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Hopf Algebras in Group Yetter-Drinfel'd Categories

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Abstract In this note we first show that if H is a finite-dimensional Hopf algebra in a group Yetter-Drinfel'd category ${}_L^L\mathcal{YD}(\pi)$ over a crossed Hopf group-coalgebra L, then its dual H^* is also a Hopf algebra in the category ${}_L^L\mathcal{YD}(\pi)$. Then we establish the fundamental theorem of Hopf modules for H in the category ${}_L^L\mathcal{YD}(\pi)$.

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1. Introduction

Hopf group-coalgebras which are generalizations of ordinary Hopf algebras, were introduced by Turaev[1] and studied in [2] and [3]. On the one hand, crossed Hopf group-coalgebras play key roles in the theory of constructing homotopical invariant of 3-manifold[1]. On the other hand, the structure of a Hopf group-coalgebra is much more complicated than that of the usual Hopf algebra. In particular, the group Yetter-Drinfel'd category introduced by Zunino[4] is more complicated than the ordinary Yetter-Drinfel'd category.

In the classical Hopf algebra theory, Sweedler showed that the dual of a finite-dimensional Hopf algebra is still a Hopf algebra and obtained the fundamental theorem of Hopf modules[5]. In 1998, Doi[6] showed that if H is a finite-dimensional Hopf algebra in the Yetter-Drinfel'd category ${}_L^L\mathcal{YD}$ over a Hopf algebra L, then its dual H^* is also a Hopf algebra in the category ${}_L^L\mathcal{YD}$ and he proved the fundamental theorem of Hopf modules in ${}_L^L\mathcal{YD}$. We remark here that although a lot of results of classical Hopf algebra theory can be generalized to Hopf group-coalgebras, we do not know why it works. This stimulates that the people are interested in some topics related to a Hopf group-coalgebra.

The main aim of this note is to generalize the Doi's results in [6] to the setting of a group Yetter-Drinfel'd category over a Hopf group-coalgebra.

The paper is organized as follows.

In Section 1, we will recall some basic notions related to a Hopf group coalgebra. In Section 2, we mainly show that if H is a finite-dimensional Hopf algebra in a group Yetter-Drinfel'd

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category ${}^L_L\mathcal{YD}(\pi)$ over a crossed Hopf group-coalgebra L, then its dual H^* is also a Hopf algebra in the category ${}^L_L\mathcal{YD}(\pi)$ (cf. Theorem 3.3). In Section 3, we establish the fundamental theorem of Hopf modules for H in the category ${}^L_L\mathcal{YD}(\pi)$ (cf. Theorem 4.3).

2. Basic definitions and results

Throughout this paper, k denotes a fixed field. We will work over k. We always let π be a discrete group, L a crossed Hopf group-coalgebra with a bijective antipode S_L, and H a Hopf algebra in the π-Yetter-Drinfel'd category ${}^L_L\mathcal{YD}(\pi)$.

We first recall from Turaev[1] that a π-coalgebra is a family of k-spaces $C=\{C_\alpha\}_{\alpha\in\pi}$ together with a family of k-linear maps $\Delta=\{\Delta_{\alpha,\beta}:C_{\alpha\beta}\longrightarrow C_\alpha\otimes C_\beta\}_{\alpha,\beta\in\pi}$ (called a comultiplication) and a k-linear map $\varepsilon: C_1\longrightarrow k$ (called a counit), such that Δ is coassociative in the sense that,

(i) $(\Delta_{\alpha,\beta}\otimes id_{C_\gamma})\Delta_{\alpha\beta,\gamma}=(id_{C_\alpha}\otimes\Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma}$, for any $\alpha,\beta,\gamma\in\pi$.

(ii) $(id_{C_\alpha}\otimes\varepsilon)\Delta_{\alpha,1}=id_{C_\alpha}=(\varepsilon\otimes id_{C_\alpha})\Delta_{1,\alpha}$, for all $\alpha\in\pi$.

We use the Sweedler-like notation[2] for the comultiplication in the following way: for any $\alpha,\beta\in\pi$ and $c\in C_{\alpha\beta}$, we write

$$\Delta_{\alpha,\beta}(c)=c_{(1,a)}\otimes c_{(2,\beta)}.$$

A Hopf π-coalgebra is a π-coalgebra $L=(\{L_\alpha\},\Delta,\varepsilon)$ endowed with a family of k-linear maps $S=\{S_\alpha:L_\alpha\longrightarrow L_{\alpha^{-1}}\}_{\alpha\in\pi}$ (called an antipode) such that

(a) each L_α is an algebra with multiplication m_α and unit element $1_\alpha\in L_\alpha$,

(b) $\varepsilon:L_1\to k$ and $\Delta_{\alpha,\beta}:L_{\alpha\beta}\to L_\alpha\otimes L_\beta$ are algebra maps, for all $\alpha,\beta\in\pi$,

(c) for each $\alpha\in\pi$, $m_\alpha(S_{\alpha^{-1}}\otimes id_{L_\alpha})\Delta_{\alpha^{-1},\alpha}=\varepsilon 1_\alpha=m_\alpha(id_{L_\alpha}\otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}$.

The antipode $S=\{S_\alpha\}_{\alpha\in\pi}$ of L is said to be bijective if each S_α is bijective. The antipode of a Hopf π-coalgebra is anti-multiplicative and anti-comultiplicative, i.e., we have

$$S_\alpha(ab)=S_\alpha(b)S_\alpha(a),\quad S_\alpha(1_\alpha)=1_{\alpha^{-1}},$$

$$\Delta_{\beta^{-1},\alpha^{-1}}S_{\alpha\beta}=T_{L_{\alpha^{-1}},L_{\beta^{-1}}}(S_\alpha\otimes S_\beta)\Delta_{\alpha,\beta},\quad \varepsilon S_1=\varepsilon$$

for all $\alpha,\beta\in\pi, a,b\in L_\alpha$.

