a 0-runs with size r to $v_f(0)$ (or $e_f(0)$) is r (or $r-1$). And, the contribution of a 1-runs with size s to $v_f(1)$ (or $e_f(1)$) is s (or $s-1$). Under a given $(0,1)$-labeling, if all vertices in a cycle C_n are labeled as 0 (or 1), then the cycle contains no runs and it will be denoted by 0-C_n (or 1-C_n).

Below, for integers $a \leq b$, denote

$$[a,b]=\{a,a+1,\ldots,b\},\ [a,b]^{-}=\{b,b-1,\ldots,a+1,a\},$$
$$[a,b]_2=\{a,a+2,\ldots,b-2,b\} \ \text{ if } \ a \equiv b \ (\mathrm{mod} 2).$$

Let f be a friend labeling of the graph $B[n,m;d]$. Then, there are the following possibilities for its two cycles, where t and $t' \geq 1$.

$f(w)=0:$ (1) $C_m(t), C_n(t')$; (2) $C_m(t)$, 0-C_n; (3) 0-C_m, 0-C_n;

$f(w)=1:$ (1) $C_m(t), C_n(t')$; (2) $C_m(t)$, 1-C_n; (3) 1-C_m, 1-C_n.

It is easy to see that the cases $f(w)=0$ and $f(w)=1$ are dual. Obviously, the difference for two cases is only to exchange of $e_f(0)$ and $e_f(1)$. Hence, the $\mathrm{BI}(B[m,n;d])=\{|e_f(0)-e_f(1)| : f$ is friend labeling of $B[m,n;d]\}$ are the same for two cases. Below, we will only discuss the case $f(w)=0$. Denote the number of 0 and 1 among $\{f(z) : z \in V(K_{1,d})\}$ to be k and $d-k$. Of course, $0 \leq k \leq d$.

2.1. Case 1: $C_m(t)$ and $C_n(t')$, $t, t' \geq 1$

Let $C_m(t)$ consist of t 0-runs (with size $r_1, \ldots, r_t$) and t 1-runs (with size $s_1, \ldots, s_t$), $C_n(t')$ consist of t' 0-runs (with size $r'_1, \ldots, r'_{t'}$) and t' 1-runs (with size $s'_1, \ldots, s'_{t'}$). Denote

$$\sum_{i=1}^{t} r_i = r,\ \sum_{i=1}^{t} s_i = s,\ \sum_{i=1}^{t'} r'_i = r',\ \sum_{i=1}^{t'} s'_i = s',\ \sigma = r + r' - s - s',$$

where all $r, s, r', s', r_i, s_i, r'_i, s'_i$ are positive integers. We solve the following equation system:

$$\begin{cases} r + s = m \\ r' + s' = n \\ r - s + r' - s' = \sigma \end{cases} \Longrightarrow \begin{cases} s = m - r \\ r' = \frac{m+\sigma+n}{2} - r \\ s' = r - \frac{m+\sigma-n}{2} \end{cases}.$$

In order to get positive integers r, s, r', s', it is necessary for a positive integer r to exist such that $\min\{m, \frac{m+\sigma+n}{2}\} = x > r > y = \frac{m+\sigma-n}{2}$, i.e.,

$*$ $\underline{x - y \geq 2}$: if $\sigma \leq m - n$ then $x = \frac{m+\sigma+n}{2}$, but $x - y = n \geq 2$ is obvious;
if $\sigma > m - n$ then $x = m$, but $x - y = \frac{m+n-\sigma}{2} \geq 2 \implies \sigma \leq m + n - 4$.

$*$ $\underline{x \geq 2}$: $x = \frac{m+n+\sigma}{2} \geq 2 \implies \sigma \geq 4 - m - n$;
$x = m \geq 2$ is obvious.

Therefore, the necessary conditions for positive integer solution for r, s, r', s' to exist are

$$\sigma \in [4 - m - n, m - n]_2 \cup [m - n + 2, m + n - 4]_2 = [4 - m - n, m + n - 4]_2. \tag{2.1}$$

On the other hand,

$$v_f(0) = (r + r') - 1 + k,\quad v_f(1) = (s + s') + d - k;$$
$$e_f(0) = (r + r') - (t + t') + k,\quad e_f(1) = (s + s') - (t + t').$$

Since σ and $m + n$ have the same parity, we have

$|v_f(0) - v_f(1)| = |\sigma + 2k - d - 1| \leq 1 \implies (d - \sigma)/2 \leq k \leq 1 + (d - \sigma)/2$

$$\implies k = \frac{d-\sigma}{2} + \varepsilon, \text{ where } \varepsilon = \begin{cases} \frac{1}{2} & \text{for } 2 \nmid (m + n - d) \\ 0, 1 & \text{for } 2 | (m + n - d) \end{cases},$$

$e_f(0) - e_f(1) = (r + r' - s - s') + k = \sigma + k = \frac{d+\sigma}{2} + \varepsilon$.

