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The Representation of Group Inverse with Affine Combination

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Abstract In this paper, we first give two equalities in the operation of determinant. Using the expression of group inverse with full-rank factorization $A_g = F(GF)^{-2}G$ and the Cramer rule of the nonsingular linear system $Ax = b$, we present a new method to prove the representation of group inverse with affine combination

$$A_g = \sum_{(I,J)\in\mathcal{N}(A)} \frac{1}{\nu^2}\det(A)_{IJ}\widehat{\mathrm{adj}A_{JI}}.$$

A numerical example is given to demonstrate that the formula is efficient.

Keywords group inverses; cramer rule; affine combination.

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1. Introduction

The generalized inverse $X \in R^{n\times m}$ of a matrix $A \in R^{m\times n}$ was defined by Penrose in 1955, which is the unique solution of the following four matrix equations. These conditions are equivalent to Moore's conditions. The unique matrix X was known as Moore-Penrose inverse (abbreviated M-P) and is denoted by $A^{\dagger}$.

Definition *Let $A \in R^{m\times n}$ be given. The M-P inverse $A^{\dagger} \in R^{n\times m}$ is the unique solution of the following matrix equations*

$$AXA = A, \tag{1}$$

$$XAX = X, \tag{2}$$

$$(AX)^* = AX, \tag{3}$$

$$(XA)^* = XA, \tag{4}$$

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where $*$ denotes "conjugate transpose".

In general, let $\phi \neq \eta \subset \{1, 2, 3, 4\}$. If X satisfies all conditions of η, then X is called a η inverse of A, denoted by $A\{\eta\}$.

Also, in the case $m = n$, consider the following equations:

$$A^{k+1}X = A^k, \tag{1'}$$

$$AX = XA \tag{5}$$

for a positive integer $k = \text{ind}(A) = \min\{p : \text{rank}(A^{p+1}) = \text{rank}(A^p)\}$.

A matrix $X = A_d$ is said to be the Drazin inverse of A if $(1')$, (2) and (5) are satisfied. When $k = 1$, A_d is called the group inverse of A and denoted by A_g.

Throughout this paper, A is an $m \times n$ real matrix of rank r, denoted by $R_r^{m\times n}$. $\text{tr}(A)$ denotes the trace of a square matrix A. $|A|$ or $\det A$ denotes the determinant of A. $R(A)$ and $N(A)$ denote the range and null space of A, respectively.

2. Preliminaries

For $k = 1, 2, \ldots, r$, the k-th compound of A, denoted by $C_k(A)$, is the $\binom{m}{k} \times \binom{n}{k}$ matrix whose elements are the $k \times k$ minors of A, i.e., the determinants of its $k \times k$ submatrices ordered lexicographically. The $r \times r$ minors of A (i.e., the elements of $C_r(A)$) are called its maximum rank minors.

Let $Q_{k,m} = \{\alpha : \alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}, 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq m\}$ denote the strictly increasing sequence of k elements from $1, 2, \ldots, m$, $A \in R^{m\times n}$. For $\alpha \in Q_{k,n}$, $\beta \in Q_{k,m}$, $A[\alpha, \beta]$ denotes the submatrix of A with row indices α and column indices β, and $A[\alpha', \beta']$ denotes the submatrix obtained from A by deleting rows indexed α and columns indexed β. For simplicity, we write $A[I, J]$ as A_{IJ} for any index sets I and J, and let A_{I*} and A_{*J} denote the submatrices of A lying in rows indexed by I and in columns indexed by J, respectively. Denote

$$\begin{aligned}
\mathcal{I}(A) &= \{I \in Q_{r,m} : \text{rank}(A_{I*}) = r\}, \\
\mathcal{J}(A) &= \{J \in Q_{r,n} : \text{rank}(A_{*J}) = r\}, \\
\mathcal{N}(A) &= \{(I, J) \in Q_{rm} \times Q_{rn} : \text{rank}(A_{IJ}) = r\}.
\end{aligned}$$

By [3], we have $\mathcal{N}(A) = \mathcal{I}(A) \times \mathcal{J}(A)$.

Let $A(i \leftarrow b)$ denote the matrix obtained by replacing the ith column of A with vector b and $\text{adj}(A)$ denote the adjoint of A.

The volume of A, denoted by $\text{Vol}(A)$, is zero if $\text{rank}(A) = 0$, otherwise

$$\text{Vol}A = \sqrt{\sum_{(I,J)\in\mathcal{N}(A)} \det{}^2 A_{IJ}}.$$

The following two formulae will be used repeatedly in the following sections.

