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On a Class of Weakly-Berwald (α,β)-Metrics

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Abstract In this paper, we study an important class of (α,β)-metrics in the form $F=(\alpha+\beta)^{m+1}/\alpha^m$ on an n-dimensional manifold and get the conditions for such metrics to be weakly-Berwald metrics, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a 1-form and m is a real number with $m\neq -1,0,-1/n$. Furthermore, we also prove that this kind of (α,β)-metrics is of isotropic mean Berwald curvature if and only if it is of isotropic S-curvature. In this case, S-curvature vanishes and the metric is weakly-Berwald metric.

Keywords mean Berwald curvature; weakly-Berwald metric; S-curvature; (α,β)-metric.

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1. Introduction

(α,β)-metrics form a very important and rich class of Finsler metrics including Randers metrics and Riemannian metrics. In the past several years, we witness a rapid development in Finsler geometry. Various curvatures have been studied and their geometric meanings are better understood. This is partially due to the study of (α,β)-metrics. Hence, it is worthy of doing study for such metrics deeply. The important applications of (α,β)-metrics in physics and biology have been found and studied[1,2,6,8,9].

An (α,β)-metric is expressed in the following form

$$F=\alpha\phi(s),\quad s=\frac{\beta}{\alpha},$$

where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta=b_i(x)y^i$ is a 1-form and $\phi=\phi(s)$ is a C^∞ positive function on an open interval $(-b_0,b_0)$. It is known that $F=\alpha\phi(\beta/\alpha)$ is a Finsler metric for any α and β with $b:=\|\beta_x\|_\alpha<b_0$ if and only if ϕ satisfies the following condition[6,11]:

$$\phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s)>0,\quad |s|\le b<b_0. \tag{1}$$

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Such metric is called an (α,β)-metric. If $\phi=1+s$, one gets a Randers metric.

Let

$$r_{ij}:=\frac{1}{2}(b_{i|j}+b_{j|i}),\quad s_{ij}:=\frac{1}{2}(b_{i|j}-b_{j|i}),$$
$$s^i{}_j:=a^{im}s_{mj},\quad s_j:=b_is^i{}_j=b^ms_{mj},\quad r_j:=b^ir_{ij},$$

where "|" denotes the covariant derivative with respect to the Levi-Civita connection of α. We will denote $r_{00}:=r_{ij}y^iy^j, s_0:=s_jy^j$, etc.

Bácsó and Yoshikawa first investigated weadly-Berwald spaces in 2002[3]. The class of weakly-Berwald metrics is larger than that of Berwald metrics. Let

$$G^i=\frac{g^{il}}{4}\{[F^2]_{x^ky^l}y^k-[F^2]_{x^l}\}$$

be the geodesic coefficients of a Finsler metric $F=F(x,y)$, where $(g_{ij}):=(\frac{1}{2}[F^2]_{y^iy^j})$ and $(g^{ij}):=(g_{ij})^{-1}$. The Berwald tensor $\mathbf{B}_y=B^i_{jkl}(x,y)\mathrm{d}x^j\otimes\mathrm{d}x^k\otimes\mathrm{d}x^l\otimes\frac{\partial}{\partial x^i}$ is defined by

$$B^i_{jkl}:=\frac{\partial^3G^i}{\partial y^j\partial y^k\partial y^l}(x,y). \tag{2}$$

Furthermore, the mean Berwald curvature tensor $\mathbf{E}_y=E_{ij}(x,y)\mathrm{d}x^i\otimes\mathrm{d}x^j$ is defined by

$$E_{ij}:=\frac{1}{2}B^r_{ijr}(x,y). \tag{3}$$

For a Finsler metric F and a volume form $\mathrm{d}V=\sigma(x)\mathrm{d}x$ on an n-dimensional manifold M, the S-curvature $\mathbf{S}$ is given by

$$\mathbf{S}=\frac{\partial G^r}{\partial x^r}-y^r\frac{\partial\ln\sigma}{\partial x^r}. \tag{4}$$

A Finsler metric is called a Berwald metric if the Berwald curvature $\mathbf{B}=0$. A Finsler metric is called a weakly-Berwald metric if the mean Berwald curvature $\mathbf{E}=0$. More general, we have the following

Definition[12] *Let F be a Finsler metric on an n-dimensional manifold M.*

(a) F is of isotropic mean Berwald curvature if

$$\mathbf{E}=\frac{n+1}{2}cF^{-1}h;$$

(b) F is of isotropic S-curvature if

$$\mathbf{S}=(n+1)cF,$$

where $c=c(x)$ is a scalar function on M and h denotes the angular metric tensor of F which is defined by $h_{ij}=FF_{y^iy^j}$.

