Journal of Mathematical Research & Exposition
Mar., 2009, Vol. 29, No. 2, pp. 293–301
DOI:10.3770/j.issn:1000-341X.2009.02.012
Http://jmre.dlut.edu.cn

On the Sharp Upper Bound of Spectral Radius of Weighted Trees

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Abstract The spectrum of weighted graphs are often used to solve the problems in the design of networks and electronic circuits. In this paper, we derive the sharp upper bound of spectral radius of all weighted trees on given order and edge independence number, and obtain all such trees that their spectral radius reach the upper bound.

Keywords weighted tree; spectral radius; matching.

Document code A
MR(2000) Subject Classification 05C50
Chinese Library Classification O157.5

1. Introduction

In this paper, we only consider simple connected graphs with positive weights. Let G be a weighted graph on vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, edge set $E(G) \neq \emptyset$ and weight set $W(G) = \{w_j > 0 : j = 1, 2, \ldots, |E(G)|\}$, where $|E(G)|$ is the number of edges in $E(G)$. The function $w_G : E(G) \to W(G)$ is called a weight function of G. It is obvious that each weighted graph corresponds to a weight function. The adjacency matrix of G is defined to be the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = w_G(v_iv_j)$ if $v_iv_j \in E(G)$, and $a_{ij} = 0$ otherwise. The characteristic polynomial of G is just $\det(\lambda I_n - A(G))$, denoted by $\phi(G, \lambda)$ or $\phi(G)$. Since $A(G)$ is a nonnegative symmetric matrix, its eigenvalues are all real numbers and its largest eigenvalue is a positive number. The largest eigenvalue of $A(G)$ is called the spectral radius of G and denoted by $\rho(G)$.

The degree of a vertex v of a weighted graph G, denoted by $d_G(v)$, is the number of edges being incident to v in G. Let H and G be two weighted graphs. If H is a subgraph of G and $w_H(e) = w_G(e)$ for each $e \in E(H)$, then H is called a weighted subgraph of G. Let H be a weighted subgraph of G. If $V(H) \neq V(G)$ or $E(H) \neq E(G)$, then H is called a weighted proper subgraph of G. If $V(H) = V(G)$, then H is called a weighted spanning subgraph of G.

Since graphs of the design of networks and electronic circuits are usually weighted, the spectrum of weighted graphs is often used to solve problems. On the other hand, a graph may be regarded as a weighted graph with weight 1 of each edge. Therefore, it is significant and necessary to investigate the spectrum of weighted graphs. Fiedler M had introduced the following

Received date: 2007-01-19; **Accepted date**: 2008-07-06
Foundation item: the National Natural Science Foundation of China (No. 10871204).

question: What is the optimal distribution of nonnegative weights (with total sum 1) among the edges of a given graph, so that the spectral radius of the resulting matrix is minimum? He himself showed that the optimum solution is achieved and Poljak presented a polynomial time algorithm which finds such optimum solution[1]. Yang et al. gave an upper bound of spectral radius of weighted trees with fixed weight set[2]. Yuan and Shu gave the second largest value of spectral radius of weighted trees with fixed weight set[3]. In this paper, we derive the sharp upper bound of spectral radius of weighted trees with fixed weight set on given order and edge independence number, and obtain all such trees that their spectral radius reach the upper bound.

Lemma 1.1[3] *Let u, v be two vertices of a weighted connected graph G with n vertices and positive weights. Suppose that $v_1, v_2, \ldots, v_s$ $(v_j \neq u)$ are some vertices of $N_G(v)\backslash N_G(u)$ $(1 \leq s \leq d_G(v))$ and $(x_1, x_2, \ldots, x_n)^t$ is the Perron vector of $A(G)$, where B^t denotes the transpose of a matrix B and x_i corresponds to the vertex v_i $(i = 1, 2, \ldots, n)$. Let G^* be the graph obtained from G by deleting the edge vv_i and adding the edge uv_i such that $w_{G^*}(uv_i) = w_G(vv_i)$ $(i = 1, 2, \ldots, s)$, respectively. If $x_u \geq x_v$, then $\rho(G) < \rho(G^*)$.*

Let A be a nonnegative irreducible matrix of order n. By Perron-Frobenius Theorem, the maximal eigenvalue of B is smaller than the maximal eigenvalue of A if $B \geq 0$, $A - B \geq 0$ and $A \neq B$ (see, e.g. [4, Theorem 0.7]). It is well known that the maximal eigenvalue of every principal submatrix of order less than n of A is smaller than the maximal eigenvalue of A (see, e.g. [4, Theorem 0.6]). Note that the adjacency matrix of a weighted connected graph with positive weights is irreducible. Therefore, it follows that the following is true.