By $0 \leq k \leq d$, the parameter σ needs to satisfy the following conditions.

$$\begin{aligned} &\varepsilon = \frac{1}{2}:\ 0 \leq d - \sigma + 1 \leq 2d \Longrightarrow \sigma \in [1 - d, 1 + d]_2; \\ &\varepsilon = 0:\ 0 \leq d - \sigma \leq 2d \Longrightarrow \sigma \in [-d, d]_2; \\ &\varepsilon = 1:\ 0 \leq d - \sigma + 2 \leq 2d \Longrightarrow \sigma \in [2 - d, 2 + d]_2. \end{aligned} \tag{2.2}$$

Summarize the conditions (2.1) and (2.2) as follows.

* When $2 \nmid (m + n - d)$,

$$[1 - d, 1 + d]_2 \cap [4 - m - n, m + n - 4]_2 = \begin{cases} [1 - d, 1 + d]_2 & d \leq m + n - 5 \\ [4 - m - n, m + n - 4]_2 & d \geq m + n - 3 \end{cases} \Longrightarrow$$

$$\left\{\frac{d+\sigma+1}{2} : \sigma \in \begin{matrix} [1 - d, 1 + d]_2 \\ [4 - m - n, m + n - 4]_2 \end{matrix}\right\} = \begin{cases} [1, d + 1] & d \leq m + n - 5 \\ [\frac{5+d-m-n}{2}, \frac{m+n+d-3}{2}] & d \geq m + n - 3 \end{cases}.$$

* When $2|(m+n-d)$,

$$[-d,d]_2 \cap [4-m-n,m+n-4]_2 = \begin{cases} [-d,d]_2 & d \le m+n-4 \\ [4-m-n,m+n-4]_2 & d \ge m+n-2 \end{cases} \Longrightarrow$$

$$\left\{\tfrac{d+\sigma}{2} : \sigma \in \begin{matrix} [-d,d]_2 \\ [4-m-n,m+n-4]_2 \end{matrix}\right\} = \begin{cases} [0,d] & d \le m+n-4 \\ [\frac{4+d-m-n}{2}, \frac{m+n+d-4}{2}] & d \ge m+n-2 \end{cases},$$

$$[2-d,2+d]_2 \cap [4-m-n,m+n-4]_2 = \begin{cases} [2-d,2+d]_2 & d \le m+n-6 \\ [2-d,d]_2 & d = m+n-4 \\ [4-m-n,m+n-4]_2 & d \ge m+n-2 \end{cases} \Longrightarrow$$

$$\left\{\tfrac{d+\sigma+2}{2} : \sigma \in \begin{matrix} [2-d,2+d]_2 \\ [2-d,d]_2 \\ [4-m-n,m+n-4]_2 \end{matrix}\right\} = \begin{cases} [2,d+2] & d \le m+n-6 \\ [2,d+1] & d = m+n-4 \\ [\frac{6+d-m-n}{2}, \frac{m+n+d-2}{2}] & d \ge m+n-2 \end{cases}.$$

Furthermore, combine the related ranges:

when $d \le m+n-6, [0,d] \cup [2,d+2] = [0,d+2]$,

when $d = m+n-4, [0,d] \cup [2,d+1] = [0,d+1]$,

when $d \ge m+n-2,\ [\frac{4+d-m-n}{2}, \frac{m+n+d-4}{2}] \cup [\frac{6+d-m-n}{2}, \frac{m+n+d-2}{2}] = [\frac{4+d-m-n}{2}, \frac{m+n+d-2}{2}]$.

Thus, for the Case 1, we get the conclusion $(*1)$:

$$\mathrm{BI}(B[m,n;d]) \subseteq \begin{cases} \begin{cases} [1,d+1] & d \le m+n-5 \\ [\frac{5+d-m-n}{2}, \frac{m+n+d-3}{2}] & d \ge m+n-3 \end{cases} & \text{for } 2 \nmid (m+n-d) \\ \begin{cases} [0,d+2] & d \le m+n-6 \\ [0,d+1] & d = m+n-4 \\ [\frac{4+d-m-n}{2}, \frac{m+n+d-2}{2}] & d \ge m+n-2 \end{cases} & \text{for } 2|(m+n-d) \end{cases}.$$

