Journal of Mathematical Research & Exposition
Mar., 2009, Vol. 29, No. 2, pp. 257–265
DOI:10.3770/j.issn:1000-341X.2009.02.008
Http://jmre.dlut.edu.cn

E-H-Unretractivity of Bipartite Graphs

LI Wei Min
(Department of Mathematics, Shanghai Jiaotong University, Shanghai 200030, China)
(E-mail: wmli@sjtu.edu.cn)

Abstract By $\mathrm{End}(G)$ and $h\mathrm{End}(G)$ we denote the set of endomorphisms and half-strong endomorphisms of a graph G respectively. A graph G is said to be E-H-unretractive if $\mathrm{End}(G) = h\mathrm{End}(G)$. A general characterization of an E-H-unretractive graph seems to be difficult. In this paper, bipartite graphs with E-H-unretractivity are characterized explicitly.

Keywords endomorphism monoid; E-H-unretractivity; bipartite graph.

Document code A
MR(2000) Subject Classification 05C25
Chinese Library Classification O157.5

1. Introduction and preliminaries

The monoid of endomorphisms of a graph has been the object of researches in the theory of semigroups for quite some time[1,2]. The motivation of these researches is to contribute to the application of semigroup theory to graph theory. The graphs for which different endomorphism classes coincide (i.e., various unretractivities) is one of the main themes in this line, and as justification for the investigation one takes the rich algebra structure which is put on a graph by its endomorphism classes and the numerous questions connected with them[3]. It was proposed in [3] as an open question to find conditions on a graph for various unretractivities. As pointed out in [4] the characterization of graphs with various unretractivities seems a difficult problem. In this paper bipartite graphs with E-H-unretractivity are explicitly presented.

In this paper, we consider only finite undirected graphs without loops and multiple edges. If G is a graph, we denote by $V(G)$ (or simply G) and $E(G)$ its vertex set and edge set, respectively. A graph G is called a bipartite graph if it is possible to partition $V(G)$ into two subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 to a vertex of V_2. By K_n we denote a complete graph with n vertices and by C_n a cycle with n vertices. It is well known that a graph is bipartite if and only if it does not contain any C_n where n is an odd number, and therefore trees is a special class of bipartite graphs. An empty graph with n vertices is denoted by $\overline{K}_n$. A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Moreover, if for any $a, b \in V(H)$, $\{a, b\} \in E(H)$ if and only if $\{a, b\} \in E(G)$, then we call H an induced subgraph of G. Let $a, b \in G$. The length of the shortest (a, b)-path (i.e., geodesic line) is called the *distance* of

Received date: 2007-03-05; **Accepted date**: 2007-07-13
Foundation item: the National Natural Science Foundation of China (No. 10671122).

the vertices a and b in G, denoted by $d_G(a,b)$. The *diameter* of a connected graph $G(\neq K_1)$, denoted by $\mathrm{diam}(G)$, is defined as the maximum of distances of all vertex pairs of G (i.e., the length of any longest geodesic lines), and define $\mathrm{diam}(K_1)=0$. A subgraph H of G is called *isometric* if for any $x,y\in H$, $d_H(x,y)=d_G(x,y)$. A maximal connected subgraph of G is called a connected component (or simply component) of G. The length of the shortest cycles (if it exists) of graph G is called the girth of G. denoted by $\mathrm{gir}(G)$. We use nG to represent a graph composed of n graphs each of which is isomorphic to a connected graph G. A completely bipartite graph $G(V_1\cup V_2,E)$ with $|V_1|=m\geq 1$ and $|V_2|=n\geq 1$ is denoted by $K_{m,n}$. A graph $K_{1,n}$ is also called a star. Let G_1 and G_2 be graphs with disjoint vertex sets. The *union* of G_1 and G_2, denoted by $G_1\cup G_2$, is a graph such that $V(G_1\cup G_2)=V(G_1)\cup V(G_2)$ and $E(G_1\cup G_2)=E(G_1)\cup E(G_2)$. A component K_1 is also called an isolated vertex of G. For a vertex $a\in G$, we denote the degree of a in G by $\deg_G(a)$ or simply $\deg(a)$ if it is clear which graph G is referred to. The usual concepts such as connected graph, completely bipartite graph, complete graph, empty graph, path, cycle and degree (of a vertex) etc., which are not defined in this paper, can be found in [5].