1) Laplace's formula

Let A be a square matrix with order n, and $\alpha, \beta \in Q_{l,n}$. Then

$$\sum_{\gamma \in Q_{l,n}} (-1)^{s(\alpha)+s(\gamma)} \det A[\alpha,\gamma] \det A[\beta',\gamma'] = \begin{cases} 0, & \alpha \neq \beta, \\ \det A, & \alpha = \beta. \end{cases} \tag{6}$$

2) Cauchy-Binet formula

Let $A \in C^{m\times p}$, $B \in C^{p\times n}$, $l \leq \min\{m,p,n\}$ and $\alpha \in Q_{l,m}, \beta \in Q_{l,n}$. Then

$$\det AB[\alpha,\beta] = \sum_{\gamma \in Q_{l,p}} \det A[\alpha,\gamma] \det B[\gamma,\beta]. \tag{7}$$

Berg[2] showed that the Moore-Penrose inverse $A^\dagger$ is a convex combination of ordinary inverse $\{A_{IJ}^{-1} : (I,J) \in \mathcal{N}(A)\}$

$$A^\dagger = \frac{1}{\mathrm{Vol}^2 A} \sum_{(I,J)\in\mathcal{N}(A)} |\det A_{IJ}|^2 \, \widehat{A_{IJ}^{-1}}, \tag{8}$$

where $\widehat{A_{IJ}^{-1}}$ is an $n \times m$ matrix with the inverse of the (I,J)th submatix of A in position (J,I) and zeros elsewhere.

In 2002, the second author first gave the combination representation of $A\{1,5\}$, then draw an affine combination representation of A_g in [4]. But in this paper we will use the full-rank factorization $A_g = F(GF)^{-2}G$ and the Cramer rule of the nonsingular linear system $Ax = b$ to give a new method to prove the affine combination representation of A_g, and point to that the representation in paper [4] is incomplete. As an application, we use an example to demonstrate that this representation is correct.

Lemma 2.1[4] *Let $A \in C^{n\times n}$ be of rank r. Then A has a group inverse if and only if*

$$\sum_I |A_{II}| \neq 0,$$

where I runs over all r-element subsets of $\{1,2,\ldots,n\}$.

Lemma 2.2 *Let $A \in C^{n\times n}$ be of* rank *r and $A = FG$ be the full-rank factorization. Then*

1) $A_g = F(GF)^{-2}G$;

2) $\nu = |GF| = \sum_I |A_{II}|$,

where I is the same as in Lemma 2.1.

Proof The first equality can be found in [1] and we omit its proof here.

2) By Cauchy-Binet formula

$$\begin{aligned} |GF| &= \sum_{I\in\mathcal{I}(F)} |G_{*I}||F_{I*}| = \sum_{I\in\mathcal{I}(F)} |F_{I*}||G_{*I}| \\ &= \sum_{I\in\mathcal{I}(F)} |(FG)_{II}| = \sum_I |A_{II}|. \end{aligned}$$

This implies $\nu = |GF| = \sum_I |A_{II}|$.

Lemma 2.3 *Let $A \in C^{n\times n}$ be of* rank *r with* ind$A = 1$. *Then the restricted linear system*

$$Ax = b, \; b \in R(A), \; x \in R(A) \tag{9}$$

has a unique $x = A_g b$.

Proof By the condition $\mathrm{ind}(A) = 1$, we have

$$R(A) \oplus N(A) = R^n \quad \text{and} \quad R(A^2) = R(A). \tag{10}$$

From which and $b \in R(A)$, we know that there exists a vector $y_0 \in R^n$, such that $b = A^2 y_0$. Denote $x_0 = Ay_0$. Then we have $x_0 \in R(A)$ and $b = Ax_0$. This implies that the restricted linear system (9) has a solution $x_0 \in R(A)$.

Let the general solution of (9) be $x = x_0 + y \in R(A)$, where $y \in N(A)$. Then $y = x - x_0 \in R(A)$. Since $R(A) \oplus N(A) = R^n$, we have $y = 0$.

Therefore, (9) has a unique solution $x = x_0 \in R(A)$.

