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An Inequality of Bohr Type on Hardy-Sobolev Classes

LI Xue Hua
(College of Science, China Agricultural University, Beijing 100083, China)
(E-mail: lixh@cau.edu.cn)

Abstract Let $\beta > 0$ and $S_\beta := \{z \in \mathbb{C} : |\mathrm{Im}\, z| < \beta\}$ be a strip in the complex plane. For an integer $r \geq 0$, let $H^r_{\infty,\beta}$ denote those real-valued functions f on $\mathbb{R}$, which are analytic in S_β and satisfy the restriction $|f^{(r)}(z)| \leq 1$, $z \in S_\beta$. For $\sigma > 0$, denote by B_σ the class of functions f which have spectra in $(-2\pi\sigma, 2\pi\sigma)$. And let $B^\perp_\sigma$ be the class of functions f which have no spectrum in $(-2\pi\sigma, 2\pi\sigma)$. We prove an inequality of Bohr type

$$\|f\|_\infty \leq \frac{\pi}{\sqrt{\lambda}\Lambda\sigma^r} \sum_{k=0}^{\infty} \frac{(-1)^{k(r+1)}}{(2k+1)^r \sinh((2k+1)2\sigma\beta)}, \qquad f \in H^r_{\infty,\beta} \cap B^\perp_\sigma,$$

where $\lambda \in (0,1)$, Λ and Λ' are the complete elliptic integrals of the first kind for the moduli λ and $\lambda' = \sqrt{1-\lambda^2}$, respectively, and λ satisfies

$$\frac{4\Lambda\beta}{\pi\Lambda'} = \frac{1}{\sigma}.$$

The constant in the above inequality is exact.

Keywords Hardy-Sobolev classes; the spectrum of a function; an inequality of Bohr type.

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1. Introduction

First we give the definition of the spectrum of a function.

Definition 1.1[1] *We denote by $\Theta(\mathbb{R})$ the totality of functions $f \in C^\infty(\mathbb{R})$ such that*

$$\sup_{x\in\mathbb{R}} |x^\gamma D^\alpha f(x)| < \infty$$

for every non-negative integers α and γ. Such functions are called rapidly decreasing (at ∞).

Definition 1.2[1] *For any $f \in \Theta(\mathbb{R})$, define its Fourier transform $\hat{f}$ by*

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x) e^{-ix\xi} \mathrm{d}x.$$

The smallest closed set outside which $\hat{f}(\xi)$ vanishes is called the spectrum of $f(x)$.

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For $\sigma > 0$, denote by B_σ the class of functions f which have spectra in $(-2\pi\sigma, 2\pi\sigma)$. And let $B_\sigma^\perp$ be the class of functions f which have no spectrum in $(-2\pi\sigma, 2\pi\sigma)$. As usual, $L_\infty(\mathbb{R})$ denotes the space of real-valued functions f on $\mathbb{R}$ with the usual norm $\|f\|_\infty := \sup_{x\in\mathbb{R}} |f(x)| < \infty$.

We now give the classes of functions studied here. Let $\beta > 0$ and $S_\beta := \{z \in \mathbb{C} : |\mathrm{Im}\, z| < \beta\}$ be a strip in the complex plane. For an integer $r \geq 0$, let $H^r_{\infty,\beta}$ be the Hardy-Sobolev class of real-valued functions f on $\mathbb{R}$, which are analytic in the strip S_β and satisfy the condition $|f^{(r)}(z)| \leq 1$, $z \in S_\beta$. Denote by $\widetilde{H}^r_{\infty,\beta}$ [2] those 2π-periodic functions in $H^r_{\infty,\beta}$.

Let f be a real function of the real variable x with a bounded derivative. Suppose that $f \in B_\sigma^\perp$. The inequality

$$\|f\|_\infty \leq (4\sigma)^{-1}\|f'\|_\infty \tag{1.1}$$

was given by Bohr[3] for almost periodic $f(x)$ with a proof based on the theory of analytic functions. The constant $(4\sigma)^{-1}$ in (1.1) is the best possible. Iteration of (1.1) gives the inequality

$$\|f\|_\infty \leq \sigma^{-n} t_n \|f^{(n)}\|_\infty \tag{1.2}$$

with $t_n = 4^{-n}$. With methods from the theory of real functions, Favard[4] found that the best possible value of t_n is

$$t_n = (2\pi)^{-n} \cdot \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k(n+1)}}{(2k+1)^{n+1}}\,.$$