The second author and Shen have proved that, for a Randers metric $F=\alpha+\beta$, the following are equivalent[5]:

(i) $\mathbf{S}=(n+1)cF$;

(ii) $\mathbf{E}=(n+1)cF^{-1}h$;

(iii) $r_{ij}+b_is_j+b_js_i=2c(a_{ij}-b_ib_j)$,

where $c=c(x)$ is scalar function on M. In particular, a Randers metric $F=\alpha+\beta$ is a weakly-Berwald metric if and only if $r_{ij}+b_is_j+b_js_i=0$, which is equivalent to $\mathbf{S}=0$. On the other hand, Yoshikawa, Okubo and Matsumoto proved that[13] a Matsumoto metric $F=\alpha^2/(\alpha-\beta)$ is a weakly-Berwald metric if and only if $r_{ij}=0,\ s_i=0$.

The main purpose of this paper is to study and characterize a special class of weakly-Berwald (α,β)-metrics in the form

$$F=\frac{(\alpha+\beta)^{m+1}}{\alpha^m},\quad \alpha=\sqrt{a_{ij}(x)y^iy^j},\quad \beta=b_i(x)y^i,$$

where m is an arbitary real number. Obviously, this class of (α,β)-metrics contains Riemannian metric $F=\alpha$ $(m=-1)$, Randers metric $F=\alpha+\beta\,(m=0)$ and the metric $F=\frac{(\alpha+\beta)^2}{\alpha}$ $(m=1)$. If we substitute β with $-\beta$ and take $m=-2$, the resulting metric is just Matsumoto metric $F=\alpha^2/(\alpha-\beta)$.

Theorem 1.1 *Let $F=(\alpha+\beta)^{m+1}/\alpha^m$ be an (α,β)-metric on an n-dimensional manifold M, where m is a real number with $m\neq -1,0,-1/n$. Then F is weakly-Berwald metric if and only if $r_{ij}=0,\ s_i=0$.*

Furthermore, we obtain the following

Theorem 1.2 *Let $F=(\alpha+\beta)^{m+1}/\alpha^m$ be an (α,β)-metric on an n-dimensional manifold M, where m is a real number with $m\neq -1,0,-1/n$. Then the following conditions are equivalent:*

(i) F is of isotropic S-curvature, $\mathbf{S}=(n+1)cF$;

(ii) F is of isotropic mean Berwald curvature, $\mathbf{E}=\frac{n+1}{2}cF^{-1}h$;

(iii) β is a Killing 1-form with $b=constant$ with respect to α, that is, $r_{ij}=0,\ s_i=0$;

(iv) $\mathbf{S}=0$;

(v) F is weakly-Berwald metric, i.e. $\mathbf{E}=0$,

where $c=c(x)$ is a scalar function on M.

2. (α,β)-metric

Let G^i and G^i_α be the spray coefficient of F and α respectively given by

$$G^i=\frac{g^{il}}{4}\{[F^2]_{x^ky^l}y^k-[F^2]_{x^l}\},\quad G^i_\alpha=\frac{a^{il}}{4}\{[\alpha^2]_{x^ky^l}y^k-[\alpha^2]_{x^l}\},$$

where $(a^{ij}):=(a_{ij})^{-1}$. We have the following formula for the spray coefficients G^i of F:

Lemma 2.1[6,10] *The geodesic coefficients G^i are related to G^i_α by*

$$G^i=G^i_\alpha+\Theta\{-2Q\alpha s_0+r_{00}\}\frac{y^i}{\alpha}+\alpha Q s^i_0+\Psi\{-2Q\alpha s_0+r_{00}\}b^i, \tag{5}$$

where G^i_α denote the spray coefficients of α and

$$\Theta:=\frac{(\phi-s\phi')\phi'-s\phi\phi''}{2\phi((\phi-s\phi')+(b^2-s^2)\phi'')},$$

$$Q:=\frac{\phi'}{\phi-s\phi'},$$

$$\Psi:=\frac{\phi''}{2((\phi-s\phi')+(b^2-s^2)\phi'')},$$

where $s := \beta/\alpha$, $b = \|\beta_x\|_\alpha$.

