Journal of Mathematical Research & Exposition
Mar., 2009, Vol. 29, No. 2, pp. 283–292
DOI:10.3770/j.issn:1000-341X.2009.02.011
Http://jmre.dlut.edu.cn

Iterative Convergence Theorems for Maximal Monotone Operators and Nonexpansive Mappings and Their Applications

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Abstract In this paper, two iterative schemes for approximating common element of the set of zero points of maximal monotone operators and the set of fixed points of a kind of generalized nonexpansive mappings in a real uniformly smooth and uniformly convex Banach space are proposed. Two strong convergence theorems are obtained and their applications on finding the minimizer of a kind of convex functional are discussed, which extend some previous work.

Keywords maximal monotone operator; iterative scheme; strong convergence; zero point; fixed point.

Document code A
MR(2000) Subject Classification 47H05; 47H17
Chinese Library Classification O177.91

1. Introduction and preliminaries

During the past 20 years or so, many efforts have been done to the construction of iterative schemes to approximate zero points of maximal monotone operators or to approximate fixed points of nonexpansive mappings in Hilbert spaces. Can we construct iterative schemes to be strongly convergent to both zero points of maximal monotone operators and the fixed points of nonexpansive mappings in more general space than Hilbert spaces? We shall give an answer in this paper, which can be regarded as an extension or complement of our previous work in [1–4].

First, we shall give a definition of generalized nonexpansive mapping in the sense of Lyapunov functional which is reduced to nonexpansive mapping in the common sense in Hilbert spaces. Then, we present two new iterative schemes with errors which are proved to be strongly convergent to common element of the set of zero points of maximal monotone operators and the set of fixed points of the generalized nonexpansive mappings in a real uniformly smooth and uniformly convex Banach space. Moreover, it is shown that some results obtained by Martinez-Yanes and Xu in [5] and Solodov and Svaiter in [6] are special cases of our results in the case of Hilbert

Received date: 2007-03-09; **Accepted date**: 2007-09-07
Foundation item: the National Natural Science Foundation of China (No. 10771050).

spaces. Finally, the new iterative schemes are applied to find the minimizer of one kind of convex functionals.

Now , let E be a real Banach space with norm $\|\cdot\|$, and E^* be its dual space. The normalized duality mapping $J : E \to 2^{E^*}$ is defined as follows:

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\| \cdot \|f\|, \|f\| = \|x\|\}, \quad x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between E and E^*. We use "$\to$" and "w-lim" to denote strong and weak convergence in E or in E^*, respectively.

A multi-valued operator $T : E \to 2^{E^*}$ with domain $D(T) = \{x \in E : Tx \neq \emptyset\}$ and range $R(T) = \bigcup\{Tx : x \in D(T)\}$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$, for $\forall x_i \in D(T)$ and $y_i \in Tx_i, i = 1, 2$. A monotone operator T is said to be maximal monotone if $R(J + rT) = E^*$, for $\forall r > 0$. For a monotone operator T, we denote by $T^{-1}0 = \{x \in D(T) : 0 \in Tx\}$ the set of zero points of T . For a single-valued mapping $S : E \to E$, we denote by $\mathrm{Fix}(S) = \{x \in E : Sx = x\}$ the set of fixed points of S.

Lemma 1.1[7,8] *The duality mapping J has the following properties:*

(i) If E is a real reflexive and smooth Banach space, then $J : E \to E^*$ is single-valued;

(ii) $\forall x \in E, \forall \lambda \in R, J(\lambda x) = \lambda Jx$;

(iii) If E is a real uniformly convex and uniformly smooth Banach space, then $J^{-1} : E^* \to E$ is also a duality mapping. Moreover, both J and J^{-1} are uniformly continuous on each bounded subset of E or E^*, respectively.

Lemma 1.2[8] *Let E be a real smooth and uniformly convex Banach space, and $T : E \to 2^{E^*}$ be a maximal monotone operator. Then $T^{-1}0$ is a closed and convex subset of E and the graph of T, $G(T)$, is demi-closed in the following sense: $\forall\{x_n\} \subset D(T)$ with $x_n \rightharpoonup x$ in E, and $\forall y_n \in Tx_n$ with $y_n \to y$ in E^* imply that $x \in D(T)$ and $y \in Tx$.*

Definition 1.1[1−4] *Let E be a real smooth and uniformly convex Banach space and $T : E \to 2^{E^*}$ be a maximal monotone operator. For $\forall r > 0$, define the operator $Q_r^T : E \to E$ by $Q_r^T x = (J + rT)^{-1}Jx$, for $x \in E$.*

Definition 1.2[1−4] *Let E be a real smooth Banach space. Then the Lyapunov functional $\varphi : E \times E \to R^+$ is defined as follows:*

$$\varphi(x, y) = \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2,$$

for $\forall x, y \in E$ and $j(y) \in J(y)$.

