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Some Properties of Morphisms in Effect Algebras

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Abstract In the present paper the properties of morphisms in effect algebras are discussed. The conditions for the morphisms in effect algebras to be join-preservation and meet-preservation are given. From the categorical point of view, some properties of ideals, filters and congruence relations under morphisms are obtained.

Keywords effect algebra; effect morphism; ideal; congruence; adjoint functor.

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1. Introduction

Effect algebras, as a useful quantum model, can deal with sharp and unsharp problems. So it has aroused the interest of many scholars in the related fields such as orthomodular lattice, orthomodular posets, orthoalgebras and so on.

Morphism is a crucial tool to reveal the relation between algebraic systems, and to research algebraic structures. The purpose of the present paper is to discuss the properties of effect morphisms.

Definition 1.1[3] *A structure* $(E;\oplus,0,1)$ *is called an effect algebra if* $0,1$ *are two distinguished elements and* $\oplus$ *is a partially defined binary operation on* E *which satisfies the following conditions for any* $a,b,c\in E$:

(Ai) If $a\oplus b$ *is defined, then* $b\oplus a$ *is defined and* $a\oplus b=b\oplus a$;

(Aii) If $a\oplus b$ *and* $(a\oplus b)\oplus c$ *are defined, then* $b\oplus c$ *and* $a\oplus(b\oplus c)$ *are defined, and* $(a\oplus b)\oplus c=a\oplus(b\oplus c)$;

(Aiii) For every $a\in E$*, there exists a unique* $b\in E$*, such that* $a\oplus b=1$ *(we put* $b=a'$*)*;

(Aiv) If $1\oplus a$ *is defined, then* $a=0$.

Effect algebras were introduced by Foulis and Bennett in [3]. Independently, Kôpka and Chovanec[7] introduced an essentially equivalent structure called D-posets. Another equivalent structure, called weak orthoalgebras, was introduced by Giuntini and Greuling in [8].

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For brevity, we denote the effect algebra $(E; \oplus, 0, 1)$ by E. In the effect algebra E, we write $a \leq b$ iff there is $c \in E$ such that $a \oplus c = b$. It is easy to check that $\leq$ is a partial order on E. If E with the defined partial order is a lattice, then E is called a lattice ordered effect algebra. If E with the order $\leq$ is a totally ordered lattice, then E is called a totally ordered lattice effect algebra. In the following we will take the liberty of using the symbol $a \perp b$ to denote $a \oplus b$ if it is defined. Moreover, it is possible to introduce a new partial operation $\ominus$ which is defined as follows: If $a \oplus c$ is defined and $a \oplus c = b$, then $b \ominus a = c$.

In this section we recall the basic concepts and properties that shall be used in the present paper.

Definition 1.2[2] *Let E, F be effect algebras and $\phi : E \to F$ be a mapping. If ϕ satisfies the following conditions, then ϕ is called an effect morphism:*

(Bi) $\phi(1) = 1$;

(Bii) For every $a, b \in E$, if $a \oplus b$ is defined in E, then $\phi(a) \oplus \phi(b)$ is defined in F, and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$.

If an effect morphism ϕ is a bijection and ϕ^{-1} is an effect morphism, then ϕ is called an effect isomorphism.

Lemma 1.3[1] *Let E be an effect algebra. Then the following assertions are true for every $a, b, c \in E$:*

(1) $a \oplus b$ *is defined iff* $a \leq b'$;

(2) If $a \oplus b$ is defined and $a \vee b$ exists, then $a \wedge b$ exists and $a \oplus b = (a \wedge b) \oplus (a \vee b)$;

(3) $a \leq a \oplus b$;

(4) $(a')' = a, 0' = 1, 1' = 0$;

(5) $a \oplus 0 = a$.

Let E be a lattice ordered effect algebras. We have

(6) If $a, b \leq c'$, then $(a \vee b) \oplus c = (a \oplus c) \vee (b \oplus c), (a \wedge b) \oplus c = (a \oplus c) \wedge (b \oplus c)$;

(7) If $c \leq a, b$, then$(a \vee b) \ominus c = (a \ominus c) \vee (b \ominus c)$, $(a \wedge b) \ominus c = (a \ominus c) \wedge (b \ominus c)$.

