Journal of Mathematical Research & Exposition
Mar., 2009, Vol. 29, No. 2, pp. 317–326
DOI:10.3770/j.issn:1000-341X.2009.02.014
Http://jmre.dlut.edu.cn

Graded and Nongraded Properties of Partial Tilting Modules and Tilting Modules

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Abstract This paper gives the relationships among partial tilting objects (tilting objects) of categories of graded left A-modules of type G, left A-modules, left A_e-modules and $A\sharp G$-modules, and then proves that for graded partial tilting modules, there exist the Bongartz complements in the category of graded A-modules.

Keywords tilting module; partial tilting module; graded module; smash product.

Document code A
MR(2000) Subject Classification 16G10; 16G99; 16W50
Chinese Library Classification O153.3

1. Introduction and preliminaries

In 1970's, arising from the quivers of hereditary algebras, Berstein, Gel'fand and Ponomarev introduced the notion of a reflection functor, and proved the BGP-theorem. Basing on it, they gave a new proof to the famous Gabriel theorem, turned on a new orientation of the development of the Morita theory, and brought a new research field—tilting theory[1]. Later, tilting theory was generalized well by Auslander, Platzek, Reiten, Bernner, Butler, Bongartz, et al. Eventually Happel and Ringel presented the notion of tilting modules with the form of axiom[2−5]. During the later twenty years, tilting theory developed well, which made algebraic representation theory progress quite well and promoted the developments of many research fields related with tilting theory[6,7]. Lately, Kleiner introduced a new grading on the preprojective algebra of a quiver[8], which provided us with a tool for our research. Combining with graded ring theory and smash product theory, this paper discusses the relationships among partial tilting objects (tilting objects) of categories of graded left A-modules, left A-modules, left A_e- modules and $A\sharp G$-modules, and then proves that for graded partial tilting modules, there exist the Bongartz complements in the category of graded A-modules.

Throughout this paper, let G be a multiplicative group with identity element e, k be an algebraically closed field, and A be a finite dimensional associative graded k-algebra of type G. The category A-gr consists of graded A-modules of type G and the morphisms are taken to be

Received date: 2007-01-20; **Accepted date**: 2008-07-06
Foundation item: the National Natural Science Foundation of China (Nos. 10371101; 10671161); the Natural Science Foundation of Fujian Province (Nos. Z0511022) and the Foundation of the Education Committee of Fujian Province (Nos. JA050206; JA06008; JB04215).

graded morphisms of degree e. We can define the restriction functor $(-)_e : A\text{-gr}\longrightarrow A_e\text{-Mod}$ by putting $(M)_e = M_e$ while a morphism $f : M \longrightarrow N$ in A-gr restricts to $(f)_e = f_e : (M)_e \longrightarrow (N)_e$. Consider $M, N \in A$-gr. Let $\mathrm{HOM}_A(M,N)_g$ be an additive subgroup of $\mathrm{Hom}_A(M,N)$ composed of graded morphisms of degree g and $\mathrm{HOM}_A(M,N) = \oplus_{g\in G}\mathrm{HOM}_A(M,N)_g$. The 1st derived functors of the functors $\mathrm{HOM}_A(-,-)_g$, $\mathrm{HOM}_A(-,-)$ and $\mathrm{Hom}_{A\text{-gr}}(-,-)$ are denoted by $\mathrm{EXT}_A(-,-)_g$, $\mathrm{EXT}_A(-,-)$ and $\mathrm{Ext}_{A\text{-gr}}(-,-)$ respectively. It is easy to show that $\mathrm{Ext}_{A\text{-gr}}(M,N) = \mathrm{EXT}_A(M,N)_e$ and $\mathrm{EXT}_A(M,N) = \oplus_{g\in G}\mathrm{EXT}_A(M,N)_g$ for $M, N \in A$-gr. Without special explanation, all our A-modules will be finitely generated left A-modules. Hence $\mathrm{HOM}_A(M,N) = \mathrm{Hom}_A(M,N)$ and for every $n \geq 0$ it follows that $\mathrm{EXT}^n_A(M,N) = \mathrm{Ext}^n_A(M,N)$, $\mathrm{HOM}_A(M,N) = \mathrm{HOM}_A(M(g), N(h))$ and $\mathrm{EXT}_A(M,N) = \mathrm{EXT}_A(M(g), N(h))$ where $g, h \in G$. We denote by $K_0(\mathrm{f.g}_{A\text{-gr}}\mathfrak{M})$ the K_0-group of $\mathrm{f.g}_{A\text{-gr}}\mathfrak{M}$ which is an additive full subcategory of A-gr composed of finitely generated graded modules. If the K_0-groups of rings (resp.exact categories) are free, then let $\mathrm{rank}K_0(-)$ be the rank of K_0-group.