Lemma 1.3[1−4] *Let E be a real reflexive, strictly convex, and smooth Banach space, let C be a nonempty closed and convex subset of E, and let $x \in E$. Then there exists a unique element $x_0 \in C$ such that $\varphi(x_0, x) = \min\{\varphi(z, x) : z \in C\}$.*

In this case, we can define the mapping Q_C of E onto C by $Q_C x = x_0$, for $\forall x \in E$. And, Q_C is called the generalized projection operator from E onto C. It is easy to see that Q_C is coincident with the metric projection P_C in a Hilbert space.

Lemma 1.4[1−4] *Let E be a real reflexive, strictly convex and smooth Banach space, let C be a nonempty closed and convex subset of E, and let $x \in E$. Then, for $\forall y \in C$, $\varphi(y, Q_C x) + \varphi(Q_C x, x) \leq \varphi(y, x)$.*

Lemma 1.5[1−4] *Let E be a real smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\varphi(x_n, y_n) \to 0$, as $n \to \infty$, then $x_n - y_n \to 0$, as $n \to \infty$.*

Lemma 1.6[1−4] *Let E be a real reflexive, strictly convex and smooth Banach space and $T : E \to 2^{E^*}$ be a maximal monotone operator with $T^{-1}0 \neq \emptyset$. Then for $\forall x \in E$, $y \in T^{-1}0$ and $r > 0$, we have*

$$\varphi(y, Q_r^T x) + \varphi(Q_r^T x, x) \leq \varphi(y, x).$$

Lemma 1.7[1−4] *Let E be a real smooth Banach space, let C be a convex subset of E, let $x \in E$ and $x_0 \in C$. Then $\varphi(x_0, x) = \inf\{\varphi(z, x) : z \in C\}$ if and only if $\langle z - x_0, Jx_0 - Jx\rangle \geq 0$, $\forall z \in C$.*

Definition 1.3 *Let E be a real Banach space. Then $S : E \to E$ is called a nonexpansive mapping in the sense of Lyapunov functional if $\varphi(Sx, Sy) \leq \varphi(x, y)$, for $\forall x, y \in E$.*

Remark 1.1 If E is reduced to a real Hilbert space H, then S is reduced to a nonexpansive mapping in the common sense that $\|Sx - Sy\| \leq \|x - y\|$, for $\forall x, y \in H$.

Lemma 1.8 *Let E be a real smooth and uniformly convex Banach space. If $S : E \to E$ is defined as that in Definition 1.3, then* $\mathrm{Fix}(S)$ *is a convex and closed subset of E.*

Proof In fact, we only need to prove the case that $\mathrm{Fix}(S) \neq \emptyset$. For $\forall x, y \in \mathrm{Fix}(S), \forall t \in [0, 1]$, let $z = tx + (1 - t)y$. Then

$$\begin{aligned}\varphi(z, Sz) =& t(\|x\|^2 - 2\langle x, JSz\rangle + \|Sz\|^2) + (1 - t)(\|y\|^2 - 2\langle y, JSz\rangle + \|Sz\|^2) - \\ & t\|x\|^2 - (1 - t)\|y\|^2 + \|z\|^2 \\ =& t\varphi(x, Sz) + (1 - t)\varphi(y, Sz) - t\|x\|^2 - (1 - t)\|y\|^2 + \|z\|^2 \\ \leq& t\varphi(x, z) + (1 - t)\varphi(y, z) - t\|x\|^2 - (1 - t)\|y\|^2 + \|z\|^2 = \varphi(z, z) = 0.\end{aligned}$$

By using Lemma 1.5, we know that $z = Sz$, which implies that $\mathrm{Fix}(S)$ is a convex subset of E.