Lemma 1.4 *Let E be an effect algebra. The following properties hold for every $a, b, c \in E$:*

(1) If $a \leq b$, then $(a \oplus b')' \oplus a = b$;

(2) If $a \leq b$, then $b' \leq a'$;

(3) $(a \vee b)' = a' \wedge b', (a \wedge b)' = a' \vee b'$;

(4) If $c \leq a \leq b'$, then $(a \oplus b) \ominus c = (a \ominus c) \oplus b$.

2. Basic properties

Proposition 2.1 *Let E, F be two effect algebras and $\phi : E \to F$ be an effect morphism. The following property holds for every $a, b, c \in E$:*

(1) If $a \geq b$, then $\phi(a) \geq \phi(b)$;

(2) $\phi(a') = (\phi(a))'$;

(3) If $b \geq a$, then $\phi(b \ominus a) = \phi(b) \ominus \phi(a)$;

(4) $\phi(0) = 0$;

(5) If $\phi(x) = 1$, then $\phi(x') = 0$.

Proof The proof is trivial and so is omitted.

Proposition 2.2 *Let $\phi : E_1 \to E_2$ be an effect morphism. If ϕ is a one-to-one mapping and E_1 is a totally ordered lattice, then $a \geq b$ if and only if $\phi(a) \geq \phi(b)$.*

Proof Necessity is clear. For the sufficiency, suppose on the contrary that $a < b$. Since ϕ is an order-preserving and one-to-one mapping, we have $\phi(a) < \phi(b)$, a contradiction, and this proves that $a \geq b$.

Proposition 2.3 *Let $\phi : E \to F$ be an effect morphism. If ϕ is a bijection and E is a totally ordered lattice effect algebra, then ϕ^{-1} is also an effect morphism.*

Proof Since $\phi(1) = 1$, we have $1 = \phi^{-1}(1)$. $\forall \overline{a}, \overline{b} \in F$, assume that $\overline{a} \oplus \overline{b}$ is defined. Since ϕ is a surjection, there exist $a, b \in E$ such that $\phi(a) = \overline{a}$, $\phi(b) = \overline{b}$. Hence $\phi(a) \leq \phi(b')$. By Proposition 2.2, we have $a \leq b'$. Thus $a \oplus b$ is defined. Since $\phi(a \oplus b) = \overline{a} \oplus \overline{b}$, we have $\phi^{-1}(\overline{a} \oplus \overline{b}) = a \oplus b = \phi^{-1}(\overline{a}) \oplus \phi^{-1}(\overline{b})$. Then $\phi^{-1} : F \to E$ is an effect morphism.

Proposition 2.4 *Let $\phi_1 : E_1 \to E_2$ and $\phi_2 : E_2 \to E_3$ be effect morphisms. Then $\phi_2 \circ \phi_1 : E_1 \to E_3$ is an effect morphism.*

Proof By the definition, it is easy to check the proposition above.

3. Monomorphism

Definition 3.1[6] *Let $\phi : E \to F$ be an effect morphism. If for every $a, b \in E$, $\phi(a) \leq \phi(b)$ implies $a \leq b$, then ϕ is called an effect monomorphism.*

Definition 3.2[6] *Let $\phi : E \to F$ be an effect morphism. For all $a, b \in E$, if $\phi(a) \leq \phi(b)$, then there exists $a_1 \in E$ such that $\phi(a_1) = \phi(a)$ and $a_1 \leq b$. We call such ϕ a strong morphism.*

Proposition 3.3 *Let $\phi : E_1 \to E_2$ be an effect morphism. Then following propositions are equivalent:*

(1) If $\phi(a) \leq \phi(b)$, $a, b \in E_1$, then $a \leq b$;

(2) If $\phi(a) \oplus \phi(b)$ is defined, $a, b \in E_1$, then $a \oplus b$ is defined.

Proof (1) $\Rightarrow$ (2). Assume that $\phi(a) \oplus \phi(b)$ is defined. Then $\phi(a) \leq \phi(b')$. Thus $a \leq b'$ by conditions. Therefore $a \oplus b$ is defined.

(2) $\Rightarrow$ (1). Assume that $\phi(a) \leq \phi(b)$. Since $\phi(a) \leq \phi'(b')$, $\phi(a) \oplus \phi(b')$ is defined. Thus $a \oplus b'$ is defined by conditions, and then $a \leq b$.

Proposition 3.4 *Let $\phi : E_1 \to E_2$ be a strong morphism and $a, b \in E_1$.*

(1) If $\phi(a) \leq \phi(b)$, then there exists $b_1 \in E_1$ such that $\phi(b_1) = \phi(b)$ and $a \leq b_1$.