For convenience, let us recall some notions and properties (the other related notions can be seen in [7, 9–11]).

Definition 1.1 *Let A be a G-graded ring. Define the smash product $A\sharp G$ as the free A-module $\oplus_{g\in G}AP_g$ with multiplication as follows:*

$$(aP_h)(bP_g) = ab_{hg^{-1}}P_g, \text{ for } g, h \in G \text{ and } a, b \in A.$$

It is clear that $A\sharp G$ is a ring.

Definition 1.2 *Let A be a G-graded ring and $M \in A$-gr. Define $M\sharp G$ as $\oplus_{g\in G}MP_g$ where P_g is just a symbol indexed by g and the elements of $\oplus_{g\in G}MP_g$ are the finite formal sums of m_gP_g for $m_g \in M$ and $g \in G$. $M\sharp G$ is an $A\sharp G$-module with its addition and $A\sharp G$-modular multiplication as follows:*

$$\Sigma_{g\in G}m_gP_g + \Sigma_{g\in G}m'_gP_g = \Sigma_{g\in G}(m_g + m'_g)P_g, \quad (a_xP_h)(m_yP_g) = \begin{cases} (a_xm_y)P_g, & h = yg, \\ 0, & h \neq yg, \end{cases}$$

where $a_x \in A_x$ and $m_y \in M_y$. We call $M\sharp G$ the smash product of M by G.

Proposition 1.3 *The functor $\sharp G$: A-gr$\longrightarrow A\sharp G$-Mod,*

$$\begin{aligned} M &\longmapsto M\sharp G \\ f : M \longrightarrow N &\longmapsto f\sharp G : M\sharp G \longrightarrow N\sharp G \\ m_xP_h &\longmapsto f(m_x)P_h, m_x \in M_x \end{aligned}$$

is a covariant exact functor and preserves the direct sum.

Proposition 1.4 *Let A be a G-graded ring. Then there is an equivalence of categories A-gr$\simeq A\sharp G$-Mod by the equivalent functors:*

$()^*$: A-gr$\longrightarrow A\sharp G$-Mod

$M = \oplus_{g\in G}M_g \longmapsto M^* = (M_e, M_g \cdots M_h \ldots)'$

and the invertible functor: $()_* : A\sharp G$-Mod$\longrightarrow A$-gr

$$N = (e_{(e,e)}N, e_{(h,h)}N, \ldots, e_{(g,g)}N, \ldots)' \longmapsto N_* = \oplus_{g\in G} e_{(g,g)}N,$$

where $e_{(g,g)} = 1_e P_g$.

2. Graded partial tilting modules and the Bongartz-complement

In this section, we introduce the notion of partial tilting objects and strongly partial tilting objects in A-gr, discuss whether the functor $(-)_e$ and $-\sharp G$ can preserve partial tilting objects, and then prove that for graded partial tilting modules, there exist the Bongartz-complements in the category of graded A-modules. Note that the projective dimension of M in A-gr will be denoted by $\text{gr.pd}_A M$.

Definition 2.1 *Assume $M \in A$-gr. Then M is called graded partial tilting module if it satisfies the following conditions:*

(1) $\text{gr.pd}_A M \leqslant 1$;

(2) $\text{Ext}_{A-\text{gr}}(M, M) = 0$.

Obviously, if a graded module is a partial tilting module, then it is certainly a graded partial tilting module. Conversely, it is always not true.