Lemma 1.2 *Let H be a weighted proper subgraph of a connected graph G with positive weights. Then $\rho(H) < \rho(G)$.*

2. Results and proofs

Lemma 2.1 *Let G be the weighted graph obtained from two weighted graphs G_1 and G_2 by joining a vertex v of G_1 to a vertex u of G_2 with a new edge $e = vu$. Then*

$$\phi(G, \lambda) = \phi(G_1, \lambda)\phi(G_2, \lambda) - w_G^2(e)\phi(G_1 - v, \lambda)\phi(G_2 - u, \lambda).$$

Proof Let $V(G_1) = \{v, v_1, v_2, \ldots, v_m\}$, $V(G_2) = \{u, u_1, u_2, \ldots, u_n\}$. Assume

$$N_{G_1}(v) = \{v_1, v_2, \ldots, v_k\}, \quad N_{G_2}(u) = \{u_1, u_2, \ldots, u_l\}.$$

Write $w_{G_1}(vv_i) = a_i$, $w_{G_2}(uu_j) = b_j$, $w_G(e) = c$, $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, l$. Let

$$a = (a_1, a_2, \ldots, a_k, 0, \ldots, 0)_{1\times m}, \quad b = (b_1, b_2, \ldots, b_l, 0 \ldots, 0)_{1\times n}.$$

Then

$$\phi(G, \lambda) = \begin{vmatrix} \lambda & -a & -c & 0 \\ -a^t & \lambda I_m - A(G_1 - v) & 0 & 0 \\ -c & 0 & \lambda & -b \\ 0 & 0 & -b^t & \lambda I_n - A(G_2 - u) \end{vmatrix}.$$

By applying Laplace expansion theorem to the first $m+1$ rows of the determinant, we obtain the three types of subdeterminant of order $m+1$ as follows.

(I) The subdeterminant $|\lambda I_{m+1} - A(G_1)| = \phi(G_1, \lambda)$, which contains the first $m+1$ columns. This subdeterminant corresponds to the algebraic cofactor $|\lambda I_{n+1} - A(G_2)| = \phi(G_2, \lambda)$.

(II) The subdeterminant Δ, which contains the $2nd$ to the $(m+2)$th columns. It is easy to compute $\Delta = (-1)^{m+1} c \cdot |\lambda I_m - A(G_1 - v)| = (-1)^{m+1} c \cdot \phi(G_1 - v, \lambda)$ and its algebraic cofactor is equal to $(-1)^{m+2} c \cdot |\lambda I_n - A(G_2 - u)| = (-1)^{m+2} c \cdot \phi(G_2 - u, \lambda)$.

(III) The other subdeterminants apart from (I) and (II). Each such subdeterminant or its algebraic cofactor always contains a zero column. Thus at least one of each such subdeterminant and its algebraic cofactor is equal to 0.

The result is obtained by combining (I), (II) and (III), and applying Laplace expansion theorem. □

Lemma 2.2 *Let H be a weighted proper spanning subgraph of a weighted tree T with positive weights. Then for $\lambda \geq \rho(T)$, we have $\phi(H, \lambda) > \phi(T, \lambda)$.*

Proof Let $E(T) \backslash E(H) = \{v_1 u_1, v_2 u_2, \ldots, v_s u_s\}$, where $s \geq 1$. Write $T_0 = T$ and

$$T_i = T_{i-1} - v_i u_i, \quad i = 1, 2, \ldots, s.$$

Then $T_s = H$. From Lemma 1.2, we have

$$\rho(T) > \rho(T_1) \geq \cdots \geq \rho(T_s).$$

Since $T_{i-1} - v_i - u_i$ is a weighted proper subgraph of T_{i-1}, by Lemma 1.2, we have

$$\rho(T) > \rho(T_0 - v_1 - u_1),$$

and for $i = 2, 3, \ldots, s$,

$$\rho(T_{i-1}) \geq \rho(T_{i-1} - v_i - u_i).$$

Therefore, for $\lambda \geq \rho(T)$, we have $\phi(T_{i-1} - v_i - u_i) > 0\,(i = 1, 2, \ldots, s)$. So by Lemma 2.1, for $\lambda \geq \rho(T)$, when $i = 1, 2, \ldots, s$, we have

$$\phi(T_{i-1}, \lambda) = \phi(T_i, \lambda) - w_{T_i}^2(v_i u_i) \phi(T_{i-1} - v_i - u_i, \lambda) < \phi(T_i, \lambda).$$