Hörmander[5] obtained the following generalization of the inequality of Bohr type: if $f(x)$ is real and

$$-M_1 \leq f^{(n)}(x) \leq M_2, \tag{1.3}$$

then

$$-\sigma^{-n}\mu_1^{(n)}(M_1, M_2) \leq f(x) \leq \sigma^{-n}\mu_2^{(n)}(M_1, M_2), \tag{1.4}$$

where $\mu_1^{(n)}$ and $\mu_2^{(n)}$ denote the best possible constants and are defined by

$$-\mu_1^{(n)}(M_1, M_2) = \min_x h_n(x; M_1, M_2), \quad \mu_2^{(n)}(M_1, M_2) = \max_x h_n(x; M_1, M_2),$$

where

$$h_n(x; M_1, M_2) = \frac{M_1 + M_2}{(n+1)!}\{\overline{B}_{n+1}(x + \frac{M_2}{2(M_1 + M_2)}) - \overline{B}_{n+1}(x - \frac{M_2}{2(M_1 + M_2)})\},$$

and the functions $\overline{B}_n(x)$ have the period 1 and coincide with the Bernoulli polynomials $B_n(x)$ in the interval $(0, 1)$.

In this paper, we get an inequality of Bohr type for the class of functions $H^r_{\infty,\beta}$.

We introduce the function $\Phi_{\lambda,r,\beta}$ which will be proved to be the extremal function of the inequality of Bohr type for some $\lambda \in (0, 1)$, and give the explicit presentation of its uniform norm $\|\Phi_{\lambda,r,\beta}\|_\infty$.

Let Λ and Λ' be the complete elliptic integrals of the first kind for the moduli $\lambda \in (0, 1)$ and

$\lambda' = \sqrt{1-\lambda^2}$, respectively. Put

$$\begin{aligned} \Phi_{\lambda,0,\beta}(z) &:= \frac{\pi}{\sqrt{\lambda}\Lambda}\sum_{k=0}^{\infty}\frac{\sin((2k+1)\frac{\pi\Lambda'}{4\Lambda\beta}z)}{\sinh((2k+1)\frac{\pi\Lambda'}{2\Lambda})}, \\ \Phi_{\lambda,2j-1,\beta}(z) &:= \int_{\frac{2\Lambda\beta}{\Lambda'}}^{z}\Phi_{\lambda,2j-2,\beta}(\mu)\mathrm{d}\mu, \qquad j=1,2,\ldots. \\ \Phi_{\lambda,2j,\beta}(z) &:= \int_0^z \Phi_{\lambda,2j-1,\beta}(\mu)\mathrm{d}\mu, \end{aligned} \tag{1.5}$$

Then from [6], we have

$$\begin{aligned} \Phi_{\lambda,r,\beta}(z) &= \frac{\pi}{\sqrt{\lambda}\Lambda}\Big(\frac{4\Lambda\beta}{\pi\Lambda'}\Big)^r\sum_{k=0}^{\infty}\frac{\sin((2k+1)\frac{\pi\Lambda'}{4\Lambda\beta}z-\pi r/2)}{(2k+1)^r\sinh((2k+1)\frac{\pi\Lambda'}{2\Lambda})}, \\ \|\Phi_{\lambda,r,\beta}\|_\infty &= \frac{\pi}{\sqrt{\lambda}\Lambda}\Big(\frac{4\Lambda\beta}{\pi\Lambda'}\Big)^r\sum_{k=0}^{\infty}\frac{(-1)^{k(r+1)}}{(2k+1)^r\sinh((2k+1)\frac{\pi\Lambda'}{2\Lambda})}, \end{aligned} \qquad r=0,1,\ldots. \tag{1.6}$$

When λ satisfies $4\Lambda\beta/(\pi\Lambda') = 1/n$ for some fixed $n\in\mathbb{N}$, we know that $\Phi_{\lambda,r,\beta}(z) = \Phi^{\beta}_{n,r}(z)$ [7].

We are now ready to state the main result.