$\forall x_n \in \mathrm{Fix}(S)$ such that $x_n \to x$, we can easily see from Definition 1.3 that $\varphi(Sx_n, Sx) \leq \varphi(x_n, x) \to 0$. Then Lemma 1.5 implies that $Sx_n \to Sx$, as $n \to \infty$. So $x \in \mathrm{Fix}(S)$. This completes the proof. □

2. Strong convergence of the new iterative schemes

Throughout this section, we assume that E is a real uniformly smooth and uniformly convex Banach space, $J : E \to E^*$ is weakly sequentially continuous, $S : E \to E$ is nonexpansive in the sense of Lyapunov functional and is weakly sequentially continuous, and $T : E \to 2^{E^*}$ is a maximal monotone operator with $T^{-1}0 \bigcap \mathrm{Fix}(S) \neq \emptyset$.

Theorem 2.1 *The sequence $\{x_n\}$ generated by the following scheme:*

$$\begin{cases} x_0 \in E, r_0 > 0 \\ y_n = Q_{r_n}^T(x_n + e_n) \\ Jz_n = \alpha_n Jx_n + (1-\alpha_n)Jy_n \\ Ju_n = \beta_n Jx_n + (1-\beta_n)JSz_n \\ H_n = \{v \in E : \varphi(v, z_n) \le \alpha_n \varphi(v, x_n) + (1-\alpha_n)\varphi(v, x_n + e_n)\} \\ V_n = \{v \in E : \varphi(v, u_n) \le \beta_n \varphi(v, x_n) + (1-\beta_n)\varphi(v, z_n)\} \\ W_n = \{z \in E : \langle z - x_n, Jx_0 - Jx_n\rangle \le 0\} \\ x_{n+1} = Q_{H_n \bigcap V_n \bigcap W_n} x_0, n = 0, 1, 2, \ldots \end{cases} \tag{2.1}$$

converges strongly to $Q_{T^{-1}0 \bigcap \mathrm{Fix}(S)} x_0$ provided

(i) $\{\alpha_n\}, \{\beta_n\} \subset [0, 1)$ with $\alpha_n \le 1 - \delta_1$ and $\beta_n \le 1 - \delta_2$, for some $\delta_1, \delta_2 \in (0, 1)$;

(ii) $\{r_n\} \subset (0, +\infty)$ with $\inf_{n \ge 0} r_n > 0$, and

(iii) the error sequence $\{e_n\} \subset E$ such that $\|e_n\| \to 0$, as $n \to \infty$.

Proof We split the proof into five steps.

Step 1. H_n, V_n and W_n are closed and convex subsets of E.

Notice the facts that

$$\begin{aligned} \varphi(v, z_n) \le \alpha_n \varphi(v, x_n) + (1-\alpha_n)\varphi(v, x_n + e_n) &\Longleftrightarrow \|z_n\|^2 - \alpha_n\|x_n\|^2 - (1-\alpha_n)\|x_n + e_n\|^2 \\ &\le 2\langle v, Jz_n - \alpha_n Jx_n - (1-\alpha_n)J(x_n + e_n)\rangle \end{aligned}$$

and

$$\begin{aligned} &\varphi(v, u_n) \le \beta_n \varphi(v, x_n) + (1-\beta_n)\varphi(v, z_n) \\ &\quad \Longleftrightarrow (1-\beta_n)\|z_n\|^2 - \|u_n\|^2 + \beta_n\|x_n\|^2 \ge 2\langle v, (1-\beta_n)Jz_n + \beta_n Jx_n - Ju_n\rangle, \end{aligned}$$

we can easily know that H_n and V_n are closed and convex subsets of E.

It is obvious that W_n is also a closed and convex subset of E.

Step 2. $T^{-1}0 \bigcap \mathrm{Fix}(S) \subset H_n \bigcap V_n \bigcap W_n$ for each nonnegative integer n.

To observe this, take $p \in T^{-1}0 \bigcap \mathrm{Fix}(S)$.

From the definition of maximal monotone operator, we know that there exists $y_0 \in E$ such that $y_0 = Q_{r_0}^T(x_0 + e_0)$. It follows from Lemma 1.6 that $\varphi(p, y_0) \le \varphi(p, x_0 + e_0)$. Then

$$\varphi(p, z_0) \le \alpha_0 \varphi(p, x_0) + (1-\alpha_0)\varphi(p, y_0) \le \alpha_0 \varphi(p, x_0) + (1-\alpha_0)\varphi(p, x_0 + e_0),$$

which implies that $p \in H_0$. Moreover, from Definition 1.3, we know that

$$\varphi(p, u_0) \le \beta_0 \varphi(p, x_0) + (1-\beta_0)\varphi(p, Sz_0) \le \beta_0 \varphi(p, x_0) + (1-\beta_0)\varphi(p, z_0),$$

which implies that $p \in V_0$. On the other hand, it is clear that $p \in W_0 = E$. Then $p \in H_0 \bigcap V_0 \bigcap W_0$, and therefore $x_1 = Q_{H_0 \bigcap V_0 \bigcap W_0} x_0$ is well defined.