It is well known that the condition for a Finsler metric to be weakly-Berwald metric is $B^r_{jkr} = 0$. This is equivalent to that $N^r_r := \partial G^r/\partial y^r$ is a 1-form. By Lemma 2.1 and (2), we have the following

Lemma 2.2[13] *An* (α,β)*-metric* $F = \alpha\phi(\beta/\alpha)$ *is a weakly-Berwald metric if and only if* N^r_r *is a 1-form.*

From Lemma 2.1, we can get

$$N^r_r = Lr_{00} + 2Mr_0 + Ns_0, \tag{6}$$

where

$$\begin{aligned} L &:= (n+1)\alpha^{-1}\Theta + \frac{\partial\Psi}{\partial y^r}b^r,\\ M &:= \Psi,\\ N &:= -\big\{2(n+1)Q\Theta + \frac{\partial(2\Psi Q\alpha)}{\partial y^r}b^r\big\} + \frac{\partial(\alpha Q)}{\partial y^r}s_0 s^r_0. \end{aligned}$$

Recently, the second author and Z. Shen have obtained a formula for the S-curvature of an (α,β)-metric on an n-dimensional manifold M as follows

Lemma 2.3[4] *The* S*-curvature of an* (α,β)*-metric is given by*

$$\mathbf{S} = \mu(r_0+s_0) + 2(\Psi + QC)s_0 - 2\Psi r_0 + \alpha^{-1}\big[(b^2-s^2)\Psi' + (n+1)\Theta\big]r_{00}, \tag{7}$$

where $\mu := -f'(b)/[bf(b)]$ *is a scalar function on* M *and* $C := -(b^2-s^2)\Psi' - (n+1)\Theta$.

3. Proof of Theorem 1.1

In this section, we consider the (α,β)-metric in the following form:

$$F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m} = \alpha\phi(s), \quad \phi = (1+s)^{m+1}, \quad s = \frac{\beta}{\alpha}. \tag{8}$$

Let $b_0 = b_0(m) > 0$ be the largest number such that

$$(1+s)^{m+1} > 0, \quad (1+s)(1-ms) + m(m+1)(b^2-s^2) > 0, \quad |s| \le b < b_0, \tag{9}$$

so that $F = (\alpha+\beta)^{m+1}/\alpha^m$ is a Finsler metric if and only if β satisfies that $b := \|\beta_x\|_\alpha < b_0$ for any $x \in M$. It is easy to see that $b_0 = b_0(m) \le 1$ for $m \neq -1$. Particularly, we have known that $b_0 = 1$ as $m = 0, 1$[5,10,12] and $b_0 = \frac{1}{2}$ as $m = -2$[7]. In general, for fixed m, we always can determine b_0 such that (9) holds. For example, when $m > 0$ and $b := \|\beta_x\|_\alpha < \min\{1, 1/m\}$ (9) holds.

For $F = (\alpha+\beta)^{m+1}/\alpha^m$, by Lemmas 2.1 and 2.2 and by using Maple programm, we can easily get the following

$$\begin{aligned} L =& \frac{(n+1)(m+1)(\alpha-2m\beta)}{2\{[1+m(m+1)b^2]\alpha^2 + (1-m)\alpha\beta - m(m+2)\beta^2\}} + \\ & \frac{m(m+1)(\beta^2-b^2\alpha^2)[(1-m)\alpha - 2m(m+2)\beta]}{2\{[1+m(m+1)b^2]\alpha^2 + (1-m)\alpha\beta - m(m+2)\beta^2\}^2}, \end{aligned}$$

$$M =\frac{m(m+1)\alpha^2}{2\{[1+m(m+1)b^2]\alpha^2+(1-m)\alpha\beta-m(m+2)\beta^2\}},$$