Therefore, the required result follows. □

Lemma 2.3 *Let v be a vertex of a weighted tree T with positive weights and at least four vertices. Suppose that there exists a path vuz of T such that u has degree 2 and z has degree 1. Let T^* be the weighted tree obtained from T by deleting the edge uz and adding the new edge vz such that $w_{T^*}(vz) = w_T(uz)$. Then $\rho(T) < \rho(T^*)$.*

Proof By Lemma 2.1, we have

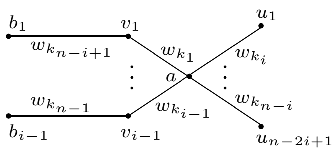
$$\begin{aligned} \phi(T, \lambda) &= \phi(T - uz, \lambda) - w_T^2(uz) \phi(T - u - z) \\ &= \lambda \phi(T - z, \lambda) - w_T^2(uz) \phi(T - u - z), \\ \phi(T^*, \lambda) &= \phi(T^* - vz, \lambda) - w_{T^*}^2(vz) \phi(T^* - v - z) \end{aligned}$$

$$= \lambda\phi(T - z, \lambda) - w_T^2(uz)\phi(T^* - v - z).$$

Therefore, we have

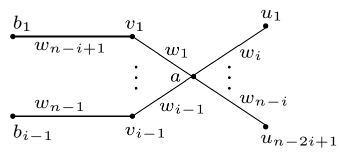
$$\phi(T, \lambda) - \phi(T^*, \lambda) = w_T^2(uz)[\phi(T^* - v - z, \lambda) - \phi(T - u - z, \lambda)].$$

It is obvious that $T - u - z$ is a weighted tree with positive weights. Since $T^* - v - z$ is a weighted proper spanning subgraph of $T - u - z$, by Lemma 2.2, for $\lambda \geq \rho(T - u - z)$, we have $\phi(T^* - v - z, \lambda) > \phi(T - u - z, \lambda)$. Note that $T - u - z$ is a weighted proper subgraph of T^*, so by Lemma 1.2, we have $\rho(T^*) > \rho(T - u - z)$. Hence for $\lambda \geq \rho(T^*)$, we have $\phi(T^* - v - z, \lambda) > \phi(T - u - z, \lambda)$, i.e., $\phi(T, \lambda) > \phi(T^*, \lambda)$. This implies the desired result. $\square$



$T_i(k_1, k_2, \ldots, k_{n-1})$

Figure 1 The special weighted tree



$T_i(1, 2, \ldots, n-1)$

Figure 2 The extreme weighted tree

Let $w_1 \geq w_2 \geq \cdots \geq w_{n-1} > 0$. For a permutation $k_1k_2\cdots k_{n-1}$ of $12\cdots(n-1)$, let $T_i(k_1, k_2, \ldots, k_{n-1}) = G$ be the weighted tree with order n and edge independence number i shown in Figure 1. Put

$$\Delta = \lambda^{n-2i} \cdot \prod_{j=1}^{i-1}(\lambda^2 - w_{k_{n-i+j}}^2); \quad J_l = \{av_1, av_2, \ldots, av_l\}, \quad l = 1, 2, \ldots, i-1.$$