Suppose that $p \in H_{n-1} \bigcap V_{n-1} \bigcap W_{n-1}$ and x_n is well defined for some $n \ge 1$. Then there exists y_n such that $y_n = Q_{r_n}^T(x_n + e_n)$. Lemma 1.6 implies that $\varphi(p, y_n) \le \varphi(p, x_n + e_n)$. Thus

$$\varphi(p, z_n) \le \alpha_n \varphi(p, x_n) + (1-\alpha_n)\varphi(p, y_n) \le \alpha_n \varphi(p, x_n) + (1-\alpha_n)\varphi(p, x_n + e_n),$$

which implies that $p \in H_n$. It can also be seen from Definition 1.3 that $p \in V_n$. It follows from Lemma 1.7 that:

$$\langle p - x_n, Jx_0 - Jx_n \rangle = \langle p - Q_{H_{n-1} \bigcap V_{n-1} \bigcap W_{n-1}} x_0, Jx_0 - JQ_{H_{n-1} \bigcap V_{n-1} \bigcap W_{n-1}} x_0 \rangle \leq 0,$$

which implies that $p \in W_n$. Hence $x_{n+1} = Q_{H_n \bigcap V_n \bigcap W_n} x_0$ is well defined. Then by induction, the sequence generated by (2.1) is well defined, and $T^{-1}0 \bigcap \mathrm{Fix}(S) \subset H_n \bigcap V_n \bigcap W_n$ for each $n \geq 0$.

Step 3. $\{x_n\}$ is a bounded sequence of E.

In fact, for $\forall p \in T^{-1}0 \bigcap \mathrm{Fix}(S) \subset H_n \bigcap V_n \bigcap W_n$, it follows from lemma 1.4 that

$$\varphi(p, Q_{W_n} x_0) + \varphi(Q_{W_n} x_0, x_0) \leq \varphi(p, x_0).$$

From the definition of W_n and Lemmas 1.3 and 1.7, we know that $x_n = Q_{W_n} x_0$, which implies that $\varphi(p, x_n) + \varphi(x_n, x_0) \leq \varphi(p, x_0)$. Therefore, $\{x_n\}$ is bounded.

Step 4. $\omega(x_n) \subset T^{-1}0 \bigcap \mathrm{Fix}(S)$, where $\omega(x_n)$ denotes the set consisting all of the weak limit points of $\{x_n\}$.

From the facts that $x_n = Q_{W_n} x_0$, $x_{n+1} \in W_n$ and Lemma 1.4, we have

$$\varphi(x_{n+1}, x_n) + \varphi(x_n, x_0) \leq \varphi(x_{n+1}, x_0).$$

Therefore, $\lim_{n\to\infty} \varphi(x_n, x_0)$ exists. Then $\varphi(x_{n+1}, x_n) \to 0$, which implies from lemma 1.5 that $x_{n+1} - x_n \to 0$, as $n \to \infty$. Since $x_{n+1} \in H_n$, then

$$\varphi(x_{n+1}, z_n) \leq \alpha_n \varphi(x_{n+1}, x_n) + (1 - \alpha_n)\varphi(x_{n+1}, x_n + e_n). \tag{2.2}$$

Notice that

$$\varphi(x_{n+1}, x_n + e_n) - \varphi(x_{n+1}, x_n) = \|x_n + e_n\|^2 - \|x_n\|^2 + 2\langle x_{n+1}, Jx_n - J(x_n + e_n)\rangle. \tag{2.3}$$

Since $J : E \to E^*$ is uniformly continuous on each bounded subset of E and $\|e_n\| \to 0$, we know from equality (2.3) that $\varphi(x_{n+1}, x_n + e_n) \to 0$, which implies that $\varphi(x_{n+1}, z_n) \to 0$ by (2.2).

Since $x_{n+1} \in V_n$, we have

$$\varphi(x_{n+1}, u_n) \leq \beta_n \varphi(x_{n+1}, x_n) + (1 - \beta_n)\varphi(x_{n+1}, z_n). \tag{2.4}$$

Therefore, $\varphi(x_{n+1}, u_n) \to 0$, as $n \to \infty$. Using Lemma 1.5, we know that $x_{n+1} - z_n \to 0$, $x_{n+1} - u_n \to 0$, as $n \to \infty$. Since both $J : E \to E^*$ and $J^{-1} : E^* \to E$ are uniformly continuous on bounded subsets, we have $x_n - y_n \to 0$, as $n \to \infty$.