2.2. Case 2: $C_m(t)$ and 0-$C_n,\ t \ge 1$

Let $C_m(t)$ consist of t 0-runs (with size $r_1, \ldots, r_t$). Denote $\sum_{i=1}^t r_i = r,\ \sum_{i=1}^t s_i = s,\ \sigma = r+n-s$, where all r, s, r_i, s_i are positive integers. Then, from $r+s=m$ and $\sigma = r+n-s$, we have $r = \frac{m+\sigma-n}{2}$ and $s = m-r$. Then, the necessary conditions for positive integers r, s to exist are

$$m+\sigma-n \ge 2 \text{ and } 2m-(m+\sigma-n) \ge 2 \Longrightarrow n+m-2 \ge \sigma \ge n-m+2$$
$$\Longrightarrow \sigma \in [n-m+2, n+m-2]_2. \tag{2.3}$$

On the other hand,

$$v_f(0) = r-1+n+k, \quad v_f(1) = s+d-k;$$
$$e_f(0) = r-t+n+k, \quad e_f(1) = s-t.$$

Since σ and $m+n$ have the same parity, taking the same notation ε in Case 1, we have

$$|v_f(0)-v_f(1)| = |\sigma+2k-d-1| \le 1 \Longrightarrow k = \frac{d-\sigma}{2} + \varepsilon,$$
$$e_f(0)-e_f(1) = \sigma + k = \frac{d+\sigma}{2} + \varepsilon.$$

By $0 \le k \le d$, the parameter σ needs to satisfy the same conditions (2.2). Summarize the conditions (2.3) and (2.2) as follows.

* When $2 \not| (m+n-d)$,

$$[1-d,1+d]_2 \cap [n-m+2,n+m-2]_2 = \begin{cases} \emptyset & d < |m-n-1|,\ m < n+1 \\ [1-d,1+d]_2 & d < |m-n-1|,\ m \ge n+1 \\ [n-m+2,1+d]_2 & |m-n-1| \le d \le m+n-3 \\ [n-m+2,n+m-2]_2 & d \ge m+n-1 \end{cases} \Longrightarrow$$

$$\{\tfrac{d+\sigma+1}{2} : \sigma \in \begin{matrix} [1-d,1+d]_2 \\ [n-m+2,1+d]_2 \\ [n-m+2,n+m-2]_2 \end{matrix}\} = \begin{cases} [1,d+1] & d < |m-n-1|,\ m \ge n+1 \\ [\frac{d+n-m+3}{2},d+1] & |m-n-1| \le d \le m+n-3 \\ [\frac{d+n-m+3}{2},\frac{d+n+m-1}{2}] & d \ge m+n-1 \end{cases}.$$

* When $2 | (m+n-d)$,

$$[-d,d]_2 \cap [n-m+2,n+m-2]_2 = \begin{cases} \emptyset & d < |m-n-2|,\ m < n+2 \\ [-d,d]_2 & d < |m-n-2|,\ m \ge n+2 \\ [n-m+2,d]_2 & |m-n-2| \le d \le m+n-2 \\ [n-m+2,n+m-2]_2 & d \ge m+n \end{cases} \Longrightarrow$$

$$\{\tfrac{d+\sigma}{2} : \sigma \in \begin{matrix} [-d,d]_2 \\ [n-m+2,d]_2 \\ [n-m+2,n+m-2]_2 \end{matrix}\} = \begin{cases} [0,d] & d < |m-n-2|,\ m \ge n+2 \\ [\frac{d+n-m+2}{2},d] & |m-n-2| \le d \le m+n-2 \\ [\frac{d+n-m+2}{2},\frac{d+n+m-2}{2}] & d \ge m+n \end{cases}.$$

$$[2-d,2+d]_2 \cap [n-m+2,n+m-2]_2 = \begin{cases} \emptyset & d < |m-n|,\ m < n \\ [2-d,2+d]_2 & d < |m-n|,\ m \ge n \\ [n-m+2,2+d]_2 & |m-n| \le d \le m+n-4 \\ [n-m+2,n+m-2]_2 & d \ge m+n-2 \end{cases} \Longrightarrow$$

$$\{\tfrac{d+\sigma+2}{2} : \sigma \in \begin{matrix} [2-d,2+d]_2 \\ [n-m+2,2+d]_2 \\ [n-m+2,n+m-2]_2 \end{matrix}\} = \begin{cases} [2,2+d] & d < |m-n|,\ m \ge n \\ [\frac{d+n-m+4}{2},d+2] & |m-n| \le d \le m+n-4 \\ [\frac{d+n-m+4}{2},\frac{d+n+m}{2}] & d \ge m+n-2 \end{cases}.$$