3. Main results

Theorem 3.1 *Let both* A *and* C *be* $n \times n$ *nonsingular matrices, and* $b = (b_1, b_2, \ldots, b_n)^{\mathrm{T}}$ *be a column vector with order* n. *Let* $c = \mathrm{adj}(A)b$. *Then*

1) $A^* b = (| A(1 \leftarrow b) |, | A(2 \leftarrow b) |, \ldots, | A(n \leftarrow b) |)^{\mathrm{T}}$;

2) $| C(i \leftarrow c) | = | AC(i \leftarrow b) |$.

Proof By Laplace's formula, we have the following equality

$$\begin{aligned} A^* b &= A^* \left(b_1 \ b_2 \ \ldots \ b_n\right)^{\mathrm{T}} \\ &= \left(\sum_{i=1}^{n} A_{i1} b_1, \sum_{i=1}^{n} A_{i2} b_2, \ldots, \sum_{i=1}^{n} A_{in} b_n \right)^{\mathrm{T}} \\ &= \left(| A(1 \leftarrow b) |, | A(2 \leftarrow b) |, \ldots, | A(n \leftarrow b) |\right)^{\mathrm{T}}. \end{aligned}$$

This means that the first equality holds. Now let $A = (a_1, a_2, \ldots, a_n)$, $C = (c_1, c_2, \ldots, c_n)$, where a_i and c_i are the i-th column vectors of matrices A and C, respectively. Then we have

$$\begin{aligned} | AC(i \leftarrow b) | &= | (Ac_1, Ac_2, \ldots, Ac_{i-1}, b, Ac_{i+1}, \ldots, Ac_n) | \\ &= | A || \left(c_1, c_2, \ldots, c_{i-1}, A^{-1} b, c_{i+1}, \ldots, c_n\right) | \\ &= | \left(c_1, c_2, \ldots, c_{i-1}, | A | A^{-1} b, c_{i+1}, \ldots, c_n\right) | \\ &= | C(i \leftarrow A^* b) | = | C(i \leftarrow c) | . \end{aligned}$$

This shows the second equality holds. □

Theorem 3.2 *Let* $A \in C^{n \times n}$ *be of rank* r *with* $\mathrm{ind}A = 1$ *and let* $A = FG$ *be an arbitrary full-decomposition of* A. *Then*

$$A_g = \frac{1}{\nu^2} \sum_{(IJ) \in \mathcal{N}(\mathcal{A})} | A_{IJ} | \, \widehat{\mathrm{adj} A_{IJ}},$$

where $\nu = \sum_{I \in \mathcal{I}(A)} | A_{II} | = \det(GF)$ *and* I *is the same as in Lemma 2.1.*

Proof From Lemmas 2.2 and 2.3, we know that the unique solution of restricted linear equations

(9) is as follows

$$x = A_g b = F(GF)^{-2}Gb. \tag{11}$$

Hence the solution (11) of the restricted linear equations (9) splits up into

$$x = F(GF)^{-1}y, \quad y = (GF)^{-1}Gb. \tag{12}$$

From the expression $y = (GF)^{-1}Gb$, we find in view of $(GF)(i \leftarrow Gb) = G[F(i \leftarrow b)]$ and Cramer's rule

$$y_i = \frac{\mid (GF)[i \leftarrow Gb] \mid}{\mid GF \mid} = \frac{1}{\nu} \mid G(F(i \leftarrow b)) \mid . \tag{13}$$

According to Cauchy-Binet formula (7), we have

$$\begin{aligned} y_i &= \frac{1}{\nu} \mid G(F(i \leftarrow b)) \mid = \frac{1}{\nu} \sum_{I \in \mathcal{J}(A)} \mid G_{*I} \mid\mid (F(i \leftarrow b))_{I*} \mid \\ &= \frac{1}{\nu} \sum_{I \in \mathcal{J}(A)} \mid G_{*I} \mid\mid F_{I*}(i \leftarrow b_I) \mid \\ &= \frac{1}{\nu} \sum_{I \in \mathcal{J}(A)} \mid G_{*I} \mid\mid F_{I*} \mid \frac{\mid F_{I*}(i \leftarrow b_I) \mid}{\mid F_{I*} \mid}. \end{aligned}$$

From which we obtain

$$y = \frac{1}{\nu} \sum_{I \in \mathcal{J}(A)} \mid G_{*I} \mid\mid F_{I*} \mid \widehat{F_{I*}^{-1}} b. \tag{14}$$