The following definitions of various types of endomorphisms are mainly based on [3]. If G and H are graphs, then a mapping $f:V(G)\to V(H)$ is called a *homomorphism* from G to H if $\{a,b\}\in E(G)$ implies that $\{f(a),f(b)\}\in E(H)$ for any $a,b\in G$. Moreover, if f is bijective and its inverse mapping is also a homomorphism (from H to G), then f is called an isomorphism from G to H. An endomorphism of G is a homomorphism from G to itself. An endomorphism is called a strong endomorphism if $\{f(a),f(b)\}\in E(G)$ implies that $\{a,b\}\in E(G)$ for any $a,b\in G$. A bijective endomorphism of a graph G is called an automorphism of G. Evidently, an automorphism of a graph G is an isomorphism from G to itself. Let f be an endomorphism of graph G and let $a\in G$. Denote $f^{-1}(a):=\{x\in G|f(x)=a\}$. An endomorphism f is said to be half-strong if $\{f(a),f(b)\}\in E(G)$ implies that there exist $c\in f^{-1}(f(a))$ and $d\in f^{-1}(f(b))$ such that $\{c,d\}\in E(G)$, where $a,b,c,d\in G$.

By $\mathrm{End}(G)$, $h\mathrm{End}(G)$, $s\mathrm{End}(G)$ and $\mathrm{Aut}(G)$ we denote the sets of endomorphisms, half-strong endomorphisms, strong endomorphisms and automorphisms of the graph G, respectively. Obviously, $\mathrm{Aut}(G)\subseteq s\mathrm{End}(G)\subseteq h\mathrm{End}(G)\subseteq \mathrm{End}(G)$. It is well-known that $\mathrm{End}(G)$ and $s\mathrm{End}(G)$ are monoids (a monoid is a semigroup with an identity element) and that $\mathrm{Aut}(G)$ is a group with respect to the composition of mappings, while $h\mathrm{End}(G)$ does not form a monoid in general. The coincidence of these endomorphism classes gives rise to various unretractivities of a graph. In particular, a graph G is called E-H-unretractive (respectively, E-S-unretractive and E-A-unretractive etc.). if $\mathrm{End}(G)=h\mathrm{End}(G)$ (respectively, $\mathrm{End}(G)=s\mathrm{End}(G)$ and $\mathrm{End}(G)=\mathrm{Aut}(G)$ etc.) If graph G is E-A-unretractive, we also call it simply *unretractive*. In [1], E-S-unretractivity, E-A-unretractivity and S-A-unretractivity of a graph were studied. In [6], E-A-unretractivity and S-A-unretractivity of joins and lexicographic products of graphs were characterized. Relationships among endomorphism classes of trees were explored in [7]. A general characterization of an E-H-unretractive graph seems to be difficult. Undoubtedly bipartite graphs is one of the most important families of graphs, and we will completely determine E-H-unretractive bipartite graphs.

Let $f \in \mathrm{End}(G)$. A subgraph of G is called the endomorphic image of G under f, denoted by I_f, if $V(I_f) = f(G)$, and $\{f(a), f(b)\} \in E(I_f)$ if and only if there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(G)$, where $a, b, c, d \in V(G)$ ([8] for the reasonableness of this definition). An element a of a semigroup S is called an idempotent if $a^2 = a$[9]. The set of idempotents of $\mathrm{End}(G)$ is denoted by $\mathrm{Idpt}(G)$. Each $f \in \mathrm{Idpt}(G)$ is also called a retraction of G. If f is a retraction of graph G, the subgraph induced by $f(G)(= \{f(x)|x \in G\})$ (i.e., the induced subgraph with vertex set $f(G)$) is called a *retract* of G[6,10,11]. Let $f \in \mathrm{End}(G)$. By ρ_f we denote the equivalence relation on $V(G)$ induced by f, i.e., for $a, b \in V(G)$, $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$. Denote by $[a]_{\rho_f}$ the equivalence class of $a \in G$ under ρ_f. The following propositions quoted from the references will be used later.

Proposition 1.1 *(1)*[1, Example 1.2] *The cycles with odd lengths are unretractive.*

(2)[7, Propositions 2.1 and 3.1] *Any tree is E-H-unretractive.*

Proposition 1.2[12, Remark 1.3] *Let G be a graph. Let $f \in \mathrm{End}(G)$ and let $a, b \in G$.*

(1) If G is connected, then I_f is connected;

(2) $d_{I_f}(f(a), f(b)) \leq d_G(a, b)$.

Proposition 1.3 *(1)*[11, Theorem 5] *Every isometric tree $T(\neq K_1)$ in a bipartite graph G is a retract of G, i.e., there exists $f \in \mathrm{Idpt}(G)$ such that T is a subgraph of G induced by $f(G)$.*

(2)[3, Proposition 2.2] *Idempotent endomorphisms of graph G are elements of $h\mathrm{End}(G)$.*

(3)[13, Lemma 2.1(1)] *Let G be a graph and let $f \in \mathrm{End}(G)$. Then $f \in h\mathrm{End}(G)$ if and only if I_f is an induced subgraph of G.*

Proposition 1.4 *Let G be a bipartite graph and let P be a path in G. If P is a geodesic line, there exists $f \in \mathrm{Idpt}(G)$ such that $I_f = P$.*

Proof Obviously, P is an isometric tree in G, and so by Proposition 1.3(1), there exists $f \in \mathrm{Idpt}(G)$ such that P is the subgraph induced by $f(G)$. By Proposition 1.3(2) $f \in h\mathrm{End}(G)$, and so by Proposition 1.3(3) I_f is an induced subgraph of G. Hence $I_f = P$. $\square$

2. E-H-unretractive bipartite graphs

In this section, we will explicitly characterize bipartite graphs with E-H-unretractivity (Theorem 2.11). First, we consider connected bipartite graphs.