Example 2.2 Consider the path algebra $A = k\overrightarrow{Q_A}$, where $\overrightarrow{Q_A}$: $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. Assume $G = \mathbb{Z}_3$. Then A is turned to be a graded k-algebra of type G according to the lengths of paths (this graded structure for a path algebra is natural). It is easy to prove that $A\sharp G = k\overrightarrow{Q_{A\sharp G}}$ where $\overrightarrow{Q_{A\sharp G}}$ is given by:

$$\begin{array}{ccccc} 1 & \longrightarrow & 2 & \longrightarrow & 3 \\ 4 & \longrightarrow & 5 & \longrightarrow & 6 \\ 7 & \longrightarrow & 8 & \longrightarrow & 9 \end{array}.$$

Hence $Ae_3 = k\{e_3\}$, $Ae_2 = k\{e_2, \beta\}$ and $Ae_1 = k\{e_1, \alpha, \beta\alpha\}$ are A-graded modules respectively with graded structures as follows:

$$(Ae_3)_{\overline{i}} = \begin{cases} k\{e_3\}, & \overline{i} = \overline{0} \\ 0, & \overline{i} \neq \overline{0} \end{cases}, \quad (Ae_2)_{\overline{i}} = \begin{cases} k\{e_2\}, & \overline{i} = \overline{0} \\ k\{\beta\}, & \overline{i} = \overline{1} \\ 0, & \overline{i} = \overline{2} \end{cases},$$

$$(Ae_1)_{\overline{i}} = \begin{cases} k\{e_1\}, & \overline{i} = \overline{0} \\ k\{\alpha\}, & \overline{i} = \overline{1} \\ k\{\beta\alpha\}, & \overline{i} = \overline{2} \end{cases}.$$

Let $M \triangleq \tau^{-1}Ae_3$. Obviously, M is graded with $(\tau^{-1}Ae_3)_{\overline{i}} = \begin{cases} k\{\beta^*\}, & \overline{i} = \overline{0} \\ 0, & \overline{i} \neq \overline{0} \end{cases}$. In fact, owing to [8], if we consider the case when $a(\gamma) = b(\gamma) = 1$ for $\forall \gamma \in Q_1$, then the (a,b)-preprojective algebra of the quiver $\overrightarrow{Q_A}$ is turned out to be $A' = (k\overrightarrow{Q'})/I$ where $\overrightarrow{Q'}$ is given by:

$$1 \underset{\alpha^*}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\beta^*}{\overset{\beta}{\rightleftarrows}} 3$$

and the two-sided ideal $I = \langle\{\alpha^*\alpha, \beta^*\beta - \alpha\alpha^*, -\beta\beta^*\}\rangle$. It follows that $M \triangleq \tau^{-1}Ae_3 = V_1^{e_3} = k\{\beta^*\}$, which can be a G-graded A-module with $M_{\overline{i}} = \begin{cases} k\{\beta^*\}, & \overline{i} = \overline{0} \\ 0, & \overline{i} \neq \overline{0} \end{cases}$.

We claim that graded module $M \oplus Ae_3$ is a graded partial tilting module but not partial tilting module. Actually, $A\sharp G$-module $MP_{\overline{0}}$ and $Ae_3P_{\overline{0}}$ can be represented by their dimension vectors as follows:

$$\begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ 0 & \longrightarrow & 1 & \longrightarrow & 0 \end{array}, \quad \begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ 0 & \longrightarrow & 0 & \longrightarrow & 1 \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 \end{array}.$$

And it is easy to check that $B \triangleq MP_{\overline{0}} \oplus Ae_3P_{\overline{0}}$ is a partial tilting $A\sharp G$-module. Applying the Proposition 1.4, we prove that $B_* = M \oplus Ae_3$ is a graded partial tilting module where M and Ae_3 are graded modules. However, since $\mathrm{Ext}_A(M, Ae_3) \neq 0$, we have $\mathrm{Ext}_A(B_*, B_*) \neq 0$. Hence A-module B_* is not a partial tilting module. □

From [9], we know that for $\forall g \in G$ the translational functor $T_g : A\text{-gr} \longrightarrow A\text{-gr}$ is an equivalent functor and if the ring A is a strongly graded ring of type G, then $()_e : A\text{-gr} \leftrightarrows A_e\text{-Mod}: A \otimes_{A_e}$-are mutually equivalent functors. We get

Proposition 2.3 *Let A be a strongly graded ring of type G and $M = \oplus_{g \in G} M_g$ be a graded A-module. Then the followings are equivalent:*

(1) M is a graded partial tilting A-module;

(2) M_e is a partial tilting A_e-module;

(3) There is $g \in G$ such that M_g is a partial tilting A_e-module;

(4) M_g is a partial tilting A_e-module for $\forall g \in G$.