In view of $x = F(GF)^{-1}y$, we find out the ith component $x_i = f_i(GF)^{-1}y$, where f_i denotes the vector of the ith row of F. Applying the identity of Magnus[8], we obtain

$$x_i = -\frac{1}{\mid GF \mid} \begin{vmatrix} 0 & f_i \\ y & GF \end{vmatrix} = -\frac{1}{\nu} \begin{vmatrix} 0 & f_i \\ y & GF \end{vmatrix}. \tag{15}$$

Using Cauchy-Binet formula (7) again changes the right side of formula (15) into

$$\begin{vmatrix} 0 & f_i \\ y & GF \end{vmatrix} = \begin{vmatrix} 0 & \delta_i \\ y & G \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 0 & F \end{vmatrix} = \sum_{J^0} \left| \begin{pmatrix} 0 & \delta_i \\ y & G \end{pmatrix}_{*J^0} \right| \left| \begin{pmatrix} 1 & 0 \\ 0 & F \end{pmatrix}_{J^0 *} \right| \tag{16}$$

where $J^0 = (j_0, j_1, j_2, \ldots, j_r), 0 \leq j_0 < j_1 < \cdots < j_r \leq m$, and δ_i denotes the ith row of the r-dimensional unit matrix. In this case $j_0 = 0$, we have

$$\det \begin{pmatrix} 1 & 0 \\ 0 & F \end{pmatrix}_{J^0 *} = \mid F_{J*} \mid . \tag{17}$$

Otherwise the determinant on the left-hand side vanishes. In the case $j_0 = 0$ and $i \in J^0$, we have

$$\det \begin{pmatrix} 0 & \delta_i \\ y & G \end{pmatrix}_{*J^0} = - \mid G_{*J}(i \leftarrow y) \mid, \tag{18}$$

whereas in the case $i \notin J^0$ the matrix on the left-hand side vanishes. Hence we obtain the result

$$x_i = \frac{1}{\nu} \sum_{J \in \mathcal{I}(A)} \mid G_{*J}(i \leftarrow y) \mid\mid F_{J*} \mid . \tag{19}$$

Substitute (14) into (19) and introduce the vector b^I with the components $\mid F_{I*}(i \leftarrow b_I) \mid$ for $i = 1, 2, \ldots, r$. It is obvious that $b^I = \mid F_{I*} \mid \operatorname{adj}(F_{I*})b$. We obtain

$$\begin{aligned} x_i &= \frac{1}{\nu^2} \sum_{I \in \mathcal{J}(A)} \sum_{J \in \mathcal{I}(A)} \mid G_{*I} \mid\mid G_{*J}(i \leftarrow b^I) \mid\mid F_{J*} \mid \\ &= \frac{1}{\nu^2} \sum_{(J,I) \in \mathcal{N}(A)} \mid A_{JI} \mid\mid G_{*J}(i \leftarrow b^I) \mid . \end{aligned}$$

By Theorem 3.1, $\mid G_{*J}(i \leftarrow b^I) \mid = \mid (FG)_{IJ}(i \leftarrow b_I) \mid = \mid A_{IJ}(i \leftarrow b_I) \mid$, then we have

$$x_i = \frac{1}{\nu^2} \sum_{(J,I) \in \mathcal{N}(A)} \mid A_{JI} \mid\mid A_{IJ}(i \leftarrow b_I) \mid \tag{20}$$

We devide the above equality into two sides:

Case 1 If $\det A_{IJ} \neq 0$, then (20) becomes

$$x_i = \frac{1}{\nu^2} \sum_{(J,I) \in \mathcal{N}(A)} \mid A_{JI} \mid\mid A_{IJ} \mid \frac{\mid A_{IJ}(i \leftarrow b_I) \mid}{\mid A_{IJ} \mid} \tag{21}$$

Case 2 If $\det A_{IJ} = 0$. Using the condition of Lemma 2.3, we know that $b_I \in R(A_{IJ})$, this implies that $\mid A_{IJ}(i \leftarrow b_I) \mid = 0$. Further, we can easy prove that $\operatorname{adj} A_{IJ} b_I = 0$, in this case we can rewrite the (20) into

$$x_i = \frac{1}{\nu^2} \sum_{(J,I) \in \mathcal{N}(A)} \mid A_{JI} \mid \operatorname{adj} A_{IJ} b_I \tag{22}$$

From (21) and (22), we know that

$$x = A_g b = \frac{1}{\nu^2} \sum_{(I,J) \in \mathcal{N}(A)} \mid A_{JI} \mid \widehat{\operatorname{adj} A_{IJ}} b \tag{23}$$