Lemma 2.1 *Let G be a connected bipartite graph with cycles. If $\mathrm{diam}(G) \leq \mathrm{gir}(G) - 2$, G is E-H-unretractive.*

Proof Assume G is not E-H-unretractive. Then there exists $f \in \mathrm{End}(G) \setminus h\mathrm{End}(G)$. Thus there exist $a, b \in G$ such that $\{f(a), f(b)\} \in E(G)$. Whereas $\{x, y\} \notin E(G)$ for any $x \in f^{-1}(f(a))$ and any $y \in f^{-1}(f(b))$, by the definition of the image of an endomorphism, $\{f(a), f(b)\} \notin E(I_f)$.

Since G is connected, by Proposition 1.2(1) I_f is connected, and so in I_f there is a geodesic line P connecting $f(a)$ and $f(b)$ with length $d_{I_f}(f(a),f(b))$. Therefore, $P\cup\{f(a),f(b)\}$ is a cycle in G with length $d_{I_f}(f(a),f(b))+1$. Then $\mathrm{gir}(G)\le d_{I_f}(f(a),f(b))+1$. Furthermore, $\mathrm{gir}(G)\le d_G(a,b)+1\le \mathrm{diam}(G)+1$ by Proposition 1.2(2), which contradicts $\mathrm{diam}(G)\le \mathrm{gir}(G)-2$. □

Lemma 2.2 *Let G be a graph, and let $f\in \mathrm{Idpt}(G)$. Then for any $a\in I_f$, $f(a)=a$.*

Proof Since $a\in I_f$ and $f^2=f$, there exists $x\in G$ such that $f(x)=a$ and so

$$f(a)=f(f(x))=f(x)=a.$$

□

The next theorem characterizes connected bipartite graphs with E-H-unretractivity.

Theorem 2.3 *Let G be a connected bipartite graph. Then G is E-H-unretractive if and only if G is a tree or* $\mathrm{diam}(G)\le \mathrm{gir}(G)-2$.

Proof Sufficiency is by Proposition 1.1(2) and Lemma 2.1.

Necessity. Now suppose G is not a tree and $\mathrm{diam}(G)\ge \mathrm{gir}(G)-1$. Let $\mathrm{gir}(G)=n$ ($=4,6,8,\ldots$) and let $\mathrm{diam}(G)=d$. So $d\ge n-1$ and there exists a geodesic line P in G with length d, denoted by $P=a_1a_2\cdots a_{d+1}$. By Proposition 1.4, there exists $f\in\mathrm{Idpt}(G)$ such that $I_f=P$. Let $C_n=b_1b_2\cdots b_n$ be a cycle with length $n(=\mathrm{gir}(G))$. Now define a mapping g from $V(G)$ to itself by the following rule:

$g(x)=b_i$ if $x\in[a_i]_{\rho_f}$ with $i\in\{1,2,\ldots,n\}$;

$g(x)=b_{n-1}$ if $x\in[a_i]_{\rho_f}$ with $i\in\{n+1,n+3,\ldots\}\subseteq\{n+1,n+2,n+3,\ldots,d+1\}$;

$g(x)=b_n$ if $x\in[a_i]_{\rho_f}$ with $i\in\{n+2,n+4,\ldots\}\subseteq\{n+1,n+2,n+3,\ldots,d+1\}$.

It remains to show $g\in\mathrm{End}(G)\setminus h\mathrm{End}(G)$. Noting $d+1\ge n$, we see the mapping g is well defined. Let $\{x_1,x_2\}\in E(G)$. Since $f\in\mathrm{End}(G)$, $\{f(x_1),f(x_2)\}\in E(I_f)=E(P)$. Without loss of generality, we suppose $f(x_1)=a_i$ and $f(x_2)=a_{i+1}$ for some $i\in\{1,2,\ldots,d\}$. As f is an idempotent, $f(x_1)=f(f(x_1))=f(a_i)$ and $f(x_2)=f(f(x_2))=f(a_{i+1})$, i.e., $x_1\in[a_i]_{\rho_f}$ and $x_2\in[a_{i+1}]_{\rho_f}$ for some $i\in\{1,2,\ldots,d\}$. By the definition of g, if $i\in\{1,2,\ldots,n-1\}$, then $g(x_1)=b_i$ and $g(x_2)=b_{i+1}$; if $i\in\{n,n+2,n+4,\ldots\}\subseteq\{n,n+1,n+2,n+3,\ldots,d\}$, then $g(x_1)=b_n$ and $g(x_2)=b_{n-1}$; if $i\in\{n+1,n+3,\ldots\}\subseteq\{n,n+1,n+2,n+3,\ldots,d\}$, then $g(x_1)=b_{n-1}$ and $g(x_2)=b_n$. In each case we see $\{g(x_1),g(x_2)\}\in E(G)$, from which it follows that $g\in\mathrm{End}(G)$.