Because of the difference between the derived functor $\mathrm{EXT}_A(M, N)$ and $\mathrm{Ext}_{A\text{-gr}}(M, N)$, we introduce the notion of graded strongly partial tilting module, that is,

Definition 2.4 *$M \in A$-gr is called graded strongly partial tilting module if it satisfies the following conditons*

(1) $\mathrm{gr.pd}_A M \leqslant 1$;

(2) $\mathrm{EXT}_A(M, M) = 0$.

Clearly, graded module M is a graded strongly partial tilting module if and only if M is a partial tilting module. In addition, graded strongly partial tilting modules are graded partial tilting modules. Conversely, it is not always true, as shown in Example 2.2.

Proposition 2.5 *Assume the multiplicative group G is finite. Then M is a graded strongly partial tilting module if and only if $M\sharp G$ is a partial tilting $A\sharp G$-module.*

Proof Since $()^* : A\text{-gr} \longrightarrow A\sharp G\text{-Mod}$ is an equivalent functor and $M\sharp G \cong (\oplus_{g \in G}(M(g)))^*$, we have

$$\begin{aligned}\mathrm{pd}_{_{A\sharp G}}M\sharp G &= \mathrm{pd}_{_{A\sharp G}}(\oplus_{g\in G}(M(g)))^* = \mathrm{gr.pd}_A(\oplus_{g\in G}(M(g)))\\ &= \mathrm{pd}_A(\oplus_{g\in G}(M(g))) = \sup_{g\in G}\{\mathrm{pd}_A M(g)\} = \mathrm{pd}_A M = \mathrm{gr.pd}_A M.\end{aligned}$$

So $\mathrm{gr.pd}_A M \leqslant 1 \Longleftrightarrow \mathrm{pd}_{_{A\sharp G}}M\sharp G \leqslant 1$.

Moreover, it is easy to show that

$$\begin{aligned}\mathrm{Ext}_{A\sharp G}(M\sharp G, M\sharp G) &= \mathrm{Ext}_{A\sharp G}((\oplus_{g\in G}(M(g)))^*, (\oplus_{g\in G}(M(g)))^*)\\ &= \mathrm{Ext}_{A\text{-gr}}(\oplus_{g\in G}(M(g)), \oplus_{g\in G}(M(g)))\\ &= \oplus_{g\in G}\oplus_{h\in G}\mathrm{Ext}_{A\text{-gr}}(M(g), M(h)) = \oplus_{g\in G}\oplus_{h\in G}\mathrm{Ext}_{A\text{-gr}}(M, M(g^{-1}h))\\ &= \oplus_{g\in G}(\oplus_{h\in G}(\mathrm{EXT}_A(M,M))_{g^{-1}h}) = \oplus_{g\in G}\mathrm{EXT}_A(M,M).\end{aligned}$$

Thus $\mathrm{Ext}_{A\sharp G}(M\sharp G, M\sharp G) = 0 \Longleftrightarrow \mathrm{EXT}_A(M,M) = 0$. □

Next let us discuss the existence of the Bongartz-complement in A-gr. For convenience, we introduce the notion of the graded tilting module firstly.

Definition 2.6 *$M \in A$-gr is called graded tilting module if it satisfies the following conditions:*

(T_1) $\mathrm{gr.pd}_A M \leqslant 1$;

(T_2) $\mathrm{Ext}_{A\text{-gr}}(M,M) = 0$;

(T_3) There exists a short exact sequence $0 \longrightarrow A \longrightarrow M' \longrightarrow M'' \longrightarrow 0$ with M' and M'' direct sums of direct summands of M in A-gr, that is, M' and $M'' \in \mathrm{add}_{A\text{-gr}}(M)$.

Remark 2.7 It is well known that one of conditions in the definition of the tilting module, similar to (T_3) above, can be replaced by (T_3)': the number of mutually non-isomorphic indecomposable summands of M equals to $\mathrm{rank}K_0(A)$. But this case is not true for graded tilting modules.