Furthermore, a Hopf π-coalgebra $L=(\{L_\alpha\},\Delta,\varepsilon,S)$ is said to be crossed if it is endowed with a family of algebra isomorphisms $\Phi=\{\Phi_\beta:L_\alpha\to L_{\beta\alpha\beta^{-1}}\}_{\alpha,\beta\in\pi}$ (the crossing) such that each Φ_β preserves the comultiplication and the counit, i.e., for all $\alpha,\beta,\gamma\in\pi$,

$$(\Phi_\beta\otimes\Phi_\beta)\circ\Delta_{\alpha,\gamma}=\Delta_{\beta\alpha\beta^{-1},\,\beta\gamma\beta^{-1}}\circ\Phi_\beta,\quad \varepsilon\Phi_\beta=\varepsilon,$$

and Φ is multiplicative in the sense that $\Phi_{\alpha\beta}=\Phi_\alpha\circ\Phi_\beta$, for all $\alpha,\beta\in\pi$.

Let L be a crossed Hopf π-coalgebra. Then one has that $\Phi_1|L_\alpha=id_{L_\alpha}$, $\Phi_\beta^{-1}=\Phi_{\beta^{-1}}$ for any $\alpha\in\pi$ and Φ preserves the antipode, i.e., $\Phi_\beta S_\alpha=S_{\beta\alpha\beta^{-1}}\Phi_\beta$ for all $\alpha,\beta\in\pi$.

Let $C=\{C_\alpha\}_{\alpha\in\pi}$ be a π-coalgebra and V a k-vector space. Then we recall from Wang[3] that a left π-C-comodulelike object is a couple $V=(V,\rho^V=\{\rho^V_\lambda\})$, where for any $\lambda\in\pi, \rho^V_\lambda:V\to C_\lambda\otimes V$ is a k-linear map (comodulelike structure), which is denoted by $\rho^V_\lambda(v)=v_{(-1,\,\lambda)}\otimes v_{(0,\,0)}$, such that the following conditions are satisfied:

(I) The couple V is coassocitative in the sense that, for any λ_1, $\lambda_2 \in \pi$, we have

$$(id_{C_{\lambda_1}} \otimes \rho^V_{\lambda_2}) \circ \rho^V_{\lambda_1} = (\Delta_{\lambda_1,\lambda_2} \otimes id_V) \circ \rho^V_{\lambda_1,\lambda_2},$$

i.e., $v_{(-1,\lambda_1)} \otimes v_{(0,0)(-1,\lambda_2)} \otimes v_{(0,0)(0,0)} = v_{(-1,\lambda_1\lambda_2)(1,\lambda_1)} \otimes v_{(-1,\lambda_1\lambda_2)(2,\lambda_2)} \otimes v_{(0,0)} \triangleq v_{(-2,\lambda_1)} \otimes v_{(-1,\lambda_2)} \otimes v_{(0,0)}$, for any $v \in V, \lambda_1,\ \lambda_2 \in \pi$.

(II) The couple V is counitary in the sense that $(\varepsilon \otimes \mathrm{id}_V) \circ \rho^V_1 = \mathrm{id}_V$.

Let L be a crossed Hopf π-coalgebra with a bijective antipode S_L. Fix $\alpha \in \pi$, a left-left α-Yetter-Drinfel'd module[3] is a left π-L-comodulelike object $V = (V, \rho^V = \{\rho^V_\lambda\})$ where V is a left L_α-module for all $\alpha \in \pi$, satisfying the compatibility condition:

$$l_{(1,\lambda)}v_{(-1,\lambda)} \otimes l_{(2,\alpha)} \rightarrow v_{(0,0)} = (l_{(1,\alpha)} \rightarrow v)_{(-1,\lambda)}\Phi_\alpha(l_{(2,\alpha^{-1}\lambda\alpha)}) \otimes (l_{(1,\alpha)} \rightarrow v)_{(0,0)}, \tag{1}$$

or equivalently,

$$\rho^V_\lambda(l \rightarrow v) = l_{(1,\lambda\alpha)(1,\lambda)}v_{(-1,\lambda)}\bar{S}_\lambda\Phi_\alpha(l_{(2,\alpha^{-1}\lambda^{-1}\alpha)}) \otimes l_{(1,\lambda\alpha)(2,\alpha)} \rightarrow v_{(0,0)}, \tag{2}$$

for all $v \in V$, $l \in L_\alpha$.

We denote the category of left-left α-Yetter-Drinfel'd modules by ${}^L_L\mathcal{YD}_\alpha$. Let ${}^L_L\mathcal{YD}(\pi)$ be the disjoint union of the categories ${}^L_L\mathcal{YD}_\alpha$ for all $\alpha \in \pi$. The category ${}^L_L\mathcal{YD}(\pi)$ admits a structure of braided T-category and is called group Yetter-Drinfel'd category (simply π-Yetter-Drinfel'd category)[4].

3. The dual in group Yetter-Drinfel'd categories

In this section, we mainly show that if H is a finite-dimensional Hopf algebra in a group Yetter-Drinfel'd category ${}^L_L\mathcal{YD}(\pi)$ over a crossed Hopf group-coalgebra L, then its dual H^* is also a Hopf algebra in the category ${}^L_L\mathcal{YD}(\pi)$.