Furthermore, combine the related ranges:

when $d \le |m-n-2|,\ [0,d] \cup [2,d+2] = [0,d+2]$,

when $|m-n| \le d \le m+n-4,\ [\frac{d+n-m+2}{2},d] \cup [\frac{d+n-m+4}{2},d+2] = [\frac{d+n-m+2}{2},d+2]$,

when $d \ge m+n-2,\ [\frac{d+n-m+2}{2},\frac{d+n+m-2}{2}] \cup [\frac{d+n-m+4}{2},\frac{d+n+m}{2}] = [\frac{d+n-m+2}{2},\frac{d+n+m}{2}]$.

Thus, for the Case 2, we get the conclusion $(*2)$:

$$\mathrm{BI}(B[m,n;d]) \subseteq \begin{cases} \begin{cases} [1,d+1] & d < |m-n-1|,\ m \ge n+1 \\ [\frac{d+n-m+3}{2},d+1] & |m-n-1| \le d \le m+n-3 \\ [\frac{d+n-m+3}{2},\frac{d+n+m-1}{2}] & d \ge m+n-1 \end{cases} & \text{for } 2 \not| (m+n-d) \\ \begin{cases} [0,d+2] & d \le |m-n-2| \\ [\frac{d+n-m+2}{2},d+2] & |m-n| \le d \le m+n-4 \\ [\frac{d+n-m+2}{2},\frac{d+n+m}{2}] & d \ge m+n-2 \end{cases} & \text{for } 2 | (m+n-d) \end{cases}.$$

2.3. Case 3: 0-C_n and 0-C_m

Using the same notation ε, we get $v_f(0) = m+n-1+k, v_f(1) = d-k, e_f(0) = m+n+k, e_f(1) = 0$.

$|v_f(0) - v_f(1)| = |2k+m+n-d-1| \le 1 \implies (d-m-n)/2 \le k \le 1+(d-m-n)/2.$

$$\implies k = \tfrac{d-m-n}{2} + \varepsilon,$$

$e_f(0) - e_f(1) = m+n+k = m+n+\frac{d-m-n}{2} + \varepsilon = \frac{d+m+n}{2} + \varepsilon.$

Thus, for the Case 3, by $0 \le k \le d$, we get the conclusion $(*3)$:

$$\mathrm{BI}(B[m,n;d]) = \begin{cases} \frac{d+m+n+1}{2} & \text{for } 2 \not| (m+n-d),\ d \ge m+n-1 \\ \frac{d+m+n+2}{2} & \text{for } 2|(m+n-d),\ d = m+n-2 \\ [\frac{d+m+n}{2}, \frac{d+m+n+2}{2}] & \text{for } 2|(m+n-d),\ d \ge m+n \end{cases}.$$

Finally, summarizing $(*1) - (*3)$ for Cases $1-3$, we conclude as follows:

$$\mathrm{BI}(B[m,n;d]) \subseteq \begin{cases} [1, d+1] & \text{for } 2 \not| (m+n-d),\ d \le m+n-3 \\ [\frac{d-m-n+5}{2}, \frac{d+m+n+1}{2}] & \text{for } 2 \not| (m+n-d),\ d \ge m+n-1 \\ [0, d+2] & \text{for } 2|(m+n-d),\ d \le m+n-4 \\ [\frac{d-m-n+4}{2}, \frac{d+m+n+2}{2}] & \text{for } 2|(m+n-d),\ d \ge m+n-2 \end{cases}. \tag{2.4}$$

3. Construction and examples

In this section, the necessary condition (2.4) for $\mathrm{BI}(B[m,n;d])$ will be proved to be also sufficient. By the discussion in §2, the values ind in $\mathrm{BI}(B[m,n;d])$ depend on the value $\sigma = (r-s)+(r'-s')$ for Case 1, or $\sigma = r+n-s$ for Case 2, where r, r' and s, s' are the sizes of the 0-runs and 1-runs given by a friendly labeling. First, give the constructing methods (M1)–(M3) for the cases $1-3$, respectively.