Example 2.8 For Example 2.2, it is clear that $A = \oplus_{i=1}^3 Ae_i$ is a graded tilting module and Ae_1, Ae_2 and Ae_3 are mutually non-isomorphic and indecomposable in A-gr. But the number of mutually non-isomorphic indecomposable summands of A 3 does not equal to $\mathrm{rank}K_0(\mathrm{f.g}_{A\text{-gr}}\mathfrak{M})$. In fact, $\mathrm{rank}K_0(\mathrm{f.g}_{A\text{-gr}}\mathfrak{M}) = \mathrm{rank}K_0(A\sharp G) = |G|\mathrm{rank}K_0(A) = 9 \neq 3$. □

Now we show the existence of Bongartz-complements of graded partial tilting modules in categories of graded modules.

Theorem 2.9 *Let T be a graded partial tilting A-module. Then there exists a graded A-module X such that $T \oplus X$ is a graded tilting A-module.*

Proof It is not hard to prove that for $\forall A', B \in A$-gr it follows that $e_{A\text{-gr}}(B, A') \cong \mathrm{Ext}_{A\text{-gr}}(B, A')$ where $e_{A\text{-gr}}(B, A') = \{$ short exact sequences $0 \longrightarrow A' \longrightarrow C \longrightarrow B \longrightarrow 0$ in A-gr $|A', B, C \in A$-gr$\}$.

Let $e_1, e_2, \ldots, e_d$ be a basis of the k-vector space $\mathrm{Ext}_{A\text{-gr}}(T, A)$, and consider the exact sequence

$$0 \longrightarrow A \xrightarrow{g_1} X \longrightarrow T^{(d)} \longrightarrow 0 - - - (*)$$

defined as the push out in A-gr along the graded morphism $f: A^{(d)} \longrightarrow A$:$(a_1, a_2, \ldots, a_d) \longmapsto \Sigma_{i=1}^{d} a_i$ of the exact sequence $\oplus_{i=1}^{d} e_i$, that is,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^{(d)} & \xrightarrow{f_1} & \oplus_{i=1}^{d} X_i & \longrightarrow & T^{(d)} & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{g_1} & X & \longrightarrow & T^{(d)} & \longrightarrow & 0 \end{array} .$$

Since $\mathrm{pd}_A T = \mathrm{gr.pd}_A T \leqslant 1$, $\mathrm{pd}_A A = 0$ and $(*)$ is exact in the A-modules category, we have $\mathrm{pd}_A X \leqslant 1$. Hence $\mathrm{gr.pd}_A(T \oplus X) = \mathrm{pd}_A(T \oplus X) \leqslant 1$.

Applying the functor $\mathrm{Hom}_{A\text{-gr}}(T, -)$ to $(*)$ gives a long exact sequence:

$$\mathrm{Hom}_{A\text{-gr}}(T, T^{(d)}) \xrightarrow{\varphi} \mathrm{Ext}_{A\text{-gr}}(T, A) \longrightarrow \mathrm{Ext}_{A\text{-gr}}(T, X) \longrightarrow \mathrm{Ext}_{A\text{-gr}}(T, T^{(d)}) = 0.$$

By construction, φ is surjective. Hence $\mathrm{Ext}_{A\text{-gr}}(T, X) = 0$.

Moreover, applying $\mathrm{Hom}_{A\text{-gr}}(-, T)$ to $(*)$ yields:

$$0 = \mathrm{Ext}_{A\text{-gr}}(T^{(d)}, T) \longrightarrow \mathrm{Ext}_{A\text{-gr}}(X, T) \longrightarrow \mathrm{Ext}_{A\text{-gr}}(A, T) = (\mathrm{EXT}_A(A, T))_e = 0.$$

Thus $\mathrm{Ext}_{A\text{-gr}}(X, T) = 0$.

And applying $\mathrm{Hom}_{A\text{-gr}}(-, X)$ to $(*)$ yields:

$$0 = \mathrm{Ext}_{A\text{-gr}}(T^{(d)}, X) \longrightarrow \mathrm{Ext}_{A\text{-gr}}(X, X) \longrightarrow \mathrm{Ext}_{A\text{-gr}}(A, X) = (\mathrm{EXT}_A(A, X))_e = 0.$$

It follows that $\mathrm{Ext}_{A\text{-gr}}(X, X) = 0$.