Theorem 1.3 *Let $\sigma>0$ and $r=0,1,2,\ldots$. Then*

$$\|f\|_\infty \le \|\Phi_{\lambda,r,\beta}\|_\infty = \frac{\pi}{\sqrt{\lambda}\Lambda\sigma^r}\sum_{k=0}^{\infty}\frac{(-1)^{k(r+1)}}{(2k+1)^r\sinh((2k+1)2\sigma\beta)}, \quad f\in H^r_{\infty,\beta}\cap B^{\perp}_{\sigma}, \tag{1.7}$$

where $\lambda\in(0,1)$ satisfying

$$4\Lambda\beta/(\pi\Lambda') = 1/\sigma. \tag{1.8}$$

The constant in the inequality (1.7) is best possible, which means that

$$\sup_{f\in H^r_{\infty,\beta}\cap B^{\perp}_{\sigma}}\|f\|_\infty = \|\Phi_{\lambda,r,\beta}\|_\infty = \frac{\pi}{\sqrt{\lambda}\Lambda\sigma^r}\sum_{k=0}^{\infty}\frac{(-1)^{k(r+1)}}{(2k+1)^r\sinh((2k+1)2\sigma\beta)}.$$

Remark 1.4 From [7, 8], we know that

$$\sup_{f\in \widetilde{H}^r_{\infty,\beta}\cap \mathcal{T}^{\perp}_{n}}\|f\|_\infty = \|\Phi^{\beta}_{n,r}\|_\infty = \frac{\pi}{\sqrt{\lambda}\Lambda n^r}\sum_{k=0}^{\infty}\frac{(-1)^{k(r+1)}}{(2k+1)^r\sinh((2k+1)2n\beta)}, \tag{1.9}$$

where $\mathcal{T}_n := \mathrm{span}\{1,\cos t,\sin t,\ldots,\cos((n-1)t),\sin((n-1)t)\}$ is the space of trigonometric polynomials with order $n-1$, and $f\in\mathcal{T}^{\perp}_n$ means that

$$\begin{aligned} \int_0^{2\pi} f(t)\sin(kt)\mathrm{d}t = 0, \\ \int_0^{2\pi} f(t)\cos(kt)\mathrm{d}t = 0, \end{aligned} \qquad k=0,1,\ldots,n-1.$$

Thus, Theorem 1.3 is the generalization of this result.

2. Proof of main result

First we give some auxiliary results.

Lemma 2.1 *For any $\sigma > 0$, there exists a $\lambda_\sigma \in (0,1)$ such that*

$$4\Lambda_\sigma \beta/(\pi \Lambda'_\sigma) = 1/\sigma, \tag{2.1}$$

where Λ_σ and Λ'_σ are the complete elliptic integrals of the first kind for the moduli $\lambda_\sigma \in (0,1)$ and $\lambda'_\sigma = \sqrt{1-\lambda_\sigma^2}$, respectively.

Proof Since $\Lambda \to \frac{\pi}{2}$ and $\Lambda' \to +\infty$ as $\lambda \to 0^+$, $\frac{4\Lambda\beta}{\pi\Lambda'} \to 0^+$ as $\lambda \to 0^+$. On the other hand, when $\lambda \to 1^-$, $\Lambda \to +\infty$ and $\Lambda' \to \frac{\pi}{2}$, i.e., when $\lambda \to 1^-$, $\frac{4\Lambda\beta}{\pi\Lambda'} \to +\infty$. So from the fact that $4\Lambda\beta/(\pi\Lambda')$ continuously depends on λ, it follows that for any $\sigma > 0$, there exists a $\lambda_\sigma \in (0,1)$ such that (2.1) holds. Lemma 2.1 is proved. □

We now consider continuous functions φ on $\mathbb{R}$ with the properties

$$\varphi(x) \geq 0, \quad \sum_{-\infty}^{+\infty} \varphi(x+n) \leq 1, \quad \varphi(0) = 1. \tag{2.2}$$

An example of such a function is $\varphi(x) = (\pi x)^{-2}\sin^2(\pi x)$. So we can take a fixed function φ on $\mathbb{R}$ having the properties (2.2). If g is a bounded function on $\mathbb{R}$, we set

$$g_h(x) = \sum_{-\infty}^{+\infty} \varphi(hx+n) g(x+nh^{-1}), \tag{2.3}$$

where $h > 0$. It is evident that the series converges on $\mathbb{R}$ and that $g_h(x)$ has the period h^{-1}.