$$\begin{aligned}N =&\frac{-(n+1)(m+1)^2\alpha^2(\alpha-2m\beta)}{(\alpha-m\beta)\{[1+m(m+1)b^2]\alpha^2+(1-m)\alpha\beta-m(m+2)\beta^2\}}+\\&\frac{-m(m+1)^2[1+m(m+1)b^2]\alpha^5\beta}{(\alpha-m\beta)^2\{[1+m(m+1)b^2]\alpha^2+(1-m)\alpha\beta-m(m+2)\beta^2\}^2}+\\&\frac{m(m+1)^2\alpha^3\beta^2[2m^2(m+1)b^2+4n-2\alpha+9m\beta]}{(\alpha-m\beta)^2\{[1+m(m+1)b^2]\alpha^2+(1-m)\alpha\beta-m(m+2)\beta^2\}^2}+\\&\frac{m(m+1)^2\alpha^2\{-4m^2(m+2)\beta^4-b^2[m^2(m+1)b^2+2n-1]\alpha^4\}}{(\alpha-m\beta)^2\{[1+m(m+1)b^2]\alpha^2+(1-m)\alpha\beta-m(m+2)\beta^2\}^2}+\\&\frac{m(m+1)^2b^2\alpha^4\beta[3m^2(m+2)\beta-\alpha]}{(\alpha-m\beta)^2\{[1+m(m+1)b^2]\alpha^2+(1-m)\alpha\beta-m(m+2)\beta^2\}^2}+\\&\frac{m(m+1)\alpha^2}{(\alpha-m\beta)^2}.\end{aligned}\tag{10}$$

Proof of Theorem 1.1 Assume F is weakly-Berwald metric. Plugging (10) into (6) yields the following equation

$$A\alpha^6+B\alpha^4+C\alpha^2+D+\alpha(E\alpha^4+F\alpha^2+G)=0,\tag{11}$$

where

$$\begin{aligned}A :=&2\big[1+m(m+1)b^2\big]^2N^r_r-2m(m+1)\big[1+m(m+1)b^2\big]r_0+\\&2(m+1)\big[1+m(m+1)b^2\big]\big\{(n+1)(m+1)-m[1+m(m+1)b^2]\big\}s_0+\\&2m(m+1)^2b^2\big[m^2(m+1)b^2+2m-1\big]s_0,\\B :=&2\big\{(1-m)^2+m\big[1+m(m+1)b^2\big]\big[2m-8+m(1+m(m+1)b^2)\big]\big\}\beta^2N^r_r+\\&\{4(n+1)m(m+1)[1+m(m+1)b^2]-(n+1)(1-m^2)-6m^2(m+1)b^2\}\beta r_{00}+\\&2m^2(m+1)\big\{4-m-m[1+m(m+1)b^2]\big\}\beta^2r_0+\\&2m(m+1)\big\{2m(mn+2m+n+3)[1+m(m+1)b^2]+1-5m^2+\\&(m+1)\big[(n+1)(2m-5)-m^2(5m+8)b^2\big]\big\}\beta^2s_0,\\C :=&2m^2\big\{9+2(m+2)[2-2m-m\big(1+m(m+1)b^2\big)]\big\}\beta^4N^r_r+\\&m^2(m+1)\big\{2(n+1)m[1+m(m+1)b^2]+(n+1)(m-13)+\\&6-2m^2(m+2)b^2\big\}\beta^3r_{00}-4m^3(m+1)(m+2)(2m+n+2)\beta^4s_0,\\D :=&2m^4(m+2)^2\beta^6N^r_r-2nm^4(m+1)(m+2)\beta^5r_{00},\\E :=&4[1+m(m+1)b^2]\big\{m[1+m(m+1)b^2]+m-1\big\}\beta N^r_r+\\&(m+1)\big\{(n+1)[1+m(m+1)b^2]+m(m-1)b^2\big\}r_{00}-\\&2m(m+1)\big\{(m-1)+m[1+m(m+1)b^2]\big\}\beta r_0+\\&2(m+1)\big\{(m+1)[(m-1)(n+1)-6m^2b^2]+\\&m[1+m(m+1)b^2](3mn+3n+4)\big\}\beta s_0,\\F :=&\big\{12m(1-m)-4m^2(m+5)[1+m(m+1)b^2]\big\}\beta^3N^r_r+\end{aligned}$$

$$m(m+1)\{5m(n+1)[1+m(m+1)b^2]+3(n+1)(m-2)+1-m-$$
$$3m^2(m+3)b^2\}\beta^2 r_{00}+2m^3(m+1)(m+5)\beta^3 r_0+$$
$$2\{-(n+1)m^2(m+1)^2(m+8)+m^2(m+1)(2m+1)(m+5)\}\beta^3 s_0,$$
$$G := \big[12m^3(m+2)\big]\beta^5 N_r^r - 3m^3(m+1)(mn+4n+1)\beta^4 r_{00}.$$