By Lemma 2.1, we have

$$\begin{aligned}
\phi(G) &= \phi(G - av_1) - w_{k_1}^2\phi(G - a - v_1) \\
&= \phi(G - J_1) - \frac{w_{k_1}^2}{\lambda^2 - w_{k_{n-i+1}}^2} \cdot \lambda^2\Delta \\
&= \phi(G - J_1 - av_2) - w_{k_2}^2\phi(G - J_1 - a - v_2) - \frac{w_{k_1}^2}{\lambda^2 - w_{k_{n-i+1}}^2} \cdot \lambda^2\Delta \\
&= \phi(G - J_2) - \lambda^2\Delta \cdot \sum_{j=1}^{2} \frac{w_{k_j}^2}{\lambda^2 - w_{k_{n-i+j}}^2} = \cdots \\
&= \phi(G - J_{i-1}) - \lambda^2\Delta \cdot \sum_{j=1}^{i-1} \frac{w_{k_j}^2}{\lambda^2 - w_{k_{n-i+j}}^2} = \cdots \\
&= \lambda^2\Delta - \Delta \cdot \sum_{j=0}^{n-2i} w_{k_{n-i-j}}^2 - \lambda^2\Delta \cdot \sum_{j=1}^{i-1} \frac{w_{k_j}^2}{\lambda^2 - w_{k_{n-i+j}}^2} \\
&= \lambda^{n-2i}\prod_{j=1}^{i-1}(\lambda^2 - w_{k_{n-i+j}}^2) \cdot f(\lambda, k_1, k_2, \ldots, k_{n-1}).
\end{aligned}$$

Where

$$f(\lambda, k_1, k_2, \ldots, k_{n-1}) = \lambda^2 - \sum_{j=i}^{n-i} w_{k_j}^2 - \sum_{j=1}^{i-1} \frac{\lambda^2 w_{k_{i-j}}^2}{\lambda^2 - w_{k_{n-j}}^2}. \tag{1}$$

Since each edge of $T_i(k_1, k_2, \ldots, k_{n-1})$ is a weighted proper subgraph of $T_i(k_1, k_2, \ldots, k_{n-1})$, by Lemma 1.2, we have

$$\rho(T_i(k_1, k_2, \ldots, k_{n-1})) > \max\{w_{k_j} : j = 1, 2, \ldots, n-1\}.$$

Therefore, $\rho(T_i(k_1, k_2, \ldots, k_{n-1}))$ is the largest root of $f(\lambda, k_1, k_2, \ldots, k_{n-1}) = 0$ on λ.

To p, l $(1 \le p < l \le n-1)$, let $\widetilde{T}_i(k_l, k_p)$ denote the weighted tree obtained from $T_i(k_1, k_2, \ldots, k_{n-1})$ by exchanging w_{k_p} and w_{k_l}, and making the weights of other edges not changed. Let $f(\lambda, k_l, k_p)$ denote the function corresponding to Equation (1) in $\phi(\widetilde{T}_i(k_l, k_p))$.

Lemma 2.4 *Suppose that two integers p, l satisfy one of the following conditions.*

(i) $1 \le p \le i-1 < l \le n-i$ *and* $w_{k_l} > w_{k_p}$.

(ii) $i \le p \le n-i < l \le n-1$ *and* $w_{k_l} > w_{k_p}$.

(iii) $n-i+1 \le p < l \le n-1$, $w_{k_l} > w_{k_p}$ *and* $w_{k_{i+l-n}} < w_{k_{i+p-n}}$.

Then $\rho(T_i(k_1, k_2, \ldots, k_{n-1})) < \rho(\widetilde{T}_i(k_l, k_p))$.

Proof (i) Suppose that the condition (i) holds. From Equation (1), we have

$$f(\lambda, k_1, k_2, \ldots, k_{n-1}) - f(\lambda, k_l, k_p) = \frac{(w_{k_l}^2 - w_{k_p}^2) w_{k_{n-i+p}}^2}{\lambda^2 - w_{k_{n-i+p}}^2}.$$

Therefore, for $\lambda \ge \rho(\widetilde{T}_i(k_l, k_p))$, we always have $f(\lambda, k_1, k_2, \ldots, k_{n-1}) > f(\lambda, k_l, k_p)$, i.e., $\phi(T_i(k_1, k_2, \ldots, k_{n-1})) > \phi(\widetilde{T}_i(k_l, k_p))$. This implies the desired result.

(ii) Suppose that the condition (ii) holds. From Equation (1), we have

$$f(\lambda, k_1, k_2, \ldots, k_{n-1}) - f(\lambda, k_l, k_p) = \frac{\lambda^4 - (w_{k_l}^2 + w_{k_p}^2 + w_{k_{i+l-n}}^2)\lambda^2 + w_{k_l}^2 w_{k_p}^2}{(w_{k_l}^2 - w_{k_p}^2)^{-1}(\lambda^2 - w_{k_l}^2)(\lambda^2 - w_{k_p}^2)}.$$