We now verify $g\notin h\mathrm{End}(G)$. Noting $g(a_1)=b_1$ and $g(a_n)=b_n$, we have $\{g(a_1),g(a_n)\}=\{b_1,b_n\}\in E(C_n)\subseteq E(G)$. Let $x\in g^{-1}(g(a_1))$ and let $y\in g^{-1}(g(a_n))$, i.e., $x\in g^{-1}(b_1)$ and $y\in g^{-1}(b_n)$. By the definition of g, it is easy to see that $g^{-1}(b_1)=[a_1]_{\rho_f}$ and $g^{-1}(b_n)=\cup\{[a_i]_{\rho_f}|i\in\{n,n+2,n+4,\ldots\}\subseteq\{n,n+1,n+2,\ldots,d+1\}\}$. Then $f(x)=f(a_1)$ and $f(y)\in\{f(a_n),f(a_{n+2}),f(a_{n+4}),\ldots\}\subseteq\{f(a_n),f(a_{n+1}),f(a_{n+2}),\ldots,f(a_{d+1})\}$. Recalling $f\in\mathrm{Idpt}(G)$ with $I_f=P=a_1a_2\cdots a_{d+1}$, by Lemma 2.2, $f(a_i)=a_i$ for any $i\in\{1,2,\ldots,d+1\}$, and so $f(x)=a_1$ and $f(y)\in\{a_n,a_{n+2},a_{n+4},\ldots\}\subseteq\{a_n,a_{n+1},a_{n+2},\ldots,a_{d+1}\}$. Since P is a geodesic line and $n\ge4$, $\{a_1,a_i\}\notin E(G)$ for any $a_i\in\{a_n,a_{n+1},a_{n+2},\ldots,a_{d+1}\}$. Hence $\{f(x),f(y)\}\notin E(G)$, and furthermore $\{x,y\}\notin E(G)$ since $f\in\mathrm{End}(G)$, which implies $g\notin h\mathrm{End}(G)$ as required.

□

Example For the following connected bipartite graphs H_i $(i = 1, 2, 3, 4)$, clearly, $\mathrm{gir}(H_i) = 6$ $(i = 1, 2, 3, 4)$. Then by Theorem 2.3,

since $\mathrm{diam}(H_1) = 5 > 4 = \mathrm{gir}(H_1) - 2$, H_1 is not E-H-unretractive;

since $\mathrm{diam}(H_2) = 5 > 4 = \mathrm{gir}(H_2) - 2$, H_2 is not E-H-unretractive;

since $\mathrm{diam}(H_3) = 4 \le 4 = \mathrm{gir}(H_3) - 2$, H_3 is E-H-unretractive;

since $\mathrm{diam}(H_4) = 4 \le 4 = \mathrm{gir}(H_4) - 2$, H_4 is E-H-unretractive.

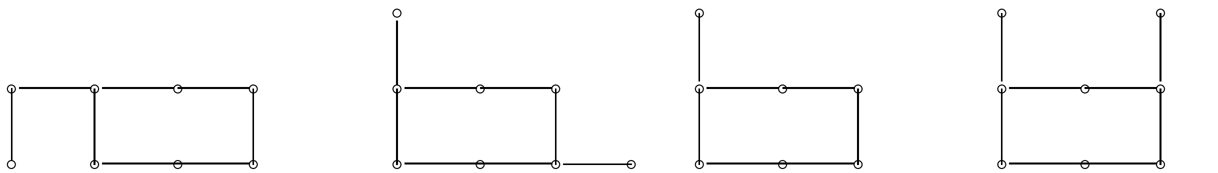
For the following connected bipartite graphs H_i $(i = 5, 6, 7, 8)$, $\mathrm{gir}(H_i) = 8$ $(i = 5, 7, 8)$ and $\mathrm{gir}(H_6) = 4$.