Definition 3.1 *Let L be a crossed Hopf π-coalgebra with a bijective antipode S_L. An object H in ${}^L_L\mathcal{YD}(\pi)$ is called a bialgebra in this category if it is both a k-algebra and a k-coalgebra satisfying the following conditions:*

$$\Delta(xy) = x_1(x_{2(-1,\lambda)} \rightarrow y_1) \otimes x_{2(0,0)}y_2, \Delta(1) = 1 \otimes 1, \varepsilon(xy) = \varepsilon(x)\varepsilon(y), \varepsilon(1) = 1, \tag{3}$$

$$\rho^H_\lambda(xy) = x_{(-1,\lambda)}y_{(-1,\lambda)} \otimes x_{(0,0)} \otimes y_{(0,0)},\ \rho^H_\lambda(1) = 1_\lambda \otimes 1_H, \tag{4}$$

i.e., H is a left π-L-comodule algebra,

$$x_{(-1,\lambda)} \otimes (x_{(0,0)})_1 \otimes (x_{(0,0)})_2 = x_{1(-1,\lambda)}x_{2(-1,\lambda)} \otimes x_{1(0,0)} \otimes x_{2(0,0)}, \tag{5}$$

$x_{(-1,\lambda)}\varepsilon_H(x_{(0,0)}) = \varepsilon_H(x)1_\lambda$, *i.e., H is a left π-L-comodule coalgebra,*

$$l \rightarrow (xy) = (l_{(1,\alpha)} \rightarrow x)(l_{(2,\beta)} \rightarrow y),\ l \rightarrow 1_H = \varepsilon(l)1_H, \tag{6}$$

i.e., H is a left π-L-module algebra,

$$\Delta(l \rightarrow x) = (l_{(1,\alpha)} \rightarrow x_1) \otimes (l_{(2,\beta)} \rightarrow x_2),\ \varepsilon(l \rightarrow x) = \varepsilon(l)\varepsilon(x), \tag{7}$$

i.e., H is a left π-L-module coalgebra.

Furthermore, we call H a Hopf algebra in ${}^L_L\mathcal{YD}(\pi)$ if there exists an antipode $S : H \to H$ (here S is both left L_α-linear and colinear, i.e., S is a morphism in the category of ${}^L_L\mathcal{YD}(\pi)$), which is a convolution inverse to id_H. We easily see that S is anti-multiplicative and anti-comultiplicative. That is, for all $x, y \in H, \lambda \in \pi$,

$$S_H(xy) = (x_{(-1,\lambda)} \to S_H(y))S_H(x_{(0,0)}) \quad \text{and} \quad S_H(1) = 1, \tag{8}$$

$$\Delta((S_H(x)) = (x_{1(-1,\lambda)} \to S_H(x_2))S_H(x_{1(0,0)}), \ \varepsilon_H S_H = \varepsilon_H. \tag{9}$$

Assume that H is a Hopf algebra in ${}^L_L\mathcal{YD}(\pi)$ and finite-dimensional over k. We will make its dual $H^* = \mathrm{Hom}(H, k)$ into a Hopf algebra in ${}^L_L\mathcal{YD}(\pi)$. First, the dual H^* has a left L_α-module structure, that is,

$$(l \to f)(h) = f(S_\alpha(l) \to h), \quad \text{for all} \ \ l \in L_\alpha, \ f \in H^*, \ h \in H. \tag{10}$$

Also, since H is a finite-dimensional left π-L-comodulelike object, its dual H^* has a left π-L-comodulelike object via

$$\rho^{H^*}_\lambda : H^* \to L_\lambda \otimes H^*, \quad \rho^{H^*}_\lambda(f) = f_{(-1,\lambda)} \otimes f_{(0,0)},$$

where

$$f_{(0,0)}(h)f_{(-1,\lambda)} = f(h_{(0,0)})\bar{S}_\lambda\Phi_\alpha(h_{(-1,\alpha^{-1}\lambda^{-1}\alpha)}), \quad \text{for all} \ \ h \in H. \tag{11}$$

Then $H^* \in^L_L \mathcal{YD}(\pi)$.

Proof We can easily prove that H^* is a left L_α-module and a left π-L-comodulelike object. Now we show the compatibility condition (1).

$$\begin{aligned}
&l_{(1,\lambda)}f_{(-1,\lambda)}(l_{(2,\alpha)} \to f_{(0,0)})(h)\\
&\quad = l_{(1,\lambda)}f_{(-1,\lambda)}f_{(0,0)}(S_\alpha(l_{(2,\alpha)}) \to h)\\
&\quad \overset{(11)}{=} l_{(1,\lambda)}f(y_{(0,0)})\bar{S}_\lambda\Phi_\alpha(y_{(-1,\alpha^{-1}\lambda^{-1}\alpha)}) \qquad (\text{here } y = S_\alpha(l_{(2,\alpha)}) \to h)\\
&\quad = l_{(1,\lambda)}f(S_\alpha(l_{(3,\alpha)}) \to h_{(0,0)})\bar{S}_\lambda\Phi_\alpha(S_{\alpha^{-1}\lambda\alpha}(l_{(4,\alpha^{-1}\lambda\alpha)})h_{(-1,\alpha^{-1}\lambda^{-1}\alpha)}S_{\alpha^{-1}\lambda\alpha}\Phi_{\alpha^{-1}}S_{\lambda^{-1}}(l_{(2,\lambda^{-1})}))\\
&\quad = l_{(1,\lambda)}S_{\lambda^{-1}}(l_{(2,\lambda^{-1})})f(S_\alpha(l_{(3,\alpha)}) \to h_{(0,0)})\bar{S}_\lambda\Phi_\alpha(h_{(-1,\alpha^{-1}\lambda^{-1}\alpha)})\Phi_\alpha(l_{(4,\alpha^{-1}\lambda\alpha)})\\
&\quad = f(S_\alpha(l_{(1,\alpha)}) \to h_{(0,0)})\bar{S}_\lambda\Phi_\alpha(h_{(-1,\alpha^{-1}\lambda^{-1}\alpha)})\Phi_\alpha(l_{(2,\alpha^{-1}\lambda\alpha)})\\
&\quad = (l_{(1,\alpha)} \to f)_{(-1,\lambda)}\Phi_\alpha(l_{(2,\alpha^{-1}\lambda\alpha)})(l_{(1,\alpha)} \to f)_{(0,0)}(h).
\end{aligned}$$