From Step 3, we know that $\omega(x_n) \neq \emptyset$. Then $\forall q \in \omega(x_n)$, there exists a subsequence of $\{x_n\}$, for simplicity we still denote it by $\{x_n\}$, such that $x_n \rightharpoonup q$, as $n \to \infty$. Therefore $u_n \rightharpoonup q$, $z_n \rightharpoonup q$ and $y_n \rightharpoonup q$, as $n \to \infty$.

Since both J and S are weakly continuous, $q = Sq$. From iterative scheme (2.1), we know that there exists $v_n \in Ty_n$ such that $r_n v_n = J(x_n + e_n) - Jy_n$. Then $v_n \to 0$, as $n \to \infty$. Lemma 1.2 implies that $q \in T^{-1}0$.

Step 5. $x_n \to q^* = Q_{T^{-1}0 \bigcap \mathrm{Fix}(S)} x_0$, as $n \to \infty$.

Let $\{x_{n_i}\}$ be any subsequence of $\{x_n\}$ which is weakly convergent to $q \in T^{-1}0 \bigcap \mathrm{Fix}(S)$. Since $x_{n+1} = Q_{H_n \bigcap V_n \bigcap W_n} x_0$ and $q^* \in T^{-1}0 \bigcap \mathrm{Fix}(S) \subset H_n \bigcap V_n \bigcap W_n$, we have $\varphi(x_{n+1}, x_0) \leq \varphi(q^*, x_0)$. Then

$$\begin{aligned}\varphi(x_n, q^*) &= \varphi(x_n, x_0) + \varphi(x_0, q^*) - 2\langle x_n - x_0, Jq^* - Jx_0\rangle \\ &\leq \varphi(q^*, x_0) + \varphi(x_0, q^*) - 2\langle x_n - x_0, Jq^* - Jx_0\rangle,\end{aligned}$$

which yields

$$\begin{aligned}\limsup_{n\to\infty} \varphi(x_{n_i}, q^*) &\leq \varphi(q^*, x_0) + \varphi(x_0, q^*) - 2\langle q - x_0, Jq^* - Jx_0\rangle \\ &= 2\langle q^* - q, Jq^* - Jx_0\rangle \leq 0.\end{aligned}$$

Hence $\varphi(x_{n_i}, q^*) \to 0$, as $i \to \infty$. It follows from Lemma 1.5 that $x_{n_i} \to q^*$, as $i \to \infty$. This means that the whole sequence $\{x_n\}$ converges weakly to q^* and that each weakly convergent subsequence of $\{x_n\}$ converges strongly to q^*. Therefore, $x_n \to q^* = Q_{T^{-1}0 \bigcap Fix(S)} x_0$, as $n \to \infty$. This completes the proof. □

Remark 2.1 If E is reduced to a real Hilbert space H, then $Q^T_{r_n}$ equals to $J^T_{r_n} = (I + r_n T)^{-1}$. If, moreover, $\beta_n \equiv 0$ and $S \equiv I$, then iterative scheme (2.1) is reduced to the following one which is introduced by Yanes and Xu in [5]:

$$\begin{cases} x_0 \in H \quad \text{chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1 - \alpha_n) J^T_{r_n}(x_n + e_n) \\ H_n = \{v \in H : \|y_n - v\|^2 \leq \|x_n - v\|^2 + 2(1 - \alpha_n)\langle x_n - v, e_n\rangle + \|e_n\|^2\}, n \geq 0 \\ W_n = \{z \in H : \langle z - x_n, x_0 - x_n\rangle \leq 0\}, n \geq 0 \\ x_{n+1} = P_{H_n \bigcap W_n} x_0, n \geq 0. \end{cases} \tag{2.5}$$

They proved that if $T^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ generated by (2.5) converges strongly to $P_{T^{-1}0} x_0$ provided (i) $\alpha_n \leq 1 - \delta$ for some $\delta \in (0, 1)$, (ii) $\inf_n r_n > 0$, and (iii) $\|e_n\| \to 0$.