Since $\mathrm{diam}(H_5) = 7 > 6 = \mathrm{gir}(H_5) - 2$, H_5 is not E-H-unretractive;

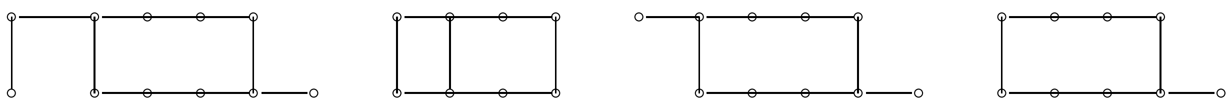
since $\mathrm{diam}(H_6) = 4 > 2 = \mathrm{gir}(H_6) - 2$, H_6 is not E-H-unretractive;

since $\mathrm{diam}(H_7) = 6 \le 6 = \mathrm{gir}(H_7) - 2$, H_7 is E-H-unretractive;

since $\mathrm{diam}(H_8) = 5 \le 6 = \mathrm{gir}(H_8) - 2$, H_8 is E-H-unretractive.



H_1 H_2 H_3 H_4



H_5 H_6 H_7 H_8

In particular, we see immediately the following:

Corollary 2.4 *All cycles are E-H-unretractive.*

Proof Notice all cycles with odd lengths are unretractive (Proposition 1.1(1)) and all cycles $C_{2m} (m \ge 2)$ satisfy the condition in Theorem 2.3, i.e., $\mathrm{diam}(C_{2m}) = m \le m+(m-2) = 2m-2 = \mathrm{gir}(C_{2m}) - 2$. The result follows immediately. □

Now, we consider E-H-unretractivity of non-connected bipartite graphs. First, we list several lemmas as follows:

Lemma 2.5 *Let G be a bipartite graph with n (≥ 2) components.*

(1) If each component is K_1, i.e., $G = \overline{K}_n$, then G is E-H-unretractive;

(2) If each component of G is K_2 or $K_{1,m}(m \ge 2)$ (i.e., a star), then G is E-H-unretractive.

Proof (1) By the definition of a half-strong endomorphism, G is trivially E-H-unretractive.

(2) Suppose $f \in \mathrm{End}(G)$. Let $a, b \in G$ such that $\{f(a), f(b)\} \in E(G)$. As G has no isolated vertices, there exist $x, y \in G$ with $\{a, x\} \in E(G)$ and $\{b, y\} \in E(G)$. So $\{f(a), f(x)\} \in E(G)$ and $\{f(b), f(y)\} \in E(G)$. If the edge $\{f(a), f(b)\}$ is exactly a component K_2 of G, then $f(x) = f(b)$,

i.e., there exist $x \in f^{-1}(f(b))$ and $a \in f^{-1}(f(a))$ such that $\{a, x\} \in E(G)$. If $\{f(a), f(b)\}$ belongs to a component $K_{1,m}$ where $m \geq 2$, then clearly either $\deg(f(a)) = 1$ and $\deg(f(b)) = m$ or $\deg(f(b)) = 1$ and $\deg(f(a)) = m$. In the former situation, we have $f(x) = f(b)$, i.e., there exist $x \in f^{-1}(f(b))$ and $a \in f^{-1}(f(a))$ such that $\{a, x\} \in E(G)$. In the latter situation, we have $f(y) = f(a)$, i.e., there exist $y \in f^{-1}(f(a))$ and $b \in f^{-1}(f(b))$ such that $\{b, y\} \in E(G)$. Hence $f \in h\mathrm{End}(G)$. □

Lemma 2.6 *Let G be a bipartite graph with n (≥ 2) components. If exactly one component is K_1 while any of the other components is K_2, then G is E-H-unretractive.*

Proof Suppose $f \in \mathrm{End}(G)$. Let $a, b \in G$ such that $\{f(a), f(b)\} \in E(G)$. Clearly at least one vertex of a and b, say a, is not an isolated vertex of G. Then there exists $c \in G$ such that $\{a, c\}$ is a component K_2 of G, and so $\{f(a), f(c)\}$ is also a component K_2 of G. Thus $f(b) = f(c)$, i.e., there exist $a \in f^{-1}(f(a))$ and $c \in f^{-1}(f(b))$ such that $\{a, c\} \in E(G)$, which implies $f \in h\mathrm{End}(G)$. □

Lemma 2.7 *Let G be a bipartite graph with n (≥ 2) components. If exactly one component is K_2 while any of the other components is K_1, then G is E-H-unretractive.*

Proof Suppose $f \in \mathrm{End}(G)$. Let $a, b \in G$ such that $\{f(a), f(b)\} \in E(G)$. Let $\{u, v\}$ be the unique component K_2 of G. Thus $\{f(a), f(b)\} = \{u, v\}$, say, $f(a) = u$ and $f(b) = v$. Thus $u \in f^{-1}(f(a))$ and $v \in f^{-1}(f(b))$ such that $\{u, v\} \in E(G)$, which implies $f \in h\mathrm{End}(G)$. □

Lemma 2.8 *Suppose G is a non-connected bipartite graph with isolated vertices. If there exists a component G_i in G such that* $\mathrm{diam}(G_i) \geq 2$ *or* $G = mK_2 \cup nK_1$ *where* $m \geq 2, n \geq 2$, *i.e.,* $G = mK_2 \cup nK_1$ $(m \geq 2, n \geq 2)$, *then* $\mathrm{End}(G) \neq h\mathrm{End}(G)$.