Since the path $P_4 = u_{p-i+1} a v_{i+l-n} b_{i+l-n}$ with weights

$$w_{P_4}(u_{p-i+1}a) = w_{k_l}, \quad w_{P_4}(a v_{i+l-n}) = w_{k_{i+l-n}}, \quad w_{P_4}(v_{i+l-n} b_{i+l-n}) = w_{k_p}$$

is a weighted proper subgraph of $\widetilde{T}_i(k_l, k_p)$, by Lemma 1.2, we have $\rho(\widetilde{T}_i(k_l, k_p)) > \rho(P_4)$. By Lemma 2.1, we easily obtain

$$\phi(P_4) = \lambda^4 - (w_{k_l}^2 + w_{k_p}^2 + w_{k_{i+l-n}}^2)\lambda^2 + w_{k_l}^2 w_{k_p}^2.$$

Therefore, for $\lambda \ge \rho(\widetilde{T}_i(k_l, k_p))$, we always have $f(\lambda, k_1, k_2, \ldots, k_{n-1}) > f(\lambda, k_l, k_p)$, i.e., $\phi(T_i(k_1, k_2, \ldots, k_{n-1})) > \phi(\widetilde{T}_i(k_l, k_p))$. This implies the desired result.

(iii) Suppose that the condition (iii) holds. From Equation (1), we have

$$f(\lambda, k_1, k_2, \ldots, k_{n-1}) - f(\lambda, k_l, k_p) = \frac{\lambda^2 (w_{k_l}^2 - w_{k_p}^2)(w_{k_{i+p-n}}^2 - w_{k_{i+l-n}}^2)}{(\lambda^2 - w_{k_l}^2)(\lambda^2 - w_{k_p}^2)}.$$

Therefore, for $\lambda \ge \rho(\widetilde{T}_i(k_l, k_p))$, we always have $f(\lambda, k_1, k_2, \ldots, k_{n-1}) > f(\lambda, k_l, k_p)$, i.e., $\phi(T_i(k_1, k_2, \ldots, k_{n-1})) > \phi(\widetilde{T}_i(k_l, k_p))$. This implies the desired result. □

Lemma 2.5 *Let $T_i(1,2,\dots,n-1)$ be the weighted tree of order n with weights $w_1 \ge w_2 \ge \cdots \ge w_{n-1} > 0$ and edge independence number i shown in Figure 2. Then*

$$\rho(T_i(k_1,k_2,\dots,k_{n-1})) \le \rho(T_i(1,2,\dots,n-1)),$$

with equality if and only if $T_i(k_1,k_2,\dots,k_{n-1}) = T_i(1,2,\dots,n-1)$.

Proof If $w_1 = w_{n-1}$, then $T_i(k_1,k_2,\dots,k_{n-1}) = T_i(1,2,\dots,n-1)$, so the results hold. Next assume $w_1 > w_{n-1}$. We prove the results by the following steps.

Step 1. Suppose that $\max\{w_{k_i}, w_{k_{i+1}},\dots,w_{k_{n-i}}\} > \min\{w_{k_1}, w_{k_2},\dots,w_{k_{i-1}}\}$. Without loss of generality, assume

$$w_{k_1} \ge w_{k_2} \ge \cdots \ge w_{k_{i-1}}, \quad w_{k_i} \ge w_{k_{i+1}} \ge \cdots \ge w_{k_{n-i}}.$$

So $w_{k_i} > w_{k_{i-1}}$. Write $T_i(l_1,l_2,\dots,l_{n-1}) = \widetilde{T}_i(k_i,k_{i-1})$. To $T_i(l_1,l_2,\dots,l_{n-1})$, repeat the above procedure until we obtain a weighted tree $T_i(s_1,s_2,\dots,s_{n-1})$ such that

$$w_{s_1} \ge w_{s_2} \ge \cdots \ge w_{s_{i-1}} \ge w_{s_i} \ge w_{s_{i+1}} \ge \cdots \ge w_{s_{n-i}}.$$

By Lemma 2.4 (i), we have $\rho(T_i(k_1,k_2,\dots,k_{n-1})) < \rho(T_i(s_1,s_2,\dots,s_{n-1}))$.