Method (M1) List the range of the parameter σ, and give corresponding k and ind.

$$2 \not| (m+n-d) : \sigma \in \begin{cases} [1-d, 1+d]_2 & d \le m+n-5 \\ [4-m-n, m+n-4]_2 & d \ge m+n-3 \end{cases},\ k = \tfrac{d-\sigma+1}{2},\ \mathrm{ind} = \tfrac{d+\sigma+1}{2};$$

$$2 | (m+n-d) : \sigma \in \begin{cases} [-d, d]_2 & d \le m+n-4 \\ [4-m-n, m+n-4]_2 & d \ge m+n-2 \end{cases},\ k = \tfrac{d-\sigma}{2},\ \mathrm{ind} = \tfrac{d+\sigma}{2},$$

$$\sigma \in \begin{cases} [2-d, 2+d]_2 & d \le m+n-6 \\ [2-d, d]_2 & d = m+n-4 \\ [4-m-n, m+n-4]_2 & d \ge m+n-2 \end{cases},\ k = \tfrac{d-\sigma+2}{2},\ \mathrm{ind} = \tfrac{d+\sigma+2}{2}.$$

As for the existence of the parameters r, s, r', s', one can refer to §2.1. They may be chosen as follows.

$$r = \max(1, \frac{m-n+\sigma+2}{2}),\ s = m-r,\ r' = \frac{m+\sigma+n}{2} - r, s' = n-r'.$$

Method (M2) For the given ind, since $r = \frac{m+\sigma-n}{2}$, $\mathrm{ind} = \frac{d+\sigma}{2} + \varepsilon$, we have

$$r = \frac{m-n-d}{2} + \mathrm{ind} - \varepsilon,\ s = m-r,\ k = \frac{d-\sigma}{2} + \varepsilon = d - \mathrm{ind} + 2\varepsilon.$$

Especially, for the indices we need to use the following:

ind$= d+1$ for $d = m+n-3$ $(\varepsilon = \frac{1}{2})$: $r = 1-n+d+1 = m-1,\ k = d-(d+1)+1 = 0$;
ind$= \frac{d+m+n-1}{2}$ for $d \geq m+n-1$ $(\varepsilon = \frac{1}{2})$: $r = m-1,\ k = d-\frac{d+m+n-1}{2}+1 = \frac{d-m-n+3}{2} \geq 1$;
ind$= d+2$ for $d = m+n-4$ $(\varepsilon = 1)$: $r = 2-n+d+2-1 = m-1,\ k = d-(d+2) = 2 = 0$;
ind$= \frac{d+m+n}{2}$ for $d \geq m+n-2$ $(\varepsilon = 1)$: $r = m-1,\ k = \frac{d-m-n}{2}+2 \geq 1$.

Method (M3) For each available ind, it is enough to show the corresponding $k = \frac{d-m-n}{2} + \varepsilon$.
ind$= \frac{d+m+n+1}{2}$ for $d \geq m+n-1$ $(\varepsilon = \frac{1}{2})$: $k = \frac{d-m-n+1}{2}$;
ind$= \frac{d+m+n}{2}$ for $d \geq m+n$ $(\varepsilon = 0)$: $k = \frac{d-m-n}{2}$;
ind$= \frac{d+m+n+2}{2}$ for $d \geq m+n-2$ $(\varepsilon = 1)$: $k = \frac{d-m-n+2}{2}$.

Then, for all indices in BI($B[m,n;d]$), the used constructions are stated as follows.

For $2 \nmid (m+n-d)$, if $d \leq m+n-5$ then use (M1);
if $d = m+n-3$ then use (M2) for $d+1$, and (M1) for other values;
if $d \geq m+n-1$ then use (M3) for $\frac{d+m+n+1}{2}$, (M2) for $\frac{d+m+n-1}{2}$, and (M1) for other values.

For $2|(m+n-d)$,
if $d \leq m+n-6$ then use (M1);
if $d = m+n-4$ then use (M2) for $d+2$, and (M1) for other values;
if $d \geq m+n-2$ then use (M3) for $\frac{d+m+n+2}{2}$, (M2) for $\frac{d+m+n}{2}$, and (M1) for other values.

Example 1 $m = 4, n = 5$ and $3 \leq d \leq 8$. First, using (M1), we have the following table.

d	$m+n-d$	σ	k	ind	BI\ind
4	5	$[-3,5]_2$	$[0,4]^-$	$[1,5]$	
6	3	$[-5,5]_2$	$[1,6]^-$	$[1,6]$	7
8	1	$[-5,5]_2$	$[2,7]^-$	$[2,7]$	8, 9
3	6	$[-3,3]_2 \cup [-1,5]_2$	$[0,3]^- \cup [0,3]^-$	$[0,3] \cup [2,5] = [0,5]$	
5	4	$[-5,5]_2 \cup [-3,5]_2$	$[0,5]^- \cup [1,5]^-$	$[0,5] \cup [2,6] = [0,6]$	7
7	2	$[-5,5]_2 \cup [-5,5]_2$	$[1,6]^- \cup [2,7]^-$	$[1,6] \cup [2,7] = [1,7]$	8, 9