Lemma 2.2[5] *If $-1 \leq g(x) \leq 1$ for all $x \in \mathbb{R}$, then $-1 \leq g_h(x) \leq 1$ and $g_h(x)$ tends to $g(x)$ as $h \longrightarrow 0^+$, uniformly on every bounded set of $\mathbb{R}$.*

From [5], we know that a function f has no spectrum in $(-2\pi\sigma, 2\pi\sigma)$ means explicitly that $\int_{-\infty}^{+\infty} f(x)\psi(x)\,dx = 0$ if $\psi(x) \in \Psi$ and $\hat{\psi}(\xi)$ vanishes outside a compact set in $(-2\pi\sigma, 2\pi\sigma)$, where Ψ is the class of all infinitely differentiable functions on $\mathbb{R}$ which vanish at infinity together with all their derivatives more rapidly than any inverse power of x.

It also follows from [5] that there exists a function φ on $\mathbb{R}$ having the properties (2.2) and the properties that $\varphi \in \Psi$ and $\hat{\varphi}(\xi)$ shall vanish outside a bounded set of $\mathbb{R}$. Denote by M a number such that $\hat{\varphi}(\xi) = 0$ for $|\xi| \geq M$. The Fourier transform of $h\varphi(hx)e^{-ikhx}$ is $\hat{\varphi}((\xi+kh)/h) = \hat{\varphi}(\xi h^{-1}+k)$. It vanishes outside an interval contained in $(-2\pi\sigma, 2\pi\sigma)$ if $|kh| < 2\pi\sigma - Mh$.

Lemma 2.3[5] *$g_h(x)$ has no spectrum in $\big(-(2\pi\sigma - Mh), (2\pi\sigma - Mh)\big)$ if $g(x)$ has no spectrum in $(-2\pi\sigma, 2\pi\sigma)$.*

Lemma 2.4 *Let $\sigma > 0$ and $r = 0, 1, 2, \ldots$. Then for any $f \in H^r_{\infty,\beta} \cap B^{\perp}_\sigma$, there exists a $F \in H^{r+1}_{\infty,\beta}$ such that*

$$F'(z) = f(z), \quad z \in S_\beta,$$

and

$$\|F\|_\infty \leq \|\Phi_{\lambda_\sigma, r+1,\beta}\|_\infty, \tag{2.4}$$

where $\lambda_\sigma \in (0,1)$ satisfying equality (2.1).

Proof Let $\sigma > 0$ and $r = 0, 1, 2, \ldots$. For any $f \in H^r_{\infty,\beta} \cap B^{\perp}_\sigma$, define the periodic approximating function (2.3) of $f^{(r)}$ by $f_h^{(r)}$ on $\mathbb{R}$. Since $f \in H^r_{\infty,\beta}$, by Lemma 2.2, we have $-1 \le f_h^{(r)}(x) \le 1$ for all $x \in \mathbb{R}$ and $f_h^{(r)}$ tends to $f^{(r)}$ as $h \longrightarrow 0^+$, uniformly on every bounded set of $\mathbb{R}$. Since $f \in B_\sigma^\perp$, $f^{(r)}$ has no spectrum in $(-2\pi\sigma, 2\pi\sigma)$. So from Lemma 2.3, it follows that $f_h^{(r)}(x)$ has no spectrum in $(-(2\pi\sigma - Mh), (2\pi\sigma - Mh))$, where the function φ on $\mathbb{R}$ has the properties (2.2) and the properties that $\varphi \in \Psi$ and $\hat{\varphi}(\xi) = 0$ for $|\xi| \ge M$, and h is so small that $2\pi\sigma - Mh > 0$. Then there exists a periodic function f_h on $\mathbb{R}$ with zero mean value satisfying the r-th derivative of $f_h(x)$ is $f_h^{(r)}(x)$ and f_h tends to f as $h \longrightarrow 0^+$, uniformly on every bounded set of $\mathbb{R}$. So there exists a periodic integral of f_h on $\mathbb{R}$, denoted by $F_h(x) := \int_0^x f_h(t)\,dt + c_0$, such that

$$F_h'(x) = f_h(x), \quad x \in \mathbb{R},$$

and

$$\|F_h\|_\infty \le \|\Phi_{\lambda_\sigma, r+1, \beta}\|_\infty,$$

where $c_0 \in \mathbb{R}$ and $\lambda_\sigma \in (0,1)$ satisfying (2.1). Denote by $F(x) := \int_0^x f(t)\,dt + c_0$ the integral of $f(x)$ on $\mathbb{R}$. Then $F'(x) = f(x)$, $x \in \mathbb{R}$ and $F_h(x)$ tends to $F(x)$ as $h \longrightarrow 0^+$, uniformly on every bounded set of $\mathbb{R}$. So from the fact that $\|F\|_\infty \le \|F - F_h\|_\infty + \|F_h\|_\infty$, we know that $\|F\|_\infty \le \|\Phi_{\lambda_\sigma, r+1, \beta}\|_\infty$, as $h \longrightarrow 0^+$. Now it follows from the uniqueness theorem of analytic functions that there exists a $F \in H^{r+1}_{\infty,\beta}$ such that $F'(z) = f(z)$, $z \in S_\beta$ and (2.4) holds. Lemma 2.4 is proved. □