Step 2. Suppose that $\max\{w_{s_{n-i+1}}, w_{s_{n-i+2}},\dots,w_{s_{n-1}}\} > w_{s_{n-i}}$. Let

$$w_{s_{n-i+j}} = \max\{w_{s_{n-i+1}}, w_{s_{n-i+2}},\dots,w_{s_{n-1}}\}.$$

Write $T_i(t_1,t_2,\dots,t_{n-1}) = \widetilde{T}_i(s_{n-i+j},s_{n-i})$. To $T_i(t_1,t_2,\dots,t_{n-1})$, repeat the above procedure until we obtain a weighted tree $T_i(p_1,p_2,\dots,p_{n-1})$ such that

$$w_{p_i} \ge w_{p_{i+1}} \ge \cdots \ge w_{p_{n-i}} \ge \max\{w_{p_{n-i+1}}, w_{p_{n-i+2}},\dots,w_{p_{n-1}}\}.$$

By Lemma 2.4 (ii), we have $\rho(T_i(s_1,s_2,\dots,s_{n-1})) < \rho(T_i(p_1,p_2,\dots,p_{n-1}))$.

Step 3. To $T_i(p_1,p_2,\dots,p_{n-1})$, repeat the procedures of step 1 until we obtain a weighted tree $T_i(q_1,q_2,\dots,q_{n-1})$ such that

$$w_{q_1} \ge \cdots \ge w_{q_{i-1}} \ge w_{q_i} \ge \cdots \ge w_{q_{n-i}} \ge \max\{w_{q_{n-i+1}}, w_{q_{n-i+2}},\dots,w_{q_{n-1}}\}.$$

By Lemma 2.4 (i), we have $\rho(T_i(p_1,p_2,\dots,p_{n-1})) \le \rho(T_i(q_1,q_2,\dots,q_{n-1}))$.

Step 4. If there are $\alpha,\beta (1 \le \alpha < \beta \le i-1)$ such that $w_{q_{n-i+\beta}} > w_{q_{n-i+\alpha}}$, then write $T_i(\delta_1,\delta_2,\dots,\delta_{n-1}) = \widetilde{T}_i(q_{n-i+\beta},q_{n-i+\alpha})$. To $T_i(\delta_1,\delta_2,\dots,\delta_{n-1})$, repeat the above procedure until we obtain a weighted tree $T_i(m_1,m_2,\dots,m_{n-1})$ such that

$$w_{m_1} \ge \cdots \ge w_{m_{i-1}} \ge w_{m_i} \ge \cdots \ge w_{m_{n-i}} \ge w_{m_{n-i+1}} \ge \cdots \ge w_{m_{n-1}}.$$

By Lemma 2.4 (iii), we have $\rho(T_i(q_1,q_2,\dots,q_{n-1})) < \rho(T_i(m_1,m_2,\dots,m_{n-1}))$.

Note that $T_i(m_1,m_2,\dots,m_{n-1}) = T_i(1,2,\dots,n-1)$ and the procedures in step 1, step 2 or step 4 carry out at least once when $T_i(k_1,k_2,\dots,k_{n-1}) \ne T_i(1,2,\dots,n-1)$. So by Steps 1–4, the desired results follow. □

Let K_t^s be the tree obtained from $K_{1,s+t}$ by adding a pendent edge for each of s vertices of degree 1. The center of $K_{1,s+t}$ is called the center of K_t^s. Let G be a weighted connected graph. A path $v_0v_1v_2\cdots v_k$ of G is called a pendent path of length k at v_0 if $d_G(v_k) = 1$, $d_G(v_0) \ge 2$

and $d_G(v_i) = 2$ for $1 \le i \le k-1$. In particular, a pendent path of length 1 is called a pendent edge. A vertex v of G is called an end-branch vertex if there are two non-negative integers s and t such that $G - v = sP_1 \bigcup tP_2 \bigcup W$, where W is a connected subgraph of G with at least three vertices, $t \neq 0$, or $t = 0$ and $s \ge 2$. Next let $S(G)$ denote the set of all end-branch vertices of G and $in(G)$ denote the edge independence number of G.

Lemma 2.6 *Let T be a weighted tree and $S(T)$ denote the set of all end-branch vertices in T.*

(i) If $S(T) = \emptyset$, then there are two non-negative integers s, t such that $T = K_t^s$.

(ii) If $S(T) \neq \emptyset$, then $|S(T)| \ge 2$.