Table 1 The parameter table for $m = 4, n = 5$ and $3 \leq d \leq 8$

For even d $(= 4,6,8)$, $k = \frac{d-\sigma+1}{2}$, ind $= \frac{d+\sigma+1}{2}$, list their constructing tables:

$d = 4$

σ	r	s	r'	s'	k	ind
−3	1	3	2	3	4	1
−1	1	3	3	2	3	2
1	1	3	4	1	2	3
3	2	2	4	1	1	4
5	3	1	4	1	0	5

$d = 6$

σ	r	s	r'	s'	k	ind
−5	1	3	1	4	6	1
−3	1	3	2	3	5	2
−1	1	3	3	2	4	3
1	1	3	4	1	3	4
3	2	2	4	1	2	5
5	3	1	4	1	1	6

$d = 8$

σ	r	s	r'	s'	k	ind
−5	1	3	1	4	7	2
−3	1	3	2	3	6	3
−1	1	3	3	2	5	4
1	1	3	4	1	4	5
3	2	2	4	1	3	6
5	3	1	4	1	2	7

Table 2 The parameter table for d $(= 4,6,8)$, $k = \frac{d-\sigma+1}{2}$, ind $= \frac{d+\sigma+1}{2}$

For odd d $(= 3,5,7)$, $k = \frac{d-\sigma}{2}$ or $\frac{d-\sigma+2}{2}$, ind $= \frac{d+\sigma}{2}$ or $\frac{d+\sigma+2}{2}$, list their constructing tables:

$d=3$

σ	r	s	r'	s'	k	ind
−3	1	3	2	3	3	0
−1	1	3	3	2	2 3	1 2
1	1	3	4	1	1 2	2 3
3	2	2	4	1	0 1	3 4
5	3	1	4	1	0	5

$d=5$

σ	r	s	r'	s'	k	ind
−5	1	3	1	4	5	0
−3	1	3	2	3	4 5	1 2
−1	1	3	3	2	3 4	2 3
1	1	3	4	1	2 3	3 4
3	2	2	4	1	1 2	4 5
5	3	1	4	1	0 1	5 6

$d=7$

σ	r	s	r'	s'	k	ind
−5	1	3	1	4	6 7	1 2
−3	1	3	2	3	5 6	2 3
−1	1	3	3	2	4 5	3 4
1	1	3	4	1	3 4	4 5
3	2	2	4	1	2 3	5 6
5	3	1	4	1	1 2	6 7

Table 3 The parameter table for d $(=3,5,7)$, $k=\frac{d-\sigma}{2}$ or $\frac{d-\sigma+2}{2}$, ind $=\frac{d+\sigma}{2}$ or $\frac{d+\sigma+2}{2}$

As for the remaining indices (in column BI\ind), we can construct them, using (M2) and (M3).

d	ind	by	r	s	k
6	7	(M2)	3	1	0
8	8	(M2)	3	1	1
8	9	(M3)			0
5	7	(M2)	3	1	0
7	8	(M2)	3	1	1
7	9	(M3)			0

Example 2 $m=6, n=4$ and $4\le d\le 9$. First, using (M1), we have the following table.

d	$m+n-d$	σ	k	ind	BI\ind
4	6	$[-4,4]_2\cup[-2,6]_2$	$[0,4]^-\cup[0,4]^-$	$[0,4]\cup[2,6]=[0,6]$	
6	4	$[-6,6]_2\cup[-4,6]_2$	$[0,6]^-\cup[1,6]^-$	$[0,6]\cup[2,7]=[0,7]$	8
8	2	$[-6,6]_2\cup[-6,6]_2$	$[1,7]^-\cup[2,8]^-$	$[1,7]\cup[2,8]=[1,8]$	$9,10$
5	5	$[-4,6]_2$	$[0,5]^-$	$[1,6]$	
7	3	$[-6,6]_2$	$[1,7]^-$	$[1,7]$	8
9	1	$[-6,6]_2$	$[2,8]^-$	$[2,8]$	$9,10$

Table 4 The parameter table for $m=6, n=4$ and $4\le d\le 9$

$d=4$

σ	r	s	r'	s'	k	ind
−4	1	5	2	2	4	0
−2	1	5	3	1	3 4	1 2
0	2	4	3	1	2 3	2 3
2	3	3	3	1	1 2	3 4
4	4	2	3	1	0 1	4 5
6	5	1	3	1	0	6