Lemma 3.2 *For any left π-L-comodulelike object $V = \{V, \rho^V_\lambda\}$, define $\theta_V : H^* \otimes V \to \mathrm{Hom}(H, V)$ by*

$$\theta_V(f \otimes v)(h) = f(v_{(-1,\lambda^{-1})} \to h)v_{(0,0)}, \quad f \in H^*, v \in V, h \in H.$$

Also, define $\theta^{(2)} : H^* \otimes H^* \to (H \otimes H)^*$ and $\theta^{(3)} : H^* \otimes H^* \otimes H^* \to (H \otimes H \otimes H)^*$ by

$$\theta^{(2)}(f \otimes g)(x \otimes y) = f(\bar{S}_\lambda(y_{(-1,\lambda^{-1})}) \to x)g(y_{(0,0)}), \quad f, g, j \in H^*, x, y, z \in H, \lambda \in \pi,$$

$$\theta^{(3)}(f \otimes g \otimes j)(x \otimes y \otimes z) = f(\bar{S}_\lambda(y_{(-1,\lambda^{-1})}z_{(-2,\lambda^{-1})}) \to x)g(\bar{S}_1(z_{(-1,1)}) \to y_{(0,0)})j(z_{(0,0)})$$

Then $\theta_V, \theta^{(2)}$ and $\theta^{(3)}$ are bijective.

Proof Define $\beta: H^*\otimes V \to H^*\otimes V$ by $\beta(f\otimes v) = (\bar{S}_\lambda(v_{(-1,\lambda^{-1})}) \to f)\otimes v_{(0,0)}$ and $\gamma: H^*\otimes V \to \mathrm{Hom}(H, V)$ by $\gamma(f\otimes v)(h) = f(h)v$. It is easy to check that $\gamma\circ\beta = \theta_V$. Note that β is bijective and the inverse is given by $\beta^{-1}(f\otimes v) = (v_{(-1,\lambda)} \to f)\otimes v_{(0,0)}$.

$$\begin{aligned}\beta\beta^{-1}(f\otimes v) &= \beta((v_{(-1,\lambda)} \to f)\otimes v_{(0,0)})\\ &= (\bar{S}_\lambda(v_{(-1,\lambda^{-1})})v_{(-2,\lambda)} \to f)\otimes v_{(0,0)}\\ &= \varepsilon(v_{(-1,1)})f\otimes v_{(0,0)} = f\otimes v.\end{aligned}$$

Similarly, we can prove $\beta^{-1}\beta = id$. The map γ is also bijective since H is finite-dimensional. Hence θ_V is bijective. The maps $\theta^{(2)}$ and $\theta^{(3)}$ are also bijective. We can refer to Lemma in [6].□

Theorem 3.3 *If H is a finite-dimensional Hopf algebra in ${}^L_L\mathcal{YD}(\pi)$, then H^* is a Hopf algebra in ${}^L_L\mathcal{YD}(\pi)$, with multiplication $m_{H^*} = (\Delta_H)^*\circ\theta^{(2)}$, unit $u_{H^*} = \varepsilon_H$, comultiplication $\Delta_{H^*} = (\theta^{(2)})^{-1}\circ(m_H)^*$, counit $\varepsilon_{H^*}: f \to f(1_H)$, and antipode $(S_H)^*$. Explicitly, multiplication is given by*

$$(fg)(x) = f(g_{(-1,\lambda^{-1})} \to x_1)g_{(0,0)}(x_2) = f(\bar{S}_\lambda(x_{2(-1,\lambda^{-1})}) \to x_1)g(x_{2(0,0)}), \tag{12}$$

for all $f, g\in H^*$, $x\in H$. Comultiplication $\Delta(f) = f_1\otimes f_2$ is given by

$$f(xy) = f_1(f_{2(-1,\lambda^{-1})} \to x)f_{2(0,0)}(y) = f_1(\bar{S}_\lambda(y_{(-1,\lambda^{-1})}) \to x)f_2(y_{(0,0)}), \tag{13}$$

or equivalently

$$f_1(x)f_2(y) = f((y_{(-1,\lambda^{-1})} \to x)y_{(0,0)}), \quad \text{for all } x, y\in H,\ \lambda\in\pi. \tag{14}$$

In particular H^{**} is a Hopf algebra in ${}^L_L\mathcal{YD}$. If $(S_{L_\alpha})^2 = id_{L_\alpha}$, then the canonical map $\iota: H \to H^{**}(\pi)$ given by $\iota(h)(f) = f(h)$ is a Hopf algebra isomorphism.