Hence $\mathrm{Ext}_{A\text{-gr}}(T \oplus X, T \oplus X) = 0$. Since $(*)$ is the sequence of (T3), $T \oplus X$ is indeed a graded tilting module. □

3. Tilting modules $M\sharp G$

Assume G is a finite multiplicative group. And in this section we mainly talk about how the functor-$\sharp G$ can preserve tilting modules. Note that according to the $A\sharp G$-modular multiplication of $M\sharp G$, MP_g is an $A\sharp G$-submodule of $M\sharp G$ for$\forall g \in G$. So we have

Theorem 3.1 *If the partial tilting graded A-module M in A-gr has an indecomposable decomposition $M = \oplus_{i=1}^{n} T_i$, where $n = \mathrm{rank} K_0(A)$, T_i are mutually non-isomorphic and $T_i \not\cong T_j(g)$ ($\forall g \in G$, $g \neq e; i, j = 1, \ldots, n$) in A-gr, then the $A\sharp G$-module $M\sharp G$ is the tilting module.*

Proof Since $()^* : A\text{-gr} \longrightarrow A\sharp G\text{-Mod}$ is an equivalent functor and $M\sharp G \cong (\oplus_{g\in G}(M(g)))^*$, we have

$$\begin{aligned} \mathrm{pd}_{A\sharp G} M\sharp G &= \mathrm{pd}_{A\sharp G}(\oplus_{g\in G}(M(g)))^* = \mathrm{gr.pd}_A(\oplus_{g\in G}(M(g))) \\ &= \mathrm{pd}_A(\oplus_{g\in G}(M(g))) = \sup_{g\in G}\{\mathrm{pd}_A M(g)\} = \mathrm{pd}_A M \leqslant 1 \end{aligned}$$

and

$$\begin{aligned} \mathrm{Ext}_{A\sharp G}(M\sharp G, M\sharp G) &= \mathrm{Ext}_{A\sharp G}((\oplus_{g\in G}(M(g)))^*, (\oplus_{g\in G}(M(g)))^*) \\ &= \mathrm{Ext}_{A-\mathrm{gr}}(\oplus_{g\in G}(M(g)), \oplus_{g\in G}(M(g))) \end{aligned}$$

$$= (\mathrm{EXT}_A((\oplus_{g\in G}(M(g)), (\oplus_{g\in G}(M(g)))))_e.$$

It is easy to show that

$$\begin{aligned}\mathrm{EXT}_A(\oplus_{g\in G}M(g), \oplus_{g\in G}M(g)) &= \mathrm{Ext}_A(\oplus_{g\in G}M(g), \oplus_{g\in G}M(g))\\ &= \oplus_{g\in G}\oplus_{g\in G}\mathrm{Ext}_A(M(g), M(h))\\ &= \oplus_{g\in G}\oplus_{g\in G}\mathrm{Ext}_A(M, M) = 0.\end{aligned}$$

Thus $\mathrm{Ext}_{A\sharp G}(M\sharp G, M\sharp G) = (\mathrm{EXT}_A(\oplus_{g\in G}M(g),\ \oplus_{g\in G}M(g)))_e = 0$.

By conditions, there exists an $A\sharp G$-modular decomposition

$$M\sharp G = (\oplus_{i=1}^n T_i)\sharp G = \oplus_{i=1}^n (T_i\sharp G) = \oplus_{i=1}^n \oplus_{g\in G} T_i P_g.$$

And we claim that T_iP_g are indecomposable $A\sharp G$-modules and mutually non-isomorphic. In fact, assume T_iP_g are decomposable, i.e., $T_iP_g = D\oplus B$. Since $D\oplus B = T_iP_g \cong (T_i(g^{-1}))^*$ and $()_*$: $A\sharp G$-Mod$\longrightarrow$ A-gr is equivalent, we have $T_i(g^{-1}) \cong D_*\oplus B_*$, that is, T_i are decomposable, a contradiction.