$d=6$

σ	r	s	r'	s'	k	ind
−6	1	5	1	3	6	0
−4	1	5	2	2	5 6	1 2
−2	1	5	3	1	4 5	2 3
0	2	4	3	1	3 4	3 4
2	3	3	3	1	2 3	4 5
4	4	2	3	1	1 2	5 6
6	5	1	3	1	0 1	6 7

$d=8$

σ	r	s	r'	s'	k	ind
−6	1	5	1	3	7 8	1 2
−4	1	5	2	2	6 7	2 3
−2	1	5	3	1	5 6	3 4
0	2	4	3	1	4 5	4 5
2	3	3	3	1	3 4	5 6
4	4	2	3	1	2 3	6 7
6	5	1	3	1	1 2	7 8

Table 5 The parameter table for d $(=4,6,8)$, $k=\frac{d-\sigma}{2}$ or $\frac{d-\sigma+2}{2}$, ind $=\frac{d+\sigma}{2}$ or $\frac{d+\sigma+2}{2}$

For even d $(=4,6,8)$, $k=\frac{d-\sigma}{2}$ or $\frac{d-\sigma+2}{2}$, ind $=\frac{d+\sigma}{2}$ or $\frac{d+\sigma+2}{2}$, list their constructing tables above.

For odd d $(=5,7,9)$, $k=\frac{d-\sigma+1}{2}$, ind $=\frac{d+\sigma+1}{2}$, list their constructing tables:

$d=5$

σ	r	s	r'	s'	k	ind
−4	1	4	2	2	5	1
−2	1	4	3	1	4	2
0	2	3	3	1	3	3
2	3	2	3	1	2	4
4	4	1	3	1	1	5
6	5	0	3	1	0	6

$d=7$

σ	r	s	r'	s'	k	ind
−6	1	5	1	3	7	1
−4	1	5	2	2	6	2
−2	1	5	3	1	5	3
0	2	4	3	1	4	4
2	3	3	3	1	3	5
4	4	2	3	1	2	6
6	5	1	3	1	1	7

$d=9$

σ	r	s	r'	s'	k	ind
−6	1	5	1	3	8	2
−4	1	5	2	2	7	3
−2	1	5	3	1	6	4
0	2	4	3	1	5	5
2	3	3	3	1	4	6
4	4	2	3	1	3	7
6	5	1	3	1	2	8

Table 6 The parameter table for d $(=5,7,9)$, $k=\frac{d-\sigma+1}{2}$, ind $=\frac{d+\sigma+1}{2}$

As for the remaining indices (in column BI\ind), we can construct them, using (M2) and (M3).

d	ind	by	r	s	k
6	8	(M2)	5	1	0
8	9	(M2)	5	1	1
8	10	(M3)			0
7	8	(M2)	5	1	0
9	9	(M2)	5	1	1
9	10	(M3)			0

Example 3 $m=5, n=7$ and $5\le d\le 10$. First, using (M1), we have the following table.

d	$m+n-d$	σ	k	ind	BI\ind
5	7	$[-4,6]_2$	$[0,5]^-$	$[1,6]$	
7	5	$[-6,8]_2$	$[0,7]^-$	$[1,8]$	
9	3	$[-8,8]_2$	$[1,9]^-$	$[1,9]$	10
6	6	$[-6,6]_2\cup[-4,8]_2$	$[0,6]^-\cup[0,6]^-$	$[0,6]\cup[2,8]=[0,8]$	
8	4	$[-8,8]_2\cup[-6,8]_2$	$[0,8]^-\cup[1,8]^-$	$[0,8]\cup[2,9]=[0,9]$	10
10	2	$[-8,8]_2\cup[-8,8]_2$	$[1,9]^-\cup[2,10]^-$	$[1,9]\cup[2,10]=[1,10]$	11, 12

Table 7 The parameter table for $m=5, n=7$ and $5\le d\le 10$

For odd d $(=5,7,9)$, $k=\frac{d-\sigma+1}{2}$, ind $=\frac{d+\sigma+1}{2}$, list their constructing tables:

$d=5$

σ	r	s	r'	s'	k	ind
-4	1	4	3	4	5	1
-2	1	4	4	3	4	2
0	1	4	5	2	3	3
2	1	4	6	1	2	4
4	2	3	6	1	1	5
6	3	2	6	1	0	6

$d=7$

σ	r	s	r'	s'	k	ind
-6	1	4	2	5	7	1
-4	1	4	3	4	6	2
-2	1	4	4	3	5	3
0	1	4	5	2	4	4
2	1	4	6	1	3	5
4	2	3	6	1	2	6
6	3	2	6	1	1	7
8	4	1	6	1	0	8