Proof (i) Let d be the diameter of T and $v_1v_2v_3 \cdots v_dv_{d+1}$ be a longest path of T. Suppose that $d \ge 5$, $N_T(v_3) = \{v_4, u_0, u_1, u_2, \ldots, u_m\}$, where $u_0 = v_2$. If a vertex u_j $(0 \le j \le m)$ has at least two pendent edges, then u_j is an end-branch vertex of T; if each of $u_0, u_1, u_2, \ldots, u_m$ has at most a pendent edge, then v_3 is an end-branch vertex of T. The above results are in contradiction with assumption. Therefore, $d \le 4$.

When $d \le 3$, T is a star or a double star obtained by joining the centers of two stars with an edge. It is obvious that there are non-negative integers t and $0 \le s \le 1$ such that $T = K_t^s$.

When $d = 4$, T can be obtained by joining the center z_j of K_{1,p_j} $(j = 1, 2, \ldots, k)$ and the center of $K_{1,q}$ with an edge. From $d = 4$, we have $k \ge 2$. If $p_j \ge 2$, then z_j is an end-branch vertex of T, this is in contradiction with assumption. Therefore, $p_1 = p_2 = \cdots = p_k = 1$, so $T = K_q^k$.

(ii) Let v be an end-branch vertex of T, and H denote the unique component of $T - v$ with at least three vertices. Then it is easy to see that H contains at least an end-branch vertex of T. □

Theorem 2.7 *Let T be a weighted tree of order n with weights $w_1 \ge w_2 \ge \cdots \ge w_{n-1} > 0$ and edge independence number i. Then*

$$\rho(T) \le \rho(T_i(1, 2, \ldots, n-1)),$$

with equality if and only if $T = T_i(1, 2, \ldots, n-1)$.

Proof Let $V(T) = \{v_1, v_2, \ldots, v_n\}$. We distinguish the following two cases.

Case 1 Suppose that $S(T) = \emptyset$.

By Lemma 2.6(i), there are two non-negative integers s, t such that $T = K_t^s$, and $s = i-1$ or $s = i$. Let v be the center of K_t^s and $vv_ju_j (j = 1, 2, \ldots, s)$ be all pendent paths of length 2 at v.

Case 1.1 Suppose that $s = i - 1$.

It is obvious that T is a weighted tree with the form shown in Figure 1. Hence there exists a permutation $k_1k_2 \cdots k_{n-1}$ of $12 \cdots (n-1)$ such that $T = T_i(k_1, k_2, \ldots, k_{n-1})$. So from Lemma 2.5, we have $\rho(T) = \rho(T_i(k_1, k_2, \ldots, k_{n-1})) \le \rho(T_i(1, 2, \ldots, n-1))$, and with equality if and only if $T = T_i(1, 2, \ldots, n-1)$.

Case 1.2 Suppose that $s = i$.

It is obvious that $t = 0$. Let $T' = K_t^s - v_1u_1 + vu_1$. Then by Lemma 2.3, we have $\rho(T) < \rho(T')$. Since T' satisfies the condition of Case 1.1, by the result of Case 1.1, we have $\rho(T) < \rho(T') \leq \rho(T_i(1, 2, \ldots, n-1))$.

Case 2 Suppose that $S(T) \neq \emptyset$.

By Lemma 2.6(ii), we have $|S(T)| \geq 2$. Let $(x_{v_1}, x_{v_2}, \ldots, x_{v_n})^t$ be the Perron vector of $A(T)$, where x_{v_j} corresponds to the vertex v_j $(j = 1, 2, \ldots, n)$. Take $v, u \in S(T)$, and without loss of generality, assume $x_u \geq x_v$. Let $vv_1, vv_2, \ldots, vv_s$ be all pendent edges at v, and $vv_1'v_1'', vv_2'v_2'', \ldots, vv_t'v_t''$ be all pendent paths of length 2 at v.

If $s \neq 0$, then set

$$\begin{aligned} T^1 =& T - vv_1 - vv_2 - \cdots - vv_{s-1} - vv_1' - vv_2' - \cdots - vv_t' + \\ & uv_1 + uv_2 + \cdots + uv_{s-1} + uv_1' + uv_2' + \cdots + uv_t', \end{aligned}$$

where $w_{T^1}(uv_j) = w_T(vv_j)(j = 1, 2, \ldots, s-1)$ and $w_{T^1}(uv_j') = w_T(vv_j')(j = 1, 2, \ldots, t)$.

If $s = 0$, then set

$$T^1 = T - vv_1' - vv_2' - \cdots - vv_t' + uv_1' + uv_2' + \cdots + uv_t',$$

where $w_{T^1}(uv_j') = w_T(vv_j')(j = 1, 2, \ldots, t)$.