Next let us divide into three steps to prove that $T_iP_g(\forall g\in G, i=1,\dots,n)$ are mutually non-isomorphic. If $T_iP_g \cong T_jP_g$ where $i\neq j$, then $(T_i(g^{-1}))^* \cong (T_j(g^{-1}))^*$. Thus $T_i(g^{-1}) \cong T_j(g^{-1})$. Hence $T_i \cong T_j$, a contradiction; if $T_iP_g \cong T_iP_h$ where $g\neq h$, then it is similar to prove that $T_i(g^{-1})) \cong T_i(h^{-1})$. And since the translational functors are equivalent functors, $T_i \cong T_i(gh^{-1})$, a contradiction; if $T_iP_g \cong T_jP_h$ with $g\neq h$ and $i\neq j$, then from $(T_i(g^{-1}))^* \cong T_iP_g \cong T_jP_h \cong (T_j(h^{-1}))^*$ it follows that there exists a graded isomorphism $T_i \cong T_j(gh^{-1})$, a contradiction.

To sum up the above-mentioned, the theorem is proved. □

Corollary 3.2 *If graded A-module M is a tilting module and in A-gr has an indecomposable decomposition $M = \overset{n}{\oplus}_{i=1} T_i$ where $T_i \ncong T_i(g)$ ($\forall g\in G$ and $g\neq e$), then $A\sharp G$-module $M\sharp G$ is a tilting module.*

Example 3.3 For Example 2.2, it is obvious that BB(APR)-tilting module $T = Ae_1\oplus Ae_2\oplus \tau^{-1}Ae_3$ satisfies the conditions of the Theorem 3.1. Hence $T\sharp G$ is a tilting $A\sharp G$-module.

In fact, for $\forall \bar{i}\in\mathbb{Z}_3$, it follows that $Ae_1P_{\bar{i}} = (A\sharp G)e_1P_{\bar{i}}, Ae_2P_{\bar{i}} = (A\sharp G)e_2P_{\bar{i}}$. Clearly, $A\sharp G$-module $MP_{\bar{0}}, MP_{\bar{1}}$ and $MP_{\bar{2}}$ can be represented respectively by the dimension vectors as follows:

$$\begin{matrix}0 & \longrightarrow & 0 & \longrightarrow & 0\\ 0 & \longrightarrow & 0 & \longrightarrow & 0\\ 0 & \longrightarrow & 1 & \longrightarrow & 0\end{matrix}\ ,\quad \begin{matrix}0 & \longrightarrow & 1 & \longrightarrow & 0\\ 0 & \longrightarrow & 0 & \longrightarrow & 0\\ 0 & \longrightarrow & 0 & \longrightarrow & 0\end{matrix}\ ,\quad \begin{matrix}0 & \longrightarrow & 0 & \longrightarrow & 0\\ 0 & \longrightarrow & 1 & \longrightarrow & 0\\ 0 & \longrightarrow & 0 & \longrightarrow & 0\end{matrix}\ .$$

Then according to the definition of the tilting module, it is proved directly that $T\sharp G$ is an $A\sharp G$-tilting module. On the other hand, although T is a BB(APR)-tilting module, $T\sharp G$ is not so. Thus $-\sharp G$ does not preserve the BB(APR)-tilting module. □

Definition 3.4 *$M\in A$-gr is called graded strongly tilting module if it satisfies the following conditions:*

(1) $\mathrm{gr.pd}_A M \leqslant 1$;

(2) $\mathrm{EXT}_A(M, M) = 0$;

(3) There exists a short exact sequence $0 \longrightarrow A \longrightarrow M' \longrightarrow M'' \longrightarrow 0$ *with* M' *and* M'' *direct sums of direct summands of* M *in A-gr, that is,* M' *and* $M'' \in \mathrm{add}_{A\text{-gr}}(M)$.

Obviously, graded strongly tilting modules are graded tilting modules. Conversely, it is not always true.