By assumption, N_r^r is a 1-form. Note that the coefficients of α in (11) must be zero (because α^{even} is a polynomial in y^i). Then (11) is equivalent to the following two equations

$$A\alpha^6 + B\alpha^4 + C\alpha^2 + D = 0, \tag{12}$$

$$E\alpha^4 + F\alpha^2 + G = 0. \tag{13}$$

If $m \neq -2$, (12) $\times$ 6 subtracting (13) $\times\, m(m+2)\beta$ yields

$$6A\alpha^6 + H\alpha^4 + I\alpha^2 - 3m^4(m+1)(m+2)(mn+1)\beta^5 r_{00} = 0, \tag{14}$$

where $H = 6B - m(m+2)\beta E$, $I = 6C - m(m+2)\beta F$. Note that $m \neq 0, -1$ and $-1/n$, we know from (14) that $\beta^5 r_{00}$ can be divided by α^2. Because β^5 and α^2 are relatively prime polynomials of (y^i), there is a scalar function $\rho(x)$ on M such that

$$r_{00} = \rho(x)\alpha^2. \tag{15}$$

Substituting (15) into (13), we get

$$E\alpha^4 + [F - 3m^3(m+1)(mn+4n+1)\rho(x)\beta^4]\alpha^2 = -12m^3(m+2)\beta^5 N_r^r.$$

It is easy to see that the left-hand side of the above equation can be divided by α^2. Hence N_r^r can be divided by α^2. However, N_r^r is a 1-form. So we obtain

$$N_r^r = 0. \tag{16}$$

By (15), we have

$$r_0 = \rho(x)\beta. \tag{17}$$

Plugging (15), (16), (17) into (12) yields

$$A\alpha^4 + \big\{B + m^2(m+1)\big[2(n+1)m\big(1+m(m+1)b^2\big)+$$
$$(n+1)(m-13)+6-2m^2(m+2)b^2\big]\rho(x)\beta^3\big\}\alpha^2$$
$$= 2m^3(m+1)(m+2)\big[2(2m+n+2)s_0 + nm\rho(x)\beta\big]\beta^4.$$

Since α^2 is not divided by β^4, from the above equation, we get

$$2(2m+n+2)s_0 + nm\rho(x)\beta = 0,$$

that is

$$2(2m+n+2)s_i + nm\rho(x)b_i = 0. \tag{18}$$

Contracting (18) with b^i yields $nm\rho(x)b^2 = 0$. Since $m \neq 0$, we obtain $\rho(x) = 0$. Thus, from (15), (17) and (18), we obtain

$$r_{00} = r_0 = s_0 = 0. \tag{19}$$

When $m = -2$, if we substitute β with $-\beta$, the resulting metric is just Matsumoto metric $F = \alpha^2/(\alpha - \beta)$. In this case, the result is still true[13].

Conversely, we suppose that the equations $r_{ij} = s_i = 0$ hold. Then from (6), we have $N^r_r = 0$. This completes the proof. □

4. Proof of Theorem 1.2

For $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m}$, by Lemma 2.1, we have

$$\begin{aligned} Q &= \frac{m+1}{1-ms}, \\ \Theta &= \frac{(m+1)(1-2ms)}{2[1+m(m+1)b^2-(m-1)s-m(m+2)s^2]}, \\ \Psi &= \frac{m(m+1)}{2[1+m(m+1)b^2-(m-1)s-m(m+2)s^2]}, \\ \Psi' &= \frac{m(m+1)[2m(m+2)s+m-1]}{2[1+m(m+1)b^2-(m-1)s-m(m+2)s^2]^2}. \end{aligned} \tag{20}$$

Proof of Theorem 1.2 The proof contains the following steps:

Step 1. (i) $\Leftrightarrow$ (ii). In fact, (i) $\Rightarrow$ (ii) is clearly true. Assume that (ii) holds, which is equivalent to

$$\mathbf{S} = (n+1)\{cF + \eta\}, \tag{21}$$

where η is a 1-form on M. So (i) is equivalent to (ii) if and only if $\eta = 0$. Plugging (20) and (21) into (7) yields

$$\begin{aligned} &J_6\alpha^5 + J_5\alpha^4 + J_4\alpha^3 + J_3\alpha^2 + J_2\alpha + J_1 \\ &= (n+1)c[K_6\alpha^5 + K_5\alpha^4 + K_4\alpha^3 + K_3\alpha^2 + K_2\alpha + K_1]\frac{(\alpha+\beta)^{m+1}}{\alpha^m}, \end{aligned} \tag{22}$$