Proof It is easy to see that H^* becomes an algebra. To show the coassociativity, we use the isomorphism $\theta^{(3)}$. For $f\in H^*$ and $x, y, z\in H$ we compute

$$\begin{aligned}f((xy)z) &\overset{(13)}{=} f_1(\bar{S}_\lambda(z_{(-1,\lambda^{-1})}) \to (xy))f_2(z_{(0,0)})\\ &= f_1((\bar{S}_\lambda(z_{(-1,\lambda^{-1})}) \to x)(\bar{S}_1(z_{(-2,1)}) \to y))f_2(z_{(0,0)})\\ &\overset{(6)(13)}{=} f_{11}(\bar{S}_\lambda(z_{(-4,\lambda^{-1})})\bar{S}_\lambda(y_{(-1,\lambda^{-1})})\bar{S}_\lambda(z_{(-1,\lambda^{-1})}\bar{S}_{\lambda^{-1}}(z_{(-2,\lambda)})) \to x)\\ &\quad f_{12}(\bar{S}_1(z_{(-3,1)}) \to y_{(0,0)})f_2(z_{(0,0)})\\ &= f_{11}(\bar{S}_\lambda(y_{(-1,\lambda^{-1})}z_{(-2,\lambda^{-1})}) \to x)f_{12}((\bar{S}_1(z_{(-1,1)}) \to y_{(0,0)}))f_2(z_{(0,0)})\\ &= \theta^{(3)}(f_{11}\otimes f_{12}\otimes f_2)(x\otimes y\otimes z),\end{aligned}$$

and

$$\begin{aligned}f(x(yz)) &\overset{(13)}{=} f_1(\bar{S}_\lambda((yz)_{(-1,\lambda^{-1})}) \to x)f_2((yz)_{(0,0)})\\ &\overset{(5)}{=} f_1(\bar{S}_\lambda(y_{(-1,\lambda^{-1})}z_{(-1,\lambda^{-1})}) \to x)f_2(y_{(0,0)}z_{(0,0)})\\ &\overset{(13)}{=} f_1(\bar{S}_\lambda(y_{(-1,\lambda^{-1})}z_{(-2,\lambda^{-1})}) \to x)f_{21}(\bar{S}_1(z_{(-1,1)}) \to y_{(0,0)})f_{22}(z_{(0,0)})\\ &= \theta^{(3)}(f_1\otimes f_{21}\otimes f_{22})(x\otimes y\otimes z).\end{aligned}$$

Thus $f_{11}\otimes f_{12}\otimes f_2 = f_1\otimes f_{21}\otimes f_{22}$ (we write it by $f_1\otimes f_2\otimes f_3$). The property of counit is easily checked. We next prove $\Delta_{H^*}(fg) = f_1(f_{2(-1,\lambda)}\rightarrow g_1)\otimes f_{2(0,0)}g_2 \in H^*\otimes H^*$ by using $\theta^{(2)}$. For all $x, y\in H$,

$$\begin{aligned}
&\theta^{(2)}(f_1(f_{2(-1,\lambda)}\rightarrow g_1)\otimes f_{2(0,0)}g_2)(x\otimes y)\\
&\quad = (f_1(f_{2(-1,\lambda)}\rightarrow g_1))(\bar{S}_\lambda(y_{(-1,\lambda^{-1})}\rightarrow x)(f_{2(0,0)}g_2)(y_{(0,0)})\\
&\quad \overset{(12)(5)}{=} (f_1(f_{2(-1,\lambda)}\rightarrow g_1))(\bar{S}_\lambda(y_{1(-1,\lambda^{-1})}y_{2(-2,\lambda^{-1})})\rightarrow x)f_{2(0,0)}\\
&\qquad (\bar{S}_1(y_{2(-1,1)})\rightarrow y_{1(0,0)})g_2(y_{2(0,0)})\\
&\quad \overset{(2)}{=} f_1(\bar{S}_\lambda(x_{2(-1,\lambda^{-1})}y_{1(-2,\lambda^{-1})}y_{2(-3,\lambda^{-1})})\rightarrow x_1)(f_{2(-1,\lambda)}\rightarrow g_1)(\bar{S}_1(y_{1(-1,1)}y_{2(-2,1)})\rightarrow x_{2(0,0)})\\
&\qquad f_{2(0,0)}(\bar{S}_1(y_{2(-1,1)})\rightarrow y_{1(0,0)})g_2(y_{2(0,0)})\\
&\quad = f_1(\bar{S}_\lambda(x_{2(-1,\lambda^{-1})}y_{1(-3,\lambda^{-1})}y_{2(-5,\lambda^{-1})})\rightarrow x_1)f_2(\bar{S}_1(y_{2(-2,1)})\rightarrow y_{1(0,0)})\\
&\qquad g_1(\bar{S}_{\lambda^{-1}}(y_{2(-1,\lambda)})y_{1(-1,\lambda^{-1})}y_{2(-3,\lambda^{-1})}\bar{S}_\lambda(y_{1(-2,\lambda^{-1})}y_{2(-4,\lambda^{-1})})\rightarrow x_{2(0,0)})g_2(y_{2(0,0)})\\
&\quad = f_1(\bar{S}_\lambda(x_{2(-1,\lambda^{-1})}y_{1(-1,\lambda^{-1})}y_{2(-3,\lambda^{-1})})\rightarrow x_1)f_2(\bar{S}_1(y_{2(-2,1)})\rightarrow y_{1(0,0)})\\
&\qquad g_1(\bar{S}_{\lambda^{-1}}(y_{2(-1,\lambda)})\rightarrow x_{2(0,0)})g_2(y_{2(0,0)}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(\theta^{(2)}\Delta_{H^*}(fg))(x\otimes y) = (fg)(xy)\\
&\quad \overset{(12)}{=} f(\bar{S}_\lambda((xy)_{2(-1,\lambda^{-1})})\rightarrow (xy)_1)g((xy)_{2(0,0)})\\
&\quad = f(\bar{S}_\lambda(x_{2(-1,\lambda^{-1})}y_{2(-1,\lambda^{-1})})\rightarrow (x_1(x_{2(-2,\lambda)}\rightarrow y_1)))g(x_{2(0,0)}y_{2(0,0)})\\
&\quad = f((\bar{S}_\lambda(x_{2(-1,\lambda^{-1})}y_{2(-1,\lambda^{-1})})\rightarrow x_1)(S_1(x_{2(-2,1)}y_{2(-2,1)})x_{2(-3,1)}\rightarrow y_1))g(x_{2(0,0)}y_{2(0,0)})\\
&\quad = f((\bar{S}_\lambda(x_{2(-1,\lambda^{-1})}y_{2(-2,\lambda^{-1})})\rightarrow x_1)(S_1(y_{2(-3,1)})\rightarrow y_1))g_1(\bar{S}_{\lambda^{-1}}(y_{2(-1,\lambda)})\rightarrow x_{2(0,0)})g_2(y_{2(0,0)})\\
&\quad = f_1(\bar{S}_\lambda(x_{2(-1,\lambda^{-1})}y_{1(-1,\lambda^{-1})}y_{2(-3,\lambda^{-1})})\rightarrow x_1)f_2(\bar{S}_1(y_{2(-2,1)})\rightarrow y_{1(0,0)})\\
&\qquad g_1(\bar{S}_{\lambda^{-1}}(y_{2(-1,\lambda)})\rightarrow x_{2(0,0)})g_2(y_{2(0,0)}).
\end{aligned}$$