(2) If $\phi(a) \oplus \phi(b)$ is defined, then there exists $a_1 \in E_1$ such that $\phi(a_1) = \phi(a)$ and $a_1 \oplus b$

is defined.

Proof (1) It follows from the hypotheses that $\phi(b') \leq \phi(a')$. Since ϕ is a strong morphism, thus there exists $c \in E_1$ such that $\phi(c) = \phi(b')$ and $c \leq a'$. Thus, there exists $b_1 \in E_1$ such that $\phi(b_1) = \phi(b)$ and $a \leq b_1$, where $b_1 = c'$.

(2) It follows from the hypotheses that $\phi(b) \leq \phi(a')$. Since ϕ is a strong morphism, there exists $a_1 \in E_1$ such that $\phi(a_1') = \phi(a)$ and $b \leq a_1$. Then $a_1' \oplus b$ is defined.

Theorem 3.5 *Let $\phi : E_1 \to E_2$ be an effect morphism and E_1, E_2 be lattice ordered effect algebras. If ϕ is a full monomorphism, then ϕ is join-preserving and meet-preserving.*

Proof $\forall a, b \in E_1$, it is easy to verify that $\phi(a \vee b)$ is a super bound of $\phi(a)$ and $\phi(b)$. Assume $\overline{c}$ is a super bound of $\phi(a)$ and $\phi(b)$, where $\overline{c} \in E_2$. There exists $c \in E_1$ such that $\phi(c) = \overline{c}$ by the condition. We have $\phi(a) \leq \phi(c)$ and $\phi(b) \leq \phi(c)$. Since ϕ is a monomorphism, we have $a \leq c$ and $b \leq c$. Then $a \vee b \leq c$, such that $\phi(a \vee b) \leq \phi(c) = \overline{c}$. We conclude that $\phi(a \vee b) = \phi(a) \vee \phi(b)$. For same reasons, it is easy to check that $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$. □

Theorem 3.6 *Let $\phi : E \to F$ be a strong morphism. Then $\phi(E)$ is a sub-effect algebra of F.*

Proof (i) Since $\phi(0) = 0$ and $\phi(1) = 1$, we have $0, 1 \in \phi(E)$.

(ii) For $\overline{a} \in \phi(E)$, set $a \in E$ such that $\phi(a) = \overline{a}$. Since $\phi(a \oplus a') = 1$, we have $\phi(a) \oplus \phi(a') = 1$, then $\overline{a}' = \phi(a') \in \phi(E)$.

(iii) For $\overline{a}, \overline{b} \in \phi(E)$ with $\overline{a} \perp \overline{b}$, set $\phi(a) = \overline{a}$ and $\phi(b) = \overline{b}, a, b \in E$. By Proposition 3.4, there exists $a_1 \in E$ such that $\phi(a_1) = \phi(a)$ and $a_1 \perp b$. Since $\phi(a_1 \oplus b) = \phi(a) \oplus \phi(b)$, we get $\overline{a} \oplus \overline{b} = \phi(a) \oplus \phi(b) \in \phi(E)$. To sum up, $\phi(E)$ is a sub-effect algebra of F. □

By Definition 3.1 and Proposition 3.3, we have the following proposition.

Proposition 3.7 *Let $\phi : E_1 \to E_2$ be an effect morphism. If ϕ is a full monomorphism, then $\phi^{-1} : E_2 \to E_1$ is effect morphism.*

4. Ideal, filter and congruence

Theorem 4.1 *Suppose that E, F are two lattice ordered effect algebras, $\phi : E \to F$ is an effect morphism, and ϕ is a surjective strong morphism. Then the following properties hold:*

(1) If I is a lattice ideal of E, then $\phi(I)$ is also a lattice ideal of F;

(2) If J is a lattice filter of E, then $\phi(J)$ is also a lattice filter of F;

(3) $\forall a \in E, \phi(\downarrow (a)) = \downarrow (\phi(a)), \phi(\uparrow (a)) = \uparrow (\phi(a))$.