Remark 2.2 If E is reduced to a real Hilbert space H, $\alpha_n \equiv 0, \beta_n \equiv 0, e_n \equiv 0$ and $S \equiv I$, then (2.1) includes the following iterative scheme introduced by Solodov and Svaiter in [6]:

$$\begin{cases} x_0 \in H, \\ 0 = v_n + \dfrac{1}{r_n}(y_n - x_n), v_n \in Ty_n \\ H_n = \{z \in H : \langle z - y_n, v_n\rangle \leq 0\} \\ W_n = \{z \in H : \langle z - x_n, x_0 - x_n\rangle \leq 0\}, n \geq 0 \\ x_{n+1} = P_{H_n \bigcap W_n} x_0, n = 0, 1, 2, \ldots . \end{cases} \tag{2.6}$$

They proved that if $T^{-1}0 \neq \emptyset$ and $\liminf_{n\to\infty} r_n > 0$, then the sequence $\{x_n\}$ generated by (2.6) converges strongly to $P_{T^{-1}0} x_0$.

Corollary 2.1 *Suppose E and S are the same as those in Theorem 2.1. For $i = 1, 2, \ldots, m$, let $T_i : E \to 2^{E^*}$ be maximal monotone operators. Denote by $D := \bigcap_{i=1}^m T_i^{-1}0 \bigcap \mathrm{Fix}(S)$ and* *suppose that* $D \neq \emptyset$. *Then the sequence* $\{x_n\}$ *generated by scheme:*

$$\begin{cases} x_0 \in E, r_{0,i} > 0, i = 1, 2, \ldots, m \\ y_{n,i} = Q_{r_{n,i}}^{T_i}(x_n + e_n), i = 1, 2, \ldots, m \\ Jz_{n,i} = \alpha_{n,i} Jx_n + (1-\alpha_{n,i}) Jy_{n,i}, i = 1, 2, \ldots, m \\ Ju_{n,i} = \beta_{n,i} Jx_n + (1-\beta_{n,i}) JSz_{n,i}, i = 1, 2, \ldots, m \\ H_{n,i} = \{v \in E : \varphi(v, z_{n,i}) \leq \alpha_{n,i}\varphi(v, x_n) + (1-\alpha_{n,i})\varphi(v, x_n + e_n)\}, i = 1, 2, \ldots, m \\ V_{n,i} = \{v \in E : \varphi(v, u_{n,i}) \leq \beta_{n,i}\varphi(v, x_n) + (1-\beta_{n,i})\varphi(v, z_{n,i})\}, i = 1, 2, \ldots, m \\ H_n := \bigcap_{i=1}^{m} H_{n,i} \bigcap V_{n,i} \\ W_n = \{z \in E : \langle z - x_n, Jx_0 - Jx_n \rangle \leq 0\} \\ x_{n+1} = Q_{H_n \bigcap V_n \bigcap W_n} x_0, n = 0, 1, 2, \ldots \end{cases} \tag{2.7}$$

converges strongly to $Q_D x_0$ *provided*

(i) $\{\alpha_{n,i}\}, \{\beta_{n,i}\} \subset [0,1)$ *with* $\alpha_{n,i} \leq 1 - \delta_1$ *and* $\beta_{n,i} \leq 1 - \delta_2$, *for some* $\delta_1, \delta_2 \in (0,1), i = 1, 2, \ldots, m$ *and* $n = 0, 1, 2, \ldots$*;*

(ii) $\{r_{n,i}\} \subset (0, +\infty)$ *with* $\inf_{n \geq 0} r_{n,i} > 0, i = 1, 2, \ldots, m$*; and*

(iii) $\|e_n\| \to 0$, *as* $n \to \infty$.

Similar to the proof of Theorem 2.1, we have the following result:

Theorem 2.2 *The sequence* $\{x_n\}$ *generated by the following scheme:*

$$\begin{cases} x_0 \in E, r_0 > 0 \\ y_n = Q_{r_n}^{T}(x_n + e_n) \\ Jz_n = \alpha_n Jx_0 + (1-\alpha_n) Jy_n \\ Ju_n = \beta_n Jx_0 + (1-\beta_n) JSz_n \\ H_n = \{v \in E : \varphi(v, z_n) \leq \alpha_n \varphi(v, x_0) + (1-\alpha_n)\varphi(v, x_n + e_n)\} \\ V_n = \{v \in E : \varphi(v, u_n) \leq \beta_n \varphi(v, x_0) + (1-\beta_n)\varphi(v, z_n)\} \\ W_n = \{z \in E : < z - x_n, Jx_0 - Jx_n > \leq 0\} \\ x_{n+1} = Q_{H_n \bigcap V_n \bigcap W_n} x_0, n = 0, 1, 2, \ldots \end{cases} \tag{2.8}$$

converges strongly to $Q_{T^{-1}0 \bigcap Fix(S)} x_0$ *provided*

(i) $\{\alpha_n\}, \{\beta_n\} \subset [0,1)$ *such that* $\alpha_n \to 0, \beta_n \to 0$, *as* $n \to \infty$,

(ii) $\inf_{n \geq 0} r_n > 0$, *and*

(iii) $\|e_n\| \to 0$, *as* $n \to \infty$.