We show $(fg)_{(-1,\alpha)}\otimes(fg)_{(0,0)} = f_{(-1,\alpha)}g_{(-1,\alpha)}\otimes f_{(0,0)}g_{(0,0)}$ in $L_\alpha\otimes H^*$, for any $x\in H$,

$$\begin{aligned}
&(f_{(0,0)}g_{(0,0)})(x)f_{(-1,\alpha)}g_{(-1,\alpha)}\\
&\quad \overset{(12)}{=} f_{(0,0)}(\bar{S}_\alpha(x_{2(-1,\alpha^{-1})})\rightarrow x_1)g_{(0,0)}(x_{2(0,0)})f_{(-1,\alpha)}g_{(-1,\alpha)}\\
&\quad \overset{(11)}{=} f(\bar{S}_\alpha(x_{2(-1,\alpha^{-1})})\rightarrow x_{1(0,0)})g(x_{2(0,0)})\bar{S}_\alpha\Phi_\alpha(x_{1(-1,\alpha)}x_{2(-2,\alpha)}),
\end{aligned}$$

and

$$\begin{aligned}
(fg)_{(0,0)}(x)(fg)_{(-1,\alpha)} &\overset{(11)}{=} fg(x_{(0,0)})\bar{S}_\alpha\Phi_\alpha(x_{(-1,\alpha)})\\
&\overset{(12)}{=} f(\bar{S}_\alpha(x_{(0,0)2(-1,\alpha^{-1})})\rightarrow x_{(0,0)})g(x_{(0,0)2(0,0)})\bar{S}_\alpha\Phi_\alpha(x_{(-1,\alpha)})\\
&\overset{(5)}{=} f(\bar{S}_\alpha(x_{2(-1,\alpha^{-1})})\rightarrow x_{1(0,0)})g(x_{2(0,0)})\bar{S}_\alpha\Phi_\alpha(x_{1(-1,\alpha)}x_{2(-2,\alpha)}).
\end{aligned}$$

We check that $f_{(-1,\alpha)}\otimes(f_{(0,0)})_1\otimes(f_{(0,0)})_2=f_{1(-1,\alpha)}f_{2(-1,\alpha)}\otimes f_{1(0,0)}\otimes f_{2(0,0)}$ in $L_\alpha\otimes H^*\otimes H^*$,

$$\begin{aligned}
&f_{1(-1,\alpha)}f_{2(-1,\alpha)}\theta^{(2)}(f_{1(0,0)}\otimes f_{2(0,0)})(x\otimes y)\\
&\quad=f_{1(-1,\alpha)}f_{2(-1,\alpha)}f_{1(0,0)}(\bar{S}_\alpha(y_{(-1,\alpha^{-1})})\rightarrow x)f_{2(0,0)}(y_{(0,0)})\\
&\quad\overset{(2)}{=}f_1(\bar{S}_\alpha(y_{(-3,\alpha^{-1})})\rightarrow x_{(0,0)})f_2(y_{(0,0)})\bar{S}_\alpha\Phi_\alpha(y_{(-1,\alpha)}\bar{S}_\alpha(y_{(-2,\alpha^{-1})})x_{(-1,\alpha)}\Phi_\alpha(y_{(-4,\alpha)}))\\
&\quad=f_1(\bar{S}_\alpha(y_{(-1,\alpha^{-1})})\rightarrow x_{(0,0)})f_2(y_{(0,0)})\bar{S}_\alpha\Phi_\alpha(x_{(-1,\alpha)}y_{(-2,\alpha)})\\
&\quad=f(x_{(0,0)}y_{(0,0)})\bar{S}_\alpha\Phi_\alpha(x_{(-1,\alpha)}y_{(-1,\alpha)})=f_{(-1,\alpha)}f_{(0,0)}(xy)\\
&\quad\overset{(13)}{=}f_{(-1,\alpha)}\theta^{(2)}((f_{(0,0)})_1\otimes(f_{(0,0)})_2)(x\otimes y).
\end{aligned}$$