Proof Firstly, suppose there exists a component being $K_1 = \{a\}$ and a component G_1 with $\mathrm{diam}(G_1) \geq 2$. Then there exist $u, u_1, u_2 \in G$ such that the path $P = uu_1u_2$ is a geodesic line of G. By Proposition 1.4, there exists $f \in \mathrm{Idpt}(G)$ such that $I_f = P$. Now define a mapping g from $V(G)$ to itself by the following rule:

$g(a) = u_2$;

$g(x) = u_1$ if $x \in f^{-1}(u_1) \setminus \{a\}$;

$g(x) = u$ if $x \in (f^{-1}(u) \cup f^{-1}(u_2)) \setminus \{a\}$.

Clearly, the mapping g is well defined. We now show $g \in \mathrm{End}(G) \setminus h\mathrm{End}(G)$. Let $x, y \in G$ such that $\{x, y\} \in E(G)$. Then $\{f(x), f(y)\} \in E(I_f)$, and so $\{f(x), f(y)\} = \{u, u_1\}$ or $\{f(x), f(y)\} = \{u_1, u_2\}$. Without loss of generality, we may suppose $f(x) = u$ and $f(y) = u_1$, or $f(x) = u_1$ and $f(y) = u_2$. In both situations, it is easy to check $\{g(x), g(y)\} = \{u, u_1\} \in E(G)$, and so $g \in \mathrm{End}(G)$.

Since $u_1 \in I_f$ and $f \in \mathrm{Idpt}(G)$, by Lemma 2.2, $f(u_1) = u_1$. Thus $u_1 \in f^{-1}(u_1) \setminus \{a\}$ and so $g(u_1) = u_1$. Then $\{g(a), g(u_1)\} = \{u_2, u_1\} \in E(G)$. Now let $x \in g^{-1}(g(a))$ and let $y \in g^{-1}(g(u_1))$. Clearly, $x = a$ and so $\{x, y\} \notin E(G)$. Hence $g \notin h\mathrm{End}(G)$ as required.

Secondly, suppose G is a union of $n(\geq 2)$ isolated vertices, say, $a_1, a_2, \ldots, a_n$, and m (≥ 2) components K_2s, say, $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_m, y_m\}$. Define a mapping g from $V(G)$ to itself by the following rule:

$g(x_i) = x_1$ and $g(y_i) = y_1$ for any $i \in \{1, 2, \ldots, m\}$;

$g(a_1) = x_2$ and $g(a_j) = y_2$ for any $j \in \{2, 3, \ldots, n\}$.

It is easy to check $g \in \mathrm{End}(G)$. Notice $\{g(a_1), g(a_2)\} = \{x_2, y_2\} \in E(G)$. However, since $g^{-1}(g(a_1)) = g^{-1}(x_2) = \{a_1\}$ and $g^{-1}(g(a_2)) = g^{-1}(y_2) = \{a_2, a_3, \ldots, a_n\}$, so for any $s \in g^{-1}(g(a_1))$ and any $t \in g^{-1}(g(a_2))$, $\{s, t\} \notin E(G)$. Thus $g \notin h\mathrm{End}(G)$. □

Lemma 2.9 *Suppose G is a non-connected bipartite graph without isolated vertices. In either of the following two cases,* $\mathrm{End}(G) \neq h\mathrm{End}(G)$*:*

Case 1. There exists a component G_i such that $\mathrm{diam}(G_i) \geq 3$*;*

Case 2. Any component G_i has $\mathrm{diam}(G_i) \leq 2$ *and there exists a component G_i such that $G_i = K_{m,n}(m, n \geq 2)$.*

Proof Case 1. Suppose G has $n(\geq 2)$ components $G_1, G_2, \ldots, G_n$ such that $G_i \neq K_1$ for any $i \in \{1, 2, ..., n\}$ and $diam(G_1) \geq 3$. Clearly there exists a geodesic line P with length 3 in G_1, say, $P = u_1u_2u_3u_4$. Since G has no isolated vertices, there exists an edge $e \in E(G_2 \cup G_3 \cup \cdots \cup G_n)$, say, $e = \{a, b\}$. By Proposition 1.4, there exists $f \in \mathrm{Idpt}(G_1)$ such that $I_f = P$ and $g \in \mathrm{Idpt}(G_2 \cup G_3 \cup \cdots G_n)$ such that $I_g = e$.

Now define a mapping h from $V(G)$ to itself by the following rule:

$h(x) = u_1$ if $x \in f^{-1}(u_1) \cup f^{-1}(u_3)$;

$h(x) = u_2$ if $x \in f^{-1}(u_2) \cup f^{-1}(u_4)$;

$h(x) = u_3$ if $x \in g^{-1}(a)$;

$h(x) = u_4$ if $x \in g^{-1}(b)$.