Remark 2.3 If E is reduced to a real Hilbert space H, $\beta_n \equiv 0$ and $S \equiv I$, then (2.8) is reduced to the following one which is similar to that in [5]:

$$\begin{cases} x_0 \in H \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_0 + (1-\alpha_n) J^T_{r_n}(x_n + e_n) \\ H_n = \{v \in H : \|y_n - v\|^2 \leq \|x_n - v\|^2 + \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, v\rangle) \\ +2(1-\alpha_n)\langle x_n - v, e_n\rangle + (1-\alpha_n)\|e_n\|^2 - \alpha_n\|x_n\|^2\} \\ W_n = \{v \in H : \langle x_n - v, x_n - x_0\rangle \leq 0\}, n \geq 0 \\ x_{n+1} = P_{H_n \bigcap W_n} x_0, n \geq 0. \end{cases} \tag{2.9}$$

Moreover, if $T^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ generated by (2.9) converges strongly to $P_{T^{-1}0}x_0$ provided

(i) $\alpha_n \to 0$, as $n \to \infty$,

(ii) $\inf_n r_n > 0$, and

(iii) $\|e_n\| \to 0$, as $n \to \infty$.

Corollary 2.2 *Suppose E, S, T_i and D are the same as those in Corollary 2.1. If $D \neq \emptyset$, then the sequence $\{x_n\}$ generated by scheme:*

$$\begin{cases} x_0 \in E, r_{0,i} > 0, i = 1,2,\ldots,m \\ y_{n,i} = Q^{T_i}_{r_{n,i}}(x_n + e_n), i = 1,2,\ldots,m \\ Jz_{n,i} = \alpha_{n,i} Jx_0 + (1-\alpha_{n,i})Jy_{n,i}, i = 1,2,\ldots,m \\ Ju_{n,i} = \beta_{n,i} Jx_0 + (1-\beta_{n,i})JSz_{n,i}, i = 1,2,\ldots,m \\ H_{n,i} = \{v \in E : \varphi(v, z_{n,i}) \leq \alpha_{n,i}\varphi(v, x_0) + (1-\alpha_{n,i})\varphi(v, x_n + e_n)\}, i = 1,2,\ldots,m \\ V_{n,i} = \{v \in E : \varphi(v, u_{n,i}) \leq \beta_{n,i}\varphi(v, x_0) + (1-\beta_{n,i})\varphi(v, z_{n,i})\}, i = 1,2,\ldots,m \\ H_n := \bigcap_{i=1}^m H_{n,i} \bigcap V_{n,i} \\ W_n = \{z \in E : \langle z - x_n, Jx_0 - Jx_n\rangle \leq 0\} \\ x_{n+1} = Q_{H_n \bigcap W_n} x_0, n = 0,1,2,\ldots \end{cases} \tag{2.10}$$

converges strongly to $Q_D x_0$ provided

(i) $\{\alpha_{n,i}\}, \{\beta_{n,i}\} \subset [0,1)$ such that $\alpha_{n,i} \to 0$ and $\beta_{n,i} \to 0$, for $i = 1,2,\ldots,m$, as $n \to \infty$;

(ii) $\{r_{n,i}\} \subset (0,+\infty)$ with $\inf_{n\geq 0} r_{n,i} > 0, i = 1,2,\ldots,m$; and

(iii) $\|e_n\| \to 0$, as $n \to \infty$.

Remark 2.4 From Theorems 2.1 and 2.2, we can see that the iterative schemes (2.1) and (2.8) are not only strongly convergent to zero point of maximal monotone operator T, but also strongly convergent to fixed point of the generalized nonexpansive mapping S.

3. Applications

Definition 3.1 *Let $f : E \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function. Then the subdifferential ∂f of f is defined by:*

$$\partial f(z) = \{v \in E^* : f(y) \geq f(z) + \langle y - z, v\rangle, \quad \forall y \in E\},$$

for $\forall z \in E$.