It is easy to see that $l\rightarrow(fg)=(l_{(1,\alpha)}\rightarrow f)(l_{(2,\beta)}\rightarrow g)$, for all $l\in L_{\alpha\beta}$, $f,g\in H^*$ and $\Delta(l\rightarrow f)=(l_{(1,\alpha)}\rightarrow f_1)\otimes(l_{(2,\beta)}\rightarrow f_2)$ in $H^*\otimes H^*$ (by using $\theta^{(2)}$). We compute that $S_{H^*}(f_1)f_2=f(1_H)\varepsilon_H=f_1S_{H^*}(f_2)$, $f\in H^*$, for all $x\in H$,

$$\begin{aligned}
(S_{H^*}(f_1)f_2)(x)&\overset{(12)}{=}S_{H^*}(f_1)(\bar{S}_\lambda(x_{2(-1,\lambda^{-1})})\rightarrow x_1)f_2(x_{2(0,0)})\\
&=f_1(\bar{S}_\lambda(x_{2(-1,\lambda^{-1})})\rightarrow S(x_1))f_2(x_{2(0,0)})\\
&\overset{(14)}{=}f((x_{2(-1,\lambda^{-1})}\bar{S}_\lambda(x_{2(-2,\lambda^{-1})})\rightarrow S(x_1))x_{2(0,0)})\\
&=f(S(x_1)x_2)=f(1_H)\varepsilon(x),
\end{aligned}$$

and

$$\begin{aligned}
f_1S_{H^*}(f_2)(x)&\overset{(12)}{=}f_1(\bar{S}_\lambda(x_{2(-1,\lambda^{-1})})\rightarrow x_1)S_{H^*}(f_2)(x_{2(0,0)})\\
&=f_1(\bar{S}_\lambda(x_{2(-1,\lambda^{-1})})\rightarrow x_1)f_2(S(x_{2(0,0)}))\\
&\overset{(14)}{=}f((x_{2(-1,\lambda^{-1})}\bar{S}_\lambda(x_{2(-2,\lambda^{-1})}))\rightarrow x_1)S(x_{2(0,0)}))\\
&=f(x_1S(x_2))=f(1_H)\varepsilon(x).
\end{aligned}$$

Thus H^* is a Hopf algebra in ${}^L_L\mathcal{YD}$.

Finally it follows from $S^2_{L_\alpha}=id_{L_\alpha}$ that the canonical map ι is both L_α-linear and colinear, since

$$\begin{aligned}
&(l\rightarrow\iota(x))(f)=\iota(x)(S_\alpha(l)\rightarrow f)=f(S^2_\alpha(l)\rightarrow x)=f(l\rightarrow x)=\iota(l\rightarrow x)(f),\\
&\iota(x)_{(-1,\alpha)}\iota(x)_{(0,0)}(f)=\iota(x)(f_{(0,0)})\bar{S}_\alpha\Phi_\alpha(f_{(-1,\alpha)})=\bar{S}_\alpha(f(x_{(0,0)})\bar{S}_\alpha(x_{(-1,\alpha)}))\\
&\quad=f(x_{(0,0)})\bar{S}^2_\alpha(x_{(-1,\alpha)})=f(x_{(0,0)})x_{(-1,\alpha)}=x_{(-1,\alpha)}\iota(x_{(0,0)})(f).
\end{aligned}$$

It is easy to see that the map ι is multiplicative and comultiplicative. □

4. The fundamental theorem in group Yetter-Drinfel'd categories

In this section, we mainly establish the fundamental theorem of Hopf modules for H in the category ${}^L_L\mathcal{YD}(\pi)$.

Definition 4.1 *Let H be a Hopf algebra in ${}^L_L\mathcal{YD}(\pi)$. A right H-Hopf module in ${}^L_L\mathcal{YD}(\pi)$ is an object $M\in{}^L_L\mathcal{YD}(\pi)$ such that it is both a right H-module and a right H-comodule via*

$\rho_M : M \to M \otimes H$, $\rho_M(m) = m_0 \otimes m_1$ *and the following (15)–(19) hold.*

(15) $\rho_M(mh) = m_0(m_{1(-1,\alpha)} \to h_1) \otimes m_{1(0,0)}h_2, \quad m \in M,\ h \in H,$

(16) $\rho_\lambda^M(mh) = m_{(-1,\lambda)}h_{(-1,\lambda)} \otimes m_{(0,0)}h_{(0,0)}, \quad m \in M,\ h \in H,$

(17) $m_{(-1,\lambda)} \otimes m_{(0,0)0} \otimes m_{(0,0)1} = m_{0(-1,\lambda)}m_{1(-1,\lambda)} \otimes m_{0(0,0)} \otimes m_{1(0,0)} \in L_\lambda \otimes M \otimes H,$

(18) $l \to (mh) = (l_{(1,\alpha)} \to m)(l_{(2,\beta)} \to h),\ l \in L_{\alpha\beta},\ m \in M, h \in H,$

(19) $\rho_M(l \to m) = (l_{(1,\alpha)} \to m_0) \otimes (l_{(2,\beta)} \to m_1),\ l \in L_{\alpha\beta},\ m \in M.$