$d=9$

σ	r	s	r'	s'	k	ind
-8	1	4	1	6	9	1
-6	1	4	2	5	8	2
-4	1	4	3	4	7	3
-2	1	4	4	3	6	4
0	1	4	5	2	5	5
2	1	4	6	1	4	6
4	2	3	6	1	3	7
6	3	2	6	1	2	8
8	4	1	6	1	1	9

Table 8 The parameter table for $d\ (=5,7,9)$, $k=\frac{d-\sigma+1}{2}$, ind $=\frac{d+\sigma+1}{2}$

For even $d\ (=6,8,10)$, $k=\frac{d-\sigma}{2}$ or $\frac{d-\sigma+2}{2}$, ind $=\frac{d+\sigma}{2}$ or $\frac{d+\sigma+2}{2}$, list their constructing tables:

$d=6$

σ	r	s	r'	s'	k	ind
-6	1	4	2	5	6	0
-4	1	4	3	4	5 6	1 2
-2	1	4	4	3	4 5	2 3
0	1	4	5	2	3 4	3 4
2	1	4	6	1	2 3	4 5
4	2	3	6	1	1 2	5 6
6	3	2	6	1	0 1	6 7
8	4	1	6	1	0	8

$d=8$

σ	r	s	r'	s'	k	ind
-8	1	4	1	6	8	0
-6	1	4	2	5	7 8	1 2
-4	1	4	3	4	6 7	2 3
-2	1	4	4	3	5 6	3 4
0	1	4	5	2	4 5	4 5
2	1	4	6	1	3 4	5 6
4	2	3	6	1	2 3	6 7
6	3	2	6	1	1 2	7 8
8	4	1	6	1	0 1	8 9

$d=10$

σ	r	s	r'	s'	k	ind
-8	1	4	1	6	9 10	1 2
-6	1	4	2	5	8 9	2 3
-4	1	4	3	4	7 8	3 4
-2	1	4	4	3	6 7	4 5
0	1	4	5	2	5 6	5 6
2	1	4	6	1	4 5	6 7
4	2	3	6	1	3 4	7 8
6	3	2	6	1	2 3	8 9
8	4	1	6	1	1 2	9 10

Table 9 The parameter table for $d\ (=6,8,10)$, $k=\frac{d-\sigma}{2}$ or $\frac{d-\sigma+2}{2}$, ind $=\frac{d+\sigma}{2}$ or $\frac{d+\sigma+2}{2}$

As for the remaining indices (in column BI\ind), we can construct them, using (M2) and (M3).

d	ind	by	r	s	k
9	10	(M2)	4	1	0
8	10	(M2)	4	1	2
10	11	(M2)	4	1	1
10	12	(M3)			1

4. Conclusion

Theorem 1 The balance index set of general butterfly graph is

$$\mathrm{BI}(B[m,n;d]) = \begin{cases} [1, d+1] & \text{for } 2 \nmid (m+n-d),\ d \le m+n-3 \\ [\frac{d-m-n+5}{2}, \frac{d+m+n+1}{2}] & \text{for } 2 \nmid (m+n-d),\ d \ge m+n-1 \\ [0, d+2] & \text{for } 2 | (m+n-d),\ d \le m+n-4 \\ [\frac{d-m-n+4}{2}, \frac{d+m+n+2}{2}] & \text{for } 2 | (m+n-d),\ d \ge m+n-2 \end{cases}.$$

Proof By the discussion in §2 and §3, the proof of this theorem completed .

Theorem 2 *For integers $m, n \ge 3$ and $d \ge 1$, the graph $B[m,n;d]$ is balanced if and only if $d \le m+n-2$.*

Proof Obviously, the intervals $[1, d+1]$ and $[0, d+2]$ contain 0 or 1. But,

when $d \ge m+n-1$, $\frac{d-m-n+5}{2} \ge 2 \Longrightarrow$ the intervals $[\frac{d-m-n+5}{2}, \frac{d+m+n+1}{2}]$ do not contain 0 or 1;

when $d \ge m+n$, $\frac{d-m-n+4}{2} \ge 2 \Longrightarrow$ the intervals $[\frac{d-m-n+4}{2}, \frac{d+m+n+2}{2}]$ do not contain 0 or 1;

when $d = m+n-2$, $\frac{d-m-n+4}{2} = 1 \Longrightarrow\ 1 \in [\frac{d-m-n+4}{2}, \frac{d+m+n+2}{2}]$.

Therefore, by the definition of balanced graph and Theorem 1, the conclusion holds.

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