Because the equalities (9) and (23) hold for any $b \in R(A)$ and $x \in R(A)$, this implies

$$A_g = \frac{1}{\nu^2} \sum_{(I,J) \in \mathcal{N}(A)} \mid A_{IJ} \mid \widehat{\operatorname{adj} A_{JI}} \tag{24}$$

Remark When $\det A_{JI} \neq 0$, there is $\widehat{\operatorname{adj} A_{JI}} = \det A_{JI} \widehat{A_{JI}^{-1}}$, so the equality (24) becomes

$$A_g = \frac{1}{\nu^2} \sum_{(I,J) \in \mathcal{N}(A)} \det A_{IJ} \det(A)_{JI} \widehat{A_{JI}^{-1}}. \tag{25}$$

Then we have

$$\begin{aligned} \sum_{(I,J) \in \mathcal{N}(A)} \mid A_{IJ} \mid\mid A_{JI} \mid &= \sum_{(I,J) \in \mathcal{N}(A)} \mid A_{II} \mid\mid A_{JJ} \mid \\ &= \sum_{J \in \mathcal{J}(\mathcal{A})} (\sum_{I \in \mathcal{I}(\mathcal{A})} A_{II}) A_{JJ} = \nu^2 \end{aligned}$$

In this case, the representation is called affine combination of A_g, which could be found in paper [4]. But $\det A_{IJ} = 0$, the representation of A_g should be as (24), which can not be included in paper [4].

Theorem 3.2 *Under the condition of Lemma 2.3, the ith component of the unique solution $x = A_g b$ of (9) is given by*

$$x_i = \frac{1}{\nu^2} \sum_{(J,I)\in\mathcal{N}(A)} \mid A_{JI} \mid\mid A_{IJ}(i \leftarrow b_I) \mid .$$

In the next section, we will use a numerical example to show that the representation in Theorem 3.2 is correct.

4. Numerical example

Example Take

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 1 & 0 & 1 \end{pmatrix}.$$

By computing, we know $A \in R_2^{3\times 3}$ and

$$A^2 = \begin{pmatrix} 3 & 3 & 6 \\ 2 & 4 & 6 \\ 2 & 1 & 3 \end{pmatrix} \in R_2^{3\times 3}.$$

This shows A_g exists. We make a list of all minors with order two, which are not equal to zero.

I	J	A_{IJ}	$\det A_{IJ}$	A_{IJ}^{-1}	$\widehat{A_{IJ}^{-1}}$
$\{1,2\}$	$\{1,2\}$	$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$	2	$\frac{1}{2}\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\{1,3\}$	$\{1,3\}$	$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$	-1	$\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$
$\{2,3\}$	$\{2,3\}$	$\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$	2	$\frac{1}{2}\begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix}$
$\{1,2\}$	$\{1,3\}$	$\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$	2	$\frac{1}{2}\begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 2 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\{1,3\}$	$\{1,2\}$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	-1	$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$
$\{1,2\}$	$\{2,3\}$	$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$	-2	$-\frac{1}{2}\begin{pmatrix} 2 & -2 \\ -2 & 1 \end{pmatrix}$	$-\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 2 & -2 & 0 \\ -2 & 1 & 0 \end{pmatrix}$

Table 1 All minors with order two, which are not equal to zero

I	J	A_{IJ}	$\det A_{IJ}$	A_{IJ}^{-1}	$\widehat{A_{IJ}^{-1}}$
$\{2,3\}$	$\{1,2\}$	$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$	-2	$\frac{1}{2}\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\{1,3\}$	$\{2,3\}$	$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	1	$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}$
$\{2,3\}$	$\{1,3\}$	$\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$	-2	$\frac{1}{2}\begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Table 2 All minors with order two, which are not equal to zero

From the above table, we have

$$\nu = \sum_{I} \mid A_{II} \mid = 2 - 1 + 2 = 3.$$

Hence

$$A_g = \frac{1}{\nu^2} \sum_{(I,J)\in\mathcal{N}(A)} \mid A_{IJ} \mid\mid A_{JI} \mid \widehat{A_{IJ}^{-1}} = \frac{1}{9}\begin{pmatrix} 1 & 1 & 2 \\ -8 & 10 & 2 \\ 5 & -4 & 1 \end{pmatrix}.$$

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