Obviously, the mapping h is well-defined. Let $x, y \in G$ with $\{x, y\} \in E(G)$. First suppose $\{x, y\} \in E(G_1)$, then $\{f(x), f(y)\} \in E(I_f) = E(P)$. If $\{f(x), f(y)\} = \{u_1, u_2\}$, say, $f(x) = u_1$ and $f(y) = u_2$, then $\{h(x), h(y)\} = \{u_1, u_2\} \in E(G_1) \subseteq E(G)$. If $\{f(x), f(y)\} = \{u_2, u_3\}$ or $\{f(x), f(y)\} = \{u_3, u_4\}$, we may similarly show $\{h(x), h(y)\} \in E(G)$. Now suppose $\{x, y\} \in E(G_2 \cup G_3 \cup \cdots G_n)$, we may prove $\{h(x), h(y)\} \in E(G)$ in an analogous manner. Hence $h \in \mathrm{End}(G)$. By Lemma 2.2, $f(u_2) = u_2$ and $g(a) = a$, and so $\{h(u_2), h(a)\} = \{u_2, u_3\} \in E(G)$. Since $h^{-1}(h(u_2)) = h^{-1}(u_2) \subseteq V(G_1)$ and $h^{-1}(h(a)) = h^{-1}(u_3) \subseteq V(G_2 \cup G_3 \cup \cdots \cup G_n)$, $\{x, y\} \notin E(G)$ for any $x \in h^{-1}(h(u_2))$ and any $y \in h^{-1}(h(a))$. Hence $h \notin h\mathrm{End}(G)$.

Case 2. Assume G has $k(\geq 2)$ components $G_1, G_2, \ldots, G_k$. Without loss of generality, suppose $G_1 = K_{m,n}$ $(m, n \geq 2)$. Let $G_1 = (A \cup B, E)$ with $|A| = m \geq 2$ and $|B| = n \geq 2$. Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$, and let $e = \{a_1, b_1\} \in E(G_1)$. Clearly, there exists an edge $e_1 = \{c, d\} \in E(G_2 \cup \cdots \cup G_k)$. Then by Proposition 1.4, there exists $f \in \mathrm{Idpt}(G_1)$ such that $I_f = e$ and $g \in \mathrm{Idpt}(G_2 \cup \cdots \cup G_k)$ such that $I_g = e_1$.

Now we define a mapping h from $V(G)$ to itself by the following rule:

$h(x) = f(x)$ if $x \in G_1$;

$h(x) = a_2$ if $x \in g^{-1}(c)$;

$h(x) = b_2$ if $x \in g^{-1}(d)$.

It is easy to see that the mapping h is well-defined. Let $x, y \in G$ with $\{x, y\} \in E(G)$. If $\{x, y\} \in E(G_1)$, then $\{h(x), h(y)\} = \{f(x), f(y)\} \in E(G_1) \subseteq E(G)$; if $\{x, y\} \in E(G_2 \cup \cdots \cup G_k)$, then $\{g(x), g(y)\} = \{c, d\}$, say, $g(x) = c$ and $g(y) = d$, and so $\{h(x), h(y)\} = \{a_2, b_2\} \in E(G)$. Thus $h \in \mathrm{End}(G)$. Since $c \in I_g$ and $g \in \mathrm{Idpt}(G_2 \cup \cdots \cup G_k)$, by Lemma 2.2, $g(c) = c$, i.e., $c \in g^{-1}(c)$; similarly, since $b_1 \in I_f$ and $f \in \mathrm{Idpt}(G_1)$, $f(b_1) = b_1$. Thus $\{h(c), h(b_1)\} = \{a_2, b_1\} \in E(G)$. However, $h^{-1}(h(c)) = h^{-1}(a_2) \subseteq V(G_2 \cup \cdots \cup G_k)$ and $h^{-1}(h(b_1)) = h^{-1}(b_1) \subseteq V(G_1)$, so for any $s \in h^{-1}(h(c))$ and any $t \in h^{-1}(h(b_1))$, $\{s, t\} \notin E(G)$. Hence $h \notin h\mathrm{End}(G)$. □

The following lemma can be proved in a routine manner.

Lemma 2.10 *Let G be a connected bipartite graph. Then*

(i) $\mathrm{diam}(G) = 0$ *if and only if* $G = K_1$*;*

(ii) $\mathrm{diam}(G) = 1$ *if and only if* $G = K_2$*;*

(iii) $\mathrm{diam}(G) = 2$ *if and only if* $G = K_{m,n}$ *with* $\max\{m, n\} \geq 2$.