Theorem 3.1 *Let $E, S, \{\alpha_n\}, \{\beta_n\}, \{r_n\}$ and $\{e_n\}$ be the same as those in Theorem2.1. Let $f : E \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function. Let $\{x_n\}$ be the sequence generated by the following scheme:*

$$\begin{cases} x_0 \in E, r_0 > 0 \\ y_n = \arg\min_{z\in E}\{f(z) + \frac{1}{2r_n}\|z_n\|^2 - \frac{1}{r_n}\langle z, J(x_n + e_n)\rangle\} \\ Jz_n = \alpha_n Jx_n + (1-\alpha_n)Jy_n \\ Ju_n = \beta_n Jx_n + (1-\beta_n)JSz_n \\ H_n = \{v \in E : \varphi(v, z_n) \le \alpha_n\varphi(v, x_n) + (1-\alpha_n)\varphi(v, x_n + e_n)\} \\ V_n = \{v \in E : \varphi(v, u_n) \le \beta_n\varphi(v, x_n) + (1-\beta_n)\varphi(v, z_n)\} \\ W_n = \{z \in E : \langle z - x_n, Jx_0 - Jx_n\rangle \le 0\} \\ x_{n+1} = Q_{H_n \bigcap V_n \bigcap W_n} x_0, n = 0, 1, 2, \ldots. \end{cases} \tag{3.1}$$

If $(\partial f)^{-1}0 \bigcap \mathrm{Fix}(S) \ne \emptyset$, then $\{x_n\}$ converges strongly to $Q_{(\partial f)^{-1}0 \bigcap \mathrm{Fix}(S)} x_0$.

Proof Since $f : E \to (-\infty, +\infty]$ is a proper convex and lower semi-continuous function, the subdifferential ∂f of f is a maximal monotone operator form E into E^*. We also know that

$$y_n = \arg\min_{z\in E}\{f(z) + \frac{1}{2r_n}\|z_n\|^2 - \frac{1}{r_n}\langle z, J(x_n + e_n)\rangle\}$$

is equivalent to

$$0 \in \partial f(y_n) + \frac{1}{r_n}Jy_n - \frac{1}{r_n}J(x_n + e_n).$$

Thus we have $y_n = Q^{\partial f}_{r_n}(x_n + e_n)$. Theorem 2.1 implies that $\{x_n\}$ strongly converges to $Q_{(\partial f)^{-1}0 \bigcap \mathrm{Fix}(S)} x_0$, as $n \to \infty$. This completes the proof. □

Similarly, we have:

Theorem 3.2 *Let $E, S, \{\alpha_n\}, \{\beta_n\}, \{r_n\}$ and $\{e_n\}$ be the same as those in Theorem 2.2. Let $f : E \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function. Let $\{x_n\}$ be the sequence generated by the following scheme:*

$$\begin{cases} x_0 \in E, r_0 > 0 \\ y_n = \arg\min_{z\in E}\{f(z) + \frac{1}{2r_n}\|z_n\|^2 - \frac{1}{r_n}\langle z, J(x_n + e_n)\rangle\} \\ Jz_n = \alpha_n Jx_0 + (1-\alpha_n)Jy_n \\ Ju_n = \beta_n Jx_0 + (1-\beta_n)JSz_n \\ H_n = \{v \in E : \varphi(v, z_n) \le \alpha_n\varphi(v, x_0) + (1-\alpha_n)\varphi(v, x_n + e_n)\} \\ V_n = \{v \in E : \varphi(v, u_n) \le \beta_n\varphi(v, x_0) + (1-\beta_n)\varphi(v, z_n)\} \\ W_n = \{z \in E : \langle z - x_n, Jx_0 - Jx_n\rangle \le 0\} \\ x_{n+1} = Q_{H_n \bigcap V_n \bigcap W_n} x_0, n = 0, 1, 2, \ldots. \end{cases} \tag{3.2}$$

If $(\partial f)^{-1}0 \bigcap \mathrm{Fix}(S) \ne \emptyset$, then the result of Theorem 3.1 is still true.

Remark 3.1 Theorems 3.1 and 3.2 are extensions of Theorem 2 in [2] and Theorem 4.1 in [4] in the sense that the sequences defined by (3.1) and (3.2) are not only strongly convergent to the minimizer of f, but also strongly convergent to a fixed point of a generalized nonexpansive mapping.

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