Proof (1) Since $0 = \phi(0) \in \phi(I)$, $\phi(I) \neq \emptyset$. $\forall \overline{b} \in F$, suppose that $\overline{b} \leq \overline{a}$ and $\overline{a} \in \phi(I)$. Then there exist $a \in I$ and $b \in E$ such that $\phi(a) = \overline{a}$ and $\phi(b) = \overline{b}$, thus $\phi(b) \leq \phi(a)$. Since ϕ is a strong morphism, we have $b_1 \in E$ such that $\phi(b_1) = \phi(b)$ and $b_1 \leq a$, and moreover I is a lattice ideal. Hence $b_1 \in I$. Thus $\overline{b} = \phi(b_1) \in \phi(I)$. $\forall \overline{a}, \overline{b} \in \phi(I)$, there exist $a, b \in I$ such that $\phi(a) = \overline{a}, \phi(b) = \overline{b}$. Because I is a lattice ideal, we have $a \vee b \in I$. Since $\phi(a \vee b) \geq \phi(a) \vee \phi(b)$, we have $\overline{a} \vee \overline{b} = \phi(a) \vee \phi(b) \in \phi(I)$. Then $\phi(I)$ is a lattice ideal of F.

(2) Since $1 \in J$, $\phi(1) = 1 \in \phi(J)$. $\forall \overline{y} \in F$, let $\overline{x} \leq \overline{y}$ and $\overline{x} \in \phi(J)$. Then there exist $x \in J$ and $y \in E$ such that $\phi(x) = \overline{x}$ and $\phi(y) = \overline{y}$. Because ϕ is a strong morphism, there exists $y_1 \in E$ such that $\phi(y_1) = \phi(y)$ and $x \leq y_1$. Since J is a lattice filter, $\overline{y} = \phi(y_1) \in \phi(J)$. $\forall \overline{x}, \overline{y} \in \phi(J)$, we have $x, y \in J$ such that $\phi(x) = \overline{x}$ and $\phi(y) = \overline{y}$. Because $x \wedge y \in J$ and $\phi(x) \wedge \phi(y) \geq \phi(x \wedge y)$, $\overline{x} \wedge \overline{y} \in \phi(I)$. Then $\phi(J)$ is a lattice filter.

(3) For any $y \in \phi(\downarrow(a))$, since there exists $x \in E$ such that $x \leq a$ and $y = \phi(x)$, we have $y = \phi(x) \leq \phi(a)$. Then $y \in \downarrow(\phi(a))$. Conversely, $\forall y \in \downarrow(\phi(a))$, since $y \leq \phi(a)$ and ϕ is a surjective morphism, there exists $x \in E$ such that $\phi(x) = y$. Because ϕ is a strong morphism, there exists $x_1 \in E$ such that $\phi(x_1) = \phi(x) = y$ and $x_1 \leq a$. Then $y \in \phi(\downarrow(a))$. For the same reason, we can get the other direction. □

Definition 4.2[8] *Let $(E; \oplus, 0, 1)$ be an effect algebra. A binary relation $\sim$ on E is a congruence if it satisfies the following conditions:*

(1) $\sim$ is an equivalence relation;

(2) $a \sim a_1$, $b \sim b_1$, $a \perp b$, $a_1 \perp b_1$ imply $a \oplus b \sim a_1 \oplus b_1$;

(3) $a \perp b$, $a \sim a_1$ imply that there exists a $b_1 \in E$ such that $b \sim b_1$, $a_1 \perp b_1$.

Proposition 4.3 *Let $\phi : E_1 \to E_2$ be an effect morphism and $\sim$ be a binary relation on E_1 satisfying $a \sim b$ iff $\phi(a) = \phi(b)$. If ϕ is a monomorphism, then:*

(1) $\sim$ is an equivalent relation in E_1;

(2) If $a \sim a_1$, $b \sim b_1$, and $a \leq b'$, then $a_1 \leq b_1'$;

(3) If $a \sim a_1$, $b \sim b_1$, and $a \oplus b$ is defined, then $a_1 \oplus b_1$ is defined and $a \oplus b \sim a_1 \oplus b_1$;

(4) If $a \sim b$, then $a' \sim b'$;

(5) If $a \oplus b \sim a_1 \oplus b_1$ and $a \sim a_1$, then $b \sim b_1$.

Proof (1) It is obvious and so is omitted.

(2) Because $\phi(a) = \phi(a_1), \phi(b) = \phi(b_1)$ and $\phi(a) \leq \phi(b')$, we have $\phi(a_1) \leq \phi(b_1')$. Since ϕ is a monomorphism, $a_1 \leq b_1'$.