Now we present the main theorem of this paper, which characterizes E-H-unretractive bipartite graphs, as follows:

Theorem 2.11 *Let G be a bipartite graph with n (≥ 1) components. Then G is E-H-unretractive if and only if G belongs to one of the following cases:*

(1) G is a tree;

(2) $n = 1$ and $\mathrm{diam}(G) \leq \mathrm{gir}(G) - 2$;

(3) $n \geq 2$ and each component is K_1, i.e., $G = \overline{K}_n$;

(4) $n \geq 2$, and each component is $= K_2$ or $K_{1,m}$ $(m \geq 2)$ (i.e., a star);

(5) $n \geq 2$, and exactly one component is K_1 while any of the other components is K_2;

(6) $n \geq 2$, and exactly one component is K_2 while any of the other components is K_1.

Proof Sufficiency is by Lemmas 2.6, 2.7, 2.8 and Theorem 2.3.

Necessity. We consider the graph G separately as containing isolated vertices and not containing any isolated vertices.

Firstly, we suppose G is a non-connected bipartite graph with isolated vertices. Then by Lemma 2.10 there are two cases to be considered as follows, where the second case may be divided into four subcases:

Case 1. There exists a component G_i such that $\mathrm{diam}(G_i) \geq 2$;

Case 2. $G = mK_2 \cup nK_1$ where $m \geq 0$, $n \geq 1$ and $m + n \geq 2$;

Subcase 1. $m = 0, n \geq 2$, i.e., $G = \overline{K}_n (n \geq 2)$;

Subcase 2. $m = 1, n \geq 1$, i.e., $G = K_2 \cup \overline{K}_n (n \geq 1)$;

Subcase 3. $m \geq 2, n = 1$, i.e., $G = mK_2 \cup K_1 (m \geq 2)$;

Subcase 4. $m \geq 2, n \geq 2$, i.e., $G = mK_2 \cup nK_1 (m \geq 2, n \geq 2)$.

By Lemma 2.9, for Case 1 or Subcase 4 of Case 2, $\mathrm{End}(G) \neq h\mathrm{End}(G)$.

Secondly, we suppose G is a non-connected bipartite graph without isolated vertices. Then there are two cases to be considered as follows, where the second case may be divided into two subcases by Lemma 2.10:

Case 1. There exists a component G_i such that $\mathrm{diam}(G_i) \geq 3$;

Case 2. Any component G_i has $\mathrm{diam}(G_i) \leq 2$:

Subcase 1. There exists a component G_i such that $G_i = K_{m,n}$ $(m, n \geq 2)$;

Subcase 2. Any component $G_i = K_2$ or $G_i = K_{1,n}$ $(n \geq 2)$.

By Lemma 2.10, for Case 1 and Subcase 1 of Case 2, $\mathrm{End}(G) \neq h\mathrm{End}(G)$.

Therefore, combining Theorem 2.3 we complete the proof. □

Acknowledgement The author would like to thank Professor Dr. U.Knauer for valuable advice and helpful comments on this paper.

References

[1] KNAUER U. *Endomorphisms of graphs. II. Various unretractive graphs* [J]. Arch. Math. (Basel), 1990, **55**(2): 193–203.

[2] KNAUER U, NIEPORTE M. *Endomorphisms of graphs. I. The monoid of strong endomorphisms* [J]. Arch. Math. (Basel), 1989, **52**(6): 607–614.

[3] KNAUER U. BÖTTCHER M. *Endomorphism spectra of graphs* [J]. Discrete Math., 1992, **109**(1-3): 45–57.

[4] WILKEIT E. *Graphs with a regular endomorphism monoid* [J]. Arch. Math. (Basel), 1996, **66**(4): 344–352.

[5] HARARY F. *Graph Theory* [M]. Addison-Wesley, Reading, 1969.

[6] KNAUER U. *Unretractive and S-unretractive joins and lexicographic products of graphs* [J]. J. Graph Theory, 1987, **11**(3): 429–440.

[7] KNAUER U. *Endomorphism types of trees* [J]. Proceedings of the Conference on Words, Languages and Combinatorics, Kyoto, Japan, August 1990, 273–287.

[8] LI Weimin. *A note on the E-S-unretractive graphs* [J]. J. Combin. Inform. System Sci., 1996, **21**(1): 27–33.

[9] HOWIE J M. *An Introduction to Semigroup Theory* [M]. Academic Press, New York-London, 1976.

[10] HELL P, NESETRIL J. *Absolute Retracts in Graphs* [M]. Springer, Berlin, 1974.

[11] NOWAKOWSKI R, RIVAL I. *Fixed-edge theorem for graphs with loops* [J]. J. Graph Theory, 1979, **3**(4): 339–350.

[12] LI Weimin. *Green's relations on the endomorphism monoid of a graph* [J]. Math. Slovaca, 1995, **45**(4): 335–347.

[13] LI Weimin, CHEN Jianfei. *Endomorphism–regularity of split graphs* [J]. European J. Combin,. 2001, **22**(2): 207–216.