Lemma 2.5[6, Corollary 3.15] *Let $r = 0, 1, 2, \ldots$. If $f \in H^r_{\infty,\beta}$ and the inequality $\|f\|_\infty \le \|\Phi_{\lambda,r,\beta}\|_\infty$ holds for some $\lambda \in (0,1)$, then for all $1 \le l \le r+1$,*

$$\|f^{(l)}\|_\infty \le \|\Phi^{(l)}_{\lambda,r,\beta}\|_\infty = \frac{\pi}{\sqrt{\lambda}\Lambda}\left(\frac{4\Lambda\beta}{\pi\Lambda'}\right)^{r-l} \sum_{k=0}^{\infty} \frac{(-1)^{k(r-l+1)}}{(2k+1)^{r-l}\sinh((2k+1)\frac{\pi\Lambda'}{2\Lambda})}. \tag{2.5}$$

Lemma 2.6 *Let $\sigma > 0$. Then $\Phi_{\lambda_\sigma,r,\beta} \in H^r_{\infty,\beta} \cap B_\sigma^\perp$, where λ_σ satisfies (2.1).*

Proof Let $\sigma > 0$ and λ_σ satisfy (2.1). It is obvious that $\Phi_{\lambda_\sigma,r,\beta} \in H^r_{\infty,\beta}$. We only need to prove that $\Phi_{\lambda_\sigma,r,\beta} \in B_\sigma^\perp$. In fact, it follows from the condition: λ_σ satisfies (2.1) that

$$\int_{-\infty}^{+\infty} \Phi_{\lambda_\sigma,r,\beta}(x)\psi(x)\mathrm{d}x = 0$$

if $\psi(x) \in \Psi$ and $\hat{\psi}(\xi)$ vanishes outside a compact set in $(-2\pi\sigma, 2\pi\sigma)$. So $\Phi_{\lambda_\sigma,r,\beta} \in B_\sigma^\perp$. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3 Let $\sigma > 0$ and $r = 0, 1, 2, \ldots$. Then from Lemma 2.4, it follows that for any $f \in H^r_{\infty,\beta} \cap B_\sigma^\perp$, there exists an $F \in H^{r+1}_{\infty,\beta}$ such that

$$F'(z) = f(z), \quad z \in S_\beta,$$

and

$$\|F\|_\infty \le \|\Phi_{\lambda_\sigma, r+1, \beta}\|_\infty,$$

which together with Lemma 2.5 gives

$$\sup_{f\in H^r_{\infty,\beta}\cap B^{\perp}_\sigma}\|f\|_\infty \le \sup_{F\in H^{r+1}_{\infty,\beta}}\|F'\|_\infty = \|\Phi'_{\lambda_\sigma,r+1,\beta}\|_\infty,$$

i.e.

$$\sup_{f\in H^r_{\infty,\beta}\cap B^{\perp}_\sigma}\|f\|_\infty \le \|\Phi_{\lambda_\sigma,r,\beta}\|_\infty,$$

where $\lambda_\sigma \in (0,1)$ satisfying (2.1). And from Lemma 2.6, we know that $\Phi_{\lambda_\sigma,r,\beta} \in H^r_{\infty,\beta}\cap B^{\perp}_\sigma$ for λ_σ satisfying (2.1). So

$$\sup_{f\in H^r_{\infty,\beta}\cap B^{\perp}_\sigma}\|f\|_\infty = \|\Phi_{\lambda_\sigma,r,\beta}\|_\infty = \frac{\pi}{\sqrt{\lambda_\sigma}\Lambda_\sigma\sigma^r}\sum_{k=0}^{\infty}\frac{(-1)^{k(r+1)}}{(2k+1)^r\sinh((2k+1)2\sigma\beta)}.$$

Theorem 1.3 is proved. □

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