(3) Since $a \leq b'$, $a_1 \leq b_1'$ by (2). Hence $\phi(a) \leq \phi'(b)$ and $\phi(a_1) \leq \phi'(b_1)$. Thus $\phi(a) \oplus_{E_2} \phi(b) = \phi(a \oplus_{E_1} b) = \phi(a_1) \oplus_{E_2} \phi(b_1) = \phi(a_1 \oplus_{E_1} b_1)$. We conclude that $a \oplus b \sim a_1 \oplus b_1$.

(4) It is easy to check the item (4) by Proposition 2.1.

(5) Because $\phi(a \oplus b) = \phi(a_1 \oplus b_1)$ and $\phi(a) = \phi(a_1)$, we have $\phi(a) \oplus \phi(b) = \phi(a_1) \oplus \phi(b_1)$. Then $\phi(b) = \phi(b_1)$, so $b \sim b_1$.

Theorem 4.4 *Let E_1 be an effect algebra and $\sim$ be a congruence which is non-trivial in E_1. Denote by $[a]$ the congruence class of a. Let $E = E_1/_\sim = \{[a] | a \in E_1\}$. Provide $\oplus_E$ in E as follows: when $a \leq b'$, then $[a] \oplus_E [b] = [a \oplus_{E_1} b]$. Then $(E_1/_\sim, \oplus_E, [0], [1])$ is an effect algebra.*

Proof By Definition 1.1, it is easy to check Theorem 4.4 above.

5. Functor and adjoint functor

Let $(E; \oplus, 0, 1)$ be a lattice ordered effect algebra. Take E as a category, whose object class is an E set. Suppose that $\mathrm{Hom}(a.b)$ is a morphism set, $\forall a, b \in E$, $\mathrm{Hom}(a,b)$ is a cell set when $a \leq b$, and $\mathrm{Hom}(a,b)$ is an empty set when $a \not\leq b$. If E_1, E_2 are lattice ordered effect algebras and $\phi : E_1 \to E_2$ is an effect morphism, then we take ϕ to be functor between categories.

Proposition 5.1 *Let E_1, E_2 be two lattice ordered effect algebras and $\phi : E_1 \to E_2$ be an effect morphism. Then following propositions are equivalent:*

(1) ϕ is an isomorphism of effect algebras;

(2) Functor ϕ is a category isomorphism.

Proof (1)⇒(2). Because $\phi : E_1 \to E_2$ is an isomorphism of effect algebras, ϕ is a bijection and ϕ^{-1} is an effect morphism. Since $\phi^{-1} \circ \phi = id_{E_1}$, $\phi \circ \phi^{-1} = id_{E_2}$, obviously, id_{E_1} and id_{E_2} are identity functors. Then ϕ is a category isomorphism.

(2)⇒(1). If ϕ is a category isomorphism, then it follows from definition that there exists the mapping $\omega : E_2 \to E_1$ which preserves order and $\phi \circ w = id_{E_2}$ and $w \circ \phi = id_{E_1}$. So ϕ is a bijection and $\phi^{-1} = w$. Because $\phi^{-1}(1_{E_2}) = \phi^{-1}(\phi(1_{E_1})) = 1_{E_1}$. $\forall \overline{x}, \overline{y} \in E_2$, assume that $\overline{x} \leq \overline{y}'$. Then $\exists\ x, y \in E_1$ such that $\phi(x) = \overline{x}$ and $\phi(y) = \overline{y}$, i.e., $\phi(x) \leq \phi(y')$. We have $x = w \circ \phi(x) \leq w \circ \phi(y') = y'$. Thus, $\phi^{-1}(\overline{x} \oplus_{E_2} \overline{y}) = \phi^{-1}(\phi(x) \oplus_{E_2} \phi(y)) = \phi^{-1}(\phi(x \oplus_{E_1} y)) = x \oplus_{E_1} y = \phi^{-1}(\overline{x}) \oplus_{E_1} \phi^{-1}(\overline{y})$. Then ϕ is an isomorphism of effect algebras. □

Theorem 5.2 *Let $\phi : E \to F$ and $\theta : F \to E$ be two effect morphisms. If ϕ is a left adjoint of θ, then ϕ is a surjection if and only if θ is an effect monomorphism.*