Example 4.2 (1) H itself is a right H-Hopf module (in ${}_L^L\mathcal{YD}(\pi)$) in the natural way. If V is an object in ${}_L^L\mathcal{YD}(\pi)$, then so is $V \otimes H$ by $l_{\alpha\beta} \to (v \otimes h) = (l_{(1,\alpha)} \to v) \otimes (l_{(2,\beta)} \to h)$ and $\rho_\lambda^{V\otimes H} = v_{(-1,\lambda)}h_{(-1,\lambda)} \otimes v_{(0,0)} \otimes h_{(0,0)}$. It is also both a right H-module and a right H-comodule by $(v \otimes h)x = v \otimes hx$ and $\rho_{V\otimes H}(v \otimes h) = v \otimes h_1 \otimes h_2$. One can easily check that $V \otimes H$ is a right H-Hopf module in ${}_L^L\mathcal{YD}(\pi)$.

(2) If H is a finite dimensional Hopf algebra in ${}_L^L\mathcal{YD}(\pi)$. We can show that H^* becomes a right H-Hopf module in ${}_L^L\mathcal{YD}(\pi)$. First, the right H-module structure is $(f \cdot h)(x) = f(hx),\ f \in H^*, h, x \in H$. Second, H^* is a right H-comodule using the identification $\theta_H : H^* \otimes H \cong \mathrm{Hom}(H, H)$ in Lemma 3.2 as follows:

$$\rho_{H^*} : H^* \to \mathrm{Hom}(H, H) \cong H^* \otimes H,\ \rho_{H^*}(f)(x) = f(x_1)S_H(x_2).$$

That is, $\rho_{H^*}(f) = f_0 \otimes f_1$ means

$$f(x_1)S_H(x_2) = f_0(f_{1(-1,\alpha)} \to x)f_{1(0,0)}, \quad \text{for all } f \in H^*, x \in H.$$

Theorem 4.3 *If H is a Hopf algebra in ${}_L^L\mathcal{YD}(\pi)$ and M a right H-Hopf module in ${}_L^L\mathcal{YD}(\pi)$, then*

a) $M^{\mathrm{coh}} = \{m \in M |\ \rho_M(m) = m \otimes 1_H\}$ is both a L_α-submodule and a π-L-subcomodulelike object. So $M^{\mathrm{coh}} \in {}_L^L\mathcal{YD}(\pi)$.

b) Let $P(m) = m_0S(m_1), m \in M$. Then $P(m) \in M^{\mathrm{coh}}$. If $n \in M^{\mathrm{coh}}$, and $h \in H$, then $\rho_M(nh) = nh_1 \otimes h_2$ and $P(nh) = n\varepsilon(h)$.

c) The map $F : M^{\mathrm{coh}} \otimes H \to M$, $F(n \otimes h) = nh$ is an isomorphism of Hopf modules. The inverse map is given by $G(m) = P(m_0) \otimes m_1$. Here $M^{\mathrm{coh}} \otimes H$ is a right H-Hopf module in ${}_L^L\mathcal{YD}(\pi)$ by Example 4.2, and the structure is given by

$$(m \otimes h)x = m \otimes hx; \quad \rho_{M^{\mathrm{coh}}\otimes H}(m \otimes h) = m \otimes h_1 \otimes h_2,$$

for all $m \in M^{\mathrm{coh}},\ \ h, x \in H$.

Proof a) Let $n \in M^{\mathrm{coh}}$. Then $\rho_M(l \to n) = (l_{(1,\alpha)} \to n) \otimes (l_{(2,1)} \to 1_H) = l_{(1,\alpha)} \to n \otimes \varepsilon(l_{(2,1)})1_H = l \to n \otimes 1_H$. Hence $l \to n \in M^{\mathrm{coh}}$. We also have $n_{(-1,\lambda)} \otimes n_{(0,0)0} \otimes n_{(0,0)1} = n_{(-1,\lambda)} \otimes n_{(0,0)} \otimes 1_H$. This implies that $n_{(-1,\lambda)} \otimes n_{(0,0)} \in L_\lambda \otimes M^{\mathrm{coh}}$.

b) Since $h_{1(-1,\lambda)}h_{2(-1,\lambda)} \otimes h_{1(0,0)}S_H(h_{2(0,0)}) = \rho^H_\lambda(h_1S(h_2)) = 1_\lambda \otimes \varepsilon(h)1_H$, we have

$$\begin{aligned}\rho_M(P(m)) &= \rho_M(m_0S(m_1)) \\ &\overset{(9)(15)}{=} m_0(m_{1(-1,\alpha)}m_{2(-1,\alpha)} \rightharpoonup S_H(m_3)) \otimes m_{1(0,0)}S_H(m_{2(0,0)}) \\ &= m_0S_H(m_1) \otimes 1_H = P(m) \otimes 1_H.\end{aligned}$$

The other is easy.

c) The map F is left L-linear, since $F(l \rightharpoonup (n \otimes h)) = (l_{(1,\alpha)} \rightharpoonup n)(l_{(2,\beta)} \rightharpoonup h) = l \rightharpoonup nh = l \rightharpoonup F(n \otimes h)$. And F is also left L-colinear by (16). Now we have

$$GF(n \otimes h) = G(nh) = P(nh_1) \otimes h_2 = n\varepsilon(h_1) \otimes h_2 = n \otimes h,$$

and

$$FG(m) = F(P(m_0) \otimes m_1) = P(m_0)m_1 = m_0S(m_1)m_2 = m.$$

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