where

$$\begin{aligned} K_1 :=& -2m^3(m+2)^2\beta^5, \\ K_2 :=& -2m^2(m+2)(m-4)\beta^4, \\ K_3 :=& 2m[3m^2+8m-5+2m^2(m+1)(m+2)b^2]\beta^3, \\ K_4 :=& -12m[1-m(m+1)b^2]\beta^2, \\ K_5 :=& -2[m^3(m+1)^2b^4+3m-6+2m(m+1)(2m-1)b^2]\beta, \\ K_6 :=& 2[1+m(m+1)b^2]^2, \\ J_1 :=& -2\mu m^3(m+2)^2\beta^5(s_0+r_0) - 2nm^3(m+1)(m+2)\beta^4 r_{00} + \\ & 2(n+1)\eta m^3(m+2)^2\beta^5, \\ J_2 :=& -2\mu m^2(m+2)(m-4)\beta^4(s_0+r_0) + m^2(m+1)[(n+1)(m+8)-m-5]\beta^3 r_{00} + \\ & 2(n+1)\eta m^2(m+2)(m-4)\beta^4, \\ J_3 :=& 2\mu m[3m^2+8m-5+2m^2(m+1)(m+2)b^2]\beta^3(s_0+r_0) + \\ & 2m^2(m+1)(m+2)[m-2n(m+1)]\beta^3 s_0 - 2m^3(m+1)(m+2)\beta^3 r_0 + \end{aligned}$$

$$m[(n+1)(4m-5)-m+1+2m^2[(n+1)(m+1)-m-2]b^2]\beta^2 r_{00}-$$
$$2(n+1)\eta m[3m^2+8m-5+2m^2(m+1)(m+2)b^2]\beta^3,$$
$$J_4 := 12\mu m[1-m(m+1)b^2]\beta^2(s_0+r_0)+2m(m+1)[3-n(m+1)(m-4)]\beta^2 s_0+$$
$$6m^2(m+1)\beta^2 r_0+(m+1)[m^2(m+5)b^2-3m^2(n+1)(m+1)b^2-$$
$$(4m-1)(n+1)]\beta r_{00}+12(n+1)\eta m[1-m(m+1)b^2]\beta^2,$$
$$J_5 := [m^3(m+1)^2b^4-3m-6+2m(m+1)(2m-1)b^2]\beta(s_0+r_0)+$$
$$2(m+1)[n(3m-1)(m+1)+m^2+3m-1-2nm^2(m+1)^2b^2]\beta s_0+$$
$$2m(m+1)[2m-1+m^2(m+1)b^2]\beta r_0+(m+1)[m(m-1)b^2+$$
$$(n+1)(1+m(m+1)b^2)]r_{00}+2(n+1)\eta[m^3(m+1)^2b^4+3m-6+$$
$$+2m(m+1)(2m-1)b^2]\beta,$$
$$J_6 := 2\mu[1+m(m+1)b^2](s_0+r_0)-2(m+1)[n(m+1)+$$
$$1+(m+1)[n(m+1)+m]b^2]s_0-2m(m+1)[1+m(m+1)b^2]r_0-$$
$$2(n+1)\eta[1+m(m+1)b^2]^2.$$

Rewrite (22) as follows

$$\begin{aligned}&\alpha^m[J_6\alpha^5+J_5\alpha^4+J_4\alpha^3+J_3\alpha^2+J_2\alpha+J_1]-\\&\quad(n+1)c[K_6\alpha^4+K_5\alpha^3+K_4\alpha^2+K_3\alpha+K_2]\alpha(\alpha+\beta)^{m+1}-\\&\quad(n+1)cK_1(\alpha+\beta)^{m+1}=0.\end{aligned} \tag{23}$$