Proof Assume that ϕ is a surjection. $\forall \overline{a}, \overline{b} \in F$, let $\theta(\overline{a}) \leq \theta(\overline{b})$. There exist $a, b \in E$ such that $\phi(a) = \overline{a}$ and $\phi(b) = \overline{b}$. Thus $\theta(\phi(a)) \leq \theta(\phi(b))$. We have $\phi(a) = \phi(\theta(\phi(a))) \leq \phi(\theta(\phi(b))) = \phi(b)$, that is, $\overline{a} \leq \overline{b}$, and so θ is an effect monomorphism. Conversely, assume that θ is an effect monomorphism. By the hypothesis, $\forall b \in F$, $\theta(b) = \theta(\phi(\theta(b)))$. Hence $b \leq \phi(\theta(b))$ and $b \geq \phi(\theta(b))$. Then $b = \phi(\theta(b))$. For any $b \in F$, we have $a = \theta(b)$ such that $\phi(a) = b$. Thus, ϕ is a surjection. □

Theorem 5.3 *Let E_1, E_2 be effect algebras and let $\phi : E_1 \to E_2$, $\theta : E_2 \to E_1$ be effect morphisms. If ϕ is a left adjoint functor of θ, then $j = \theta \circ \phi$ has the following properties for every $a \in E$:*

(1) $j(a) \geq a$;

(2) $j(j(a)) \leq j(a)$;

(3) When ϕ is a monomorphism, $L_j = \{x \in E_1 | j(x) = x\}$ is a sub-effect algebra of E_1.

Proof The proofs of (1) and (2) are trivial.

(3) Since $j(0_{E_1}) = \theta \circ \phi(0_{E_1}) = 0_{E_1}$ and $j(1_{E_1}) = \theta \circ \phi(1_{E_1}) = 1_{E_1}$, we have $0_{E_1}, 1_{E_1} \in L_j$. $\forall a \in L_j$, because $j(a') = \theta \circ \phi(a') = \theta(\phi'(a)) = \theta'(\phi(a)) = j'(a) = a'$, $a' \in L_j$. $\forall a, b \in L_j$, if $a \leq b'$, then $j(a \oplus_{E_1} b) = \theta \circ \phi(a \oplus_{E_1} b) = \theta \circ \phi(a) \oplus_{E_1} \theta \circ \phi(b) = a \oplus_{E_1} b$. We have $a \oplus b \in L_j$.

(Ai) If $a \oplus b$ is defined and $a \oplus b \in L_j$, since $a \oplus b = b \oplus a$ in E, we have $b \oplus a \in L_j$.

(Aii) Aussme that $b \oplus c$ and $a \oplus (b \oplus c)$ are defined, and $b \oplus c, a \oplus (b \oplus c) \in L_j$. Since $L_j \subseteq E_1$,

$a \oplus b$ is defined in E_1 and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ in E_1, we have $(a \oplus b) \oplus c \in L_j$. Because $j(a \oplus b) \geq a \oplus b$, $\phi(\theta \circ \phi(a \oplus b)) = \phi \circ \theta(\phi(a \oplus b)) \leq \phi(a \oplus b)$ and ϕ is a monomorphism, we obtain $\theta \circ \phi(a \oplus b) \leq a \oplus b$, that is, $j(a \oplus b) \leq a \oplus b$. Thus $j(a \oplus b) = a \oplus b$ and so $a \oplus b \in L_j$.

(Aiii) $\forall a \in L_j$, since $a' \in L_j$, $a \oplus a' = 1$.

(Aiv) If $1 \oplus a \in L_j$, since $L_j \subseteq E_1$, we have $a = 0_{E_1}$.

To sum up, L_j is a sub-effect algebra of E_1. □

Corollary 5.4 *Let E_1, E_2 be two effect algebras and let $\phi : E_1 \to E_2$, $\theta : E_2 \to E_1$ be two effect morphisms. If ϕ is a left adjoint functor of θ, then*

(1) $j = \theta \circ \phi$ is a closure mapping.

(2) When E_1 is a meet-semi-lattice, L_j is also a meet-semi-lattice.

Proof (1) By Theorem 5.3, it is clear that $j \circ j = j$.

(2) $\forall x_1, x_2 \in L_j$, it follows from that j is isotone that $j(x_1 \wedge x_2) \leq j(x_1) \wedge j(x_2)$. We can get $j(x_1 \wedge x_2) = x_1 \wedge x_2$ by Theorem 5.3, and so $x_1 \wedge x_2 \in L_j$. Then L_j is a meet-semi-lattice too.