It is obvious that $in(T) \leq in(T^1)$, and from Lemma 1.1, we have $\rho(T) < \rho(T^1)$. If $S(T^1) \neq \emptyset$, then repeat the above steps to T^1 until we obtain a weighted tree T^r with $S(T^r) = \emptyset$. So we get trees $T, T^1, \ldots, T^r$ such that

$$\begin{aligned} i = in(T) \leq in(T^1) \leq \cdots \leq in(T^r), \\ \rho(T) < \rho(T^1) < \cdots < \rho(T^r). \end{aligned} \tag{2}$$

By Lemma 2.6(i), there are two non-negative integers s, t such that $T^r = K_t^s$. Let v be the center of K_t^s and vv_ju_j $(j = 1, 2, \ldots, s)$ be all pendent paths of length 2 at v. If $in(T^r) > i$, then set $T^* = K_t^s - v_su_s + vu_s$. It is obvious that $in(T^*) = in(T^r)$ if $t = 0$, $in(T^*) = in(T^r) - 1$ if $t \neq 0$, and by Lemma 2.3, we have $\rho(T^r) < \rho(T^*)$. If $in(T^*) > i$, then repeat the above steps to T^* until we obtain a weighted tree T^{**} with $in(T^{**}) = i$. So we get trees $T^r, T^*, \ldots, T^{**}$ such that

$$\rho(T^r) < \rho(T^*) < \cdots < \rho(T^{**}). \tag{3}$$

Since $S(T^{**}) = S(T^r) = \emptyset$, by Equations (2), (3) and the results of Case 1, we have

$$\rho(T) < \rho(T^r) \leq \rho(T^{**}) \leq \rho(T_i(1, 2, \ldots, n-1)).$$

By the above two cases, the proof is completed.

From Lemmas 2.3 and 2.5, when $i \geq 2$, there exists a permutation $k_1k_2 \cdots k_{n-1}$ of $12 \cdots (n-1)$ such that

$$\rho(T_i(1, 2, \ldots, n-1)) < \rho(T_{i-1}(k_1, k_2, \ldots, k_{n-1})) \leq \rho(T_{i-1}(1, 2, \ldots, n-1)).$$

On the other hand, $in(K_{1,n-1}) = 1$ and $in(T) \geq 2$ if $T \neq K_{1,n-1}$. So by Theorem 2.7, we immediately obtain the following two corollaries.

Corollary 2.8[2] *Let T be a weighted tree with order n and weights $w_1 \geq w_2 \geq \cdots \geq w_{n-1} > 0$. Then $\rho(T) \leq \rho(K_{1,n-1})$, with equality if and only if $T = K_{1,n-1}$.*

Corollary 2.9[3] *Let $T \neq K_{1,n-1}$ be a weighted tree with order n and weights $w_1 \geq w_2 \geq \cdots \geq w_{n-1} > 0$. Then $\rho(T) \leq \rho(T_2(1, 2, \ldots, n-1))$, with equality if and only if $T = T_2(1, 2, \ldots, n-1)$.*

A tree may be regarded as a weighted tree with weight 1 of each edge. Let $T_{n,i} = T_i(1, 2, \ldots, n-1)$ for $w_1 = w_2 = \cdots = w_{n-1} = 1$. So from Theorem 2.7, we have

Corollary 2.10[5] *Let T be a tree with order n and edge independence number i. Then $\rho(T) \leq \rho(T_{n,i})$, with equality if and only if $T = T_{n,i}$.*

References

[1] POLJAK S. *Minimum spectral radius of a weighted graph* [J]. Linear Algebra Appl., 1992, **171**: 53–63.
[2] YANG Huazhong, HU Guanzhang, HONG Yuan. *Bounds of spectral radii of weighted trees* [J]. Tsinghua Sci. Technol., 2003, **8**(5): 517–520.
[3] YUAN Jinsong, SHU Jinlong. *On the weighted trees which have the second largest spectral radius* [J]. OR Transactions, 2006, **10**(1): 81–87.
[4] CVETKOVIĆ D M, DOOB M, SACHS H. *Spectra of Graphs* [M]. New York, Academic Press, 1980.
[5] GUO Jiming, TAN Shangwang. *On the spectral radius of trees* [J]. Linear Algebra Appl., 2001, **329**(1-3): 1–8.