When m is a positive integer, it is easy to see that the term which does not includ α in (23) is just $-(n+1)cK_1\beta^{m+1}$. Because α^2 is not divided by β, we get $c=0$. So

$$J_6\alpha^5+J_5\alpha^4+J_4\alpha^3+J_3\alpha^2+J_2\alpha+J_1=0.$$

When m is a non-zero real number but not a positive integer, we know that the left-hand side of (22) is a polynomial in α, but the the term $(\alpha+\beta)^{m+1}/\alpha^m$ is not a polynomial in α. Hence we also have

$$J_6\alpha^5+J_5\alpha^4+J_4\alpha^3+J_3\alpha^2+J_2\alpha+J_1=0.$$

Therefore, when m is a non-zero real number, we always have

$$J_5\alpha^4+J_3\alpha^2+J_1+\alpha(J_6\alpha^4+J_4\alpha^2+J_2)=0. \tag{24}$$

Note that the coefficients of α in (24) must be zero (because α^{even} is a polynomial in y^i). Then (24) is equivalent to the following two equations

$$J_5\alpha^4+J_3\alpha^2+J_1=0, \tag{25}$$

$$J_6\alpha^4+J_4\alpha^2+J_2=0. \tag{26}$$

If $m\neq -2$, (25) $\times$ $(m-4)$ $-$ (26) $\times$ $m(m+2)\beta$ yields

$$X\alpha^4+Y\alpha^2-3m^3(m+1)(m+2)(mn+1)\beta^4 r_{00}=0,$$

where $X = (m-4)J_5 - m(m+2)\beta J_6$, $Y = (m-4)J_3 - m(m+2)\beta J_4$. Note $m \neq 0, -1$ and $-1/n$, and β^4 and α^2 are relatively prime polynomials of (y^i), we know that r_{00} can be divided by α^2. That is, there is a scalar function $\tau(x)$ on M such that

$$r_{00} = \tau(x)\alpha^2. \tag{27}$$

Substituting (27) into (25), we get

$$J_5\alpha^4 + [J_3 - 2nm^3(m+1)(m+2)\tau(x)\beta^4]\alpha^2 + 2m^3(m+2)^2[(n+1)\eta - \mu(s_0 + r_0)]\beta^5 = 0. \tag{28}$$

This implies that $[(n+1)\eta - \mu(s_0+r_0)]\beta^5$ can be divided by α^2. Because β^5 and α^2 are relatively prime polynomials of (y^i), we know that $(n+1)\eta - \mu(s_0 + r_0)$ can be divided by α^2, which is impossible unless

$$(n+1)\eta - \mu(s_0 + r_0) = 0. \tag{29}$$

From (27), we have

$$r_0 = \tau(x)\beta. \tag{30}$$

Plugging (27), (29), (30) into (25) yields

$$\begin{aligned} &J_5\alpha^2 + m\big\{(n+1)(4m-5) - (m-1) + 2m^2\big[(n+1)(m+1) - (m+2)\big]b^2\big\}\tau(x)\beta^2\alpha^2 \\ &= 2m^2(m+1)(m+2)\big\{[m - 2n(m+1)]s_0 - m(n+1)\tau(x)\beta\big\}\beta^3. \end{aligned} \tag{31}$$

Since β^3 is not divided by α^2, we get

$$[m - 2n(m+1)]s_0 - m(n+1)\tau(x)\beta = 0,$$

that is,

$$[m - 2n(m+1)]s_i - m(n+1)\tau(x)b_i = 0. \tag{32}$$

Contracting (32) with b_i yields

$$-m(n+1)\tau(x)b^2 = 0.$$

Because $b^2 \neq 0$ and $m \neq 0$, $\tau(x) = 0$. From (27), (30) and (32), we obtain

$$r_{00} = 0,\ r_0 = 0,\ s_0 = 0. \tag{33}$$

Thus, from (29), we obtain $\eta = 0$.

When $m = -2$, if we substitute β with $-\beta$, the resulting metric is just Matsumoto metric $F = \alpha^2/(\alpha - \beta)$. In this case, the result is still true[4].

Step 2. (ii) $\Rightarrow$ (iii). The proof has been contained in the Step 1.

Step 3. (iii) $\Rightarrow$ (iv). When $r_{00} = 0$, $s_0 = 0$, by Lemma 2.3, we have $\mathbf{S} = 0$.

Step 4. (iv) $\Rightarrow$ (v). $\mathbf{S} = 0$ implies that F is of isotropic S-curvature with $c = 0$. Thus we obtain $\mathbf{E} = 0$ by the equivalence of (i) and (ii).

Step 5. (v) $\Rightarrow$ (i). $\mathbf{E} = 0$ is equivalent to that F is of isotropic mean Berwald curvature with $c = 0$, that is, (ii) holds with $c = 0$. By the equivalence of (i) and (ii), we know that F has isotropic S-curvature with $c = 0$. This completes the proof. □

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