Theorem 5.5 *Let $\phi_1 : E_1 \to E_2$, $\phi_2 : E_2 \to E_1$ be two effect morphisms, E_1, E_2 be two lattice order effect algebras, and ϕ_1 be a left adjoint of ϕ_2. Then*

(1) If ϕ_1 is a meet-preserving mapping and ϕ_2 is a surjection, then ϕ_2 is a prime element-preserving.

(2) If ϕ_2 is a join-preserving mapping and ϕ_1 is a surjection, then ϕ_1 is a coprime element-preserving.

(3) If F is a prime filter of E_1 and ϕ_1 is a bijection, then $\phi_1(F)$ is also a prime filter of E_2.

(4) If I is a prime ideal of E_2 and ϕ_2 is a bijection, then $\phi_2(I)$ is also a prime ideal of E_1.

Proof (1) $\forall a, b \in E_1$, let $\overline{c}$ be a prime element of E_2 and $a \wedge b \leq \phi_2(\overline{c})$. Since ϕ_2 is a surjection, there exist $\overline{a}, \overline{b} \in E_2$ such that $\phi_2(\overline{a}) = a$ and $\phi_2(\overline{b}) = b$. It follows from that ϕ_1 is a left adjoint of ϕ_2 and ϕ_1 is a meet-preserving mapping that $\phi_1(\phi_2(\overline{a})) \wedge \phi_1(\phi_2(\overline{b})) \leq \overline{c}$. Thus $a = \phi_2(\overline{a}) \leq \phi_2(\overline{c})$ or $b = \phi_2(\overline{b}) \leq \phi_2(\overline{c})$. We conclude that $\phi_2(\overline{c})$ is a prime element.

(2) $\forall \overline{a}, \overline{b} \in E_2$, let c be a coprime element of E_1 and $\overline{a} \vee \overline{b} \geq \phi_1(c)$. Since ϕ_1 is a surjection, there exist $a, b \in E_1$ such that $\phi_1(a) = \overline{a}$ and $\phi_1(b) = \overline{b}$. Because ϕ_1 is a left adjoint of ϕ_2 and ϕ_2 is a join-preserving, $c \leq \phi_2(\phi_1(a)) \vee \phi_2(\phi_1(b))$. Thus $\phi_1(c) \leq \overline{a}$ or $\phi_1(c) \leq \overline{b}$. Then $\phi_1(c)$ is a coprime element.

(3) Since $0_{E_1} \notin F$, $0_{E_2} \notin \phi_1(F)$. $\forall \overline{b} \in E_2, \forall \overline{a} \in \phi_1(F)$, let $\overline{a} \leq \overline{b}$. Since ϕ_1 is a bijection, there exist $b \in E_1$ and $a \in F$ such that $\overline{b} = \phi_1(b)$ and $\overline{a} = \phi_1(a)$. It follows from that ϕ_1 is a left adjoint of ϕ_2 that $a \leq \phi_2(\phi_1(b))$. Thus, $\phi_2(\phi_1(b)) \in F$. Then $\overline{b} = \phi_1(\phi_2(\phi_1(b))) \in \phi_1(F)$. Therefore $\phi_1(F)$ is a supper-set.

$\forall \overline{a}, \overline{b} \in \phi_1(F)$, let $\phi_1(a) = \overline{a}$ and $\phi_1(b) = \overline{b}$, where $a, b \in F$. Since F is a filter, we have $c \in F$ such that $a \geq c$ and $b \geq c$. Thus, there exists $\phi_1(c) \in \phi_1(F)$ such that $\overline{a} \geq \phi_1(c)$ and

$\overline{b} \geq \phi_1(c)$, so $\phi_1(F)$ is a proper filter.

$\forall \overline{a}, \overline{b} \in E_2$, let $\overline{a} \vee \overline{b} \in \phi_1(F)$. By the condition, $\exists a, b \in E_1$ such that $\phi_1(a) = \overline{a}, \phi_1(b) = \overline{b}$, thus $\exists c \in F$ such that $\phi_1(a) \vee \phi_1(b) = \phi_1(c)$. Since ϕ_1 is a left adjoint of ϕ_2, ϕ_1 is join-preserving. Thus $a \vee b = c$, that is, $a \vee b \in F$. So $a \in F$ or $b \in F$. Then $\overline{a} \in \phi_1(F)$ or $\overline{b} \in \phi_1(F)$. We conclude that $\phi_1(F)$ is a prime filter.

(4) In the same way as in (3), it is easy to show that $\phi_2(I)$ is a prime ideal. □

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