Example 3.5 For example 2.2, it follows that $Ae_1P_{\bar{i}} = (A\sharp G)e_1P_{\bar{i}}, Ae_2P_{\bar{i}} = (A\sharp G)e_2P_{\bar{i}}$ and $Ae_3P_{\bar{i}} = (A\sharp G)e_3P_{\bar{i}}$ are all indecomposable $A\sharp G$-projective modules for $\forall \bar{i} \in \mathbb{Z}_3$. It is easy to prove that $A\sharp G$-module

$$C \triangleq MP_{\bar{0}} \oplus Ae_2P_{\bar{0}} \oplus Ae_1P_{\bar{2}} \oplus Ae_1P_{\bar{0}} \oplus Ae_2P_{\bar{1}} \oplus Ae_3P_{\bar{2}} \oplus Ae_1P_{\bar{1}} \oplus Ae_2P_{\bar{2}} \oplus Ae_3P_{\bar{0}}$$

is an $A\sharp G$-tilting module. Applying the Proposition 1.4 yields that C_* is a graded tilting module. But since $\mathrm{Ext}_A(M, Ae_3) \neq 0$, $\mathrm{EXT}_A(C_*, C_*) = \mathrm{Ext}_A(C_*, C_*) \neq 0$. Hence C_* is a graded tilting module but not graded strongly tilting module. □

Applying the Proposition 2.5 and the exactness of the functor $-\sharp G$, we have

Proposition 3.6 *If* M *is a graded strongly tilting A-module, then* $M\sharp G$ *is an* $A\sharp G$*-tilting module.*

4. A_e-tilting modules

In this section, we discuss the relationships among tilting objects (partial tilting objects) of categories A-gr, A-Mod and A_e-Mod.

From [9], we know that for $\forall g \in G$ the translational functor $T_g : A\text{-gr} \longrightarrow A\text{-gr}$ is an equivalent functor and if the ring A is a strongly graded ring of type G, then $()_e : A\text{-gr} \leftrightharpoons A_e\text{-Mod} : A\otimes_{A_e}$-are mutually equivalent functors. We have

Proposition 4.1 *Let* A *be a strongly graded ring of type* G *and* $M = \oplus_{g\in G} M_g$ *be a graded A-module. Then the followings are equivalent:*

(1) M *is a graded tilting A-module;*

(2) M_e *is a tilting* A_e*-module;*

(3) There exists $g \in G$ *such that* M_g *is a tilting* A_e*-module;*

(4) M_g *is a tilting* A_e*-module for* $\forall g \in G$.

Theorem 4.2 *Let* A *be a strongly graded ring of type* G*. If the partial tilting graded A-module* M *in A-gr has an indecomposable decomposition* $M = \oplus_{i=1}^n N_i^{k_i}$ *where* $n = \mathrm{rank} K_0(A_e)$, $k_i \in \mathbb{Z}^+$ *and* N_i *are mutually non-isomorphic, then* M_g *is a tilting* A_e*-module for* $\forall g \in G$ *(*M_g *is g-component of* M*).*

Proof Since A is a strongly graded ring of type G, then $()_e : A\text{-gr} \leftrightharpoons A_e\text{-Mod} : A\otimes_{A_e}$-are mutually equivalent functors. Thus $\mathrm{pd} M_g \leqslant 1$. Observe that $\mathrm{Ext}_{A_e}(M_g, M_g) \cong \mathrm{Ext}_{A_e}((M(g))_e, (M(g))_e)$ $\cong \mathrm{Ext}_{A\text{-gr}}(M(g), M(g)) = (\mathrm{EXT}_A(M(g), M(g)))_e = 0$.

Next, we claim that N_i is a strongly graded module. Since A is a strongly graded ring of type G, we have $(N_i)_g \neq 0$. In addition, applying the functor $(T_g(-))_e$ to $M = \oplus_{i=1}^n N_i^{k_i}$ yields $(M(g))_e = \oplus_{i=1}^n (N_i(g))_e^{k_i}$, that is, $M_g = \oplus_{i=1}^n (N_i)_g^{k_i}$. Since N_i is indecomposable in A-gr, $(N_i)_g$ is also indecomposable in A-gr. (If not, then there is a decomposition of A_e-modules $(N_i)_g = B \oplus D$, which leads to that $N_i(g) \cong A \otimes_{A_e} (N_i)_g = A \otimes_{A_e} B \oplus A \otimes_{A_e} D$, that is, N_i is decomposable in A-gr. It yields the contradiction). Then since $N_i \ncong N_j (i \neq j)$, it is easy to prove that $(N_i)_g \ncong (N_j)_g$. So the number of mutually non-isomorphic indecomposable summands of M_g equals to rank$K_0(A_e)$. □

Remark 4.3 The condition that A is a strongly graded ring of type G in Theorem 4.2 cannot be lost but is not necessary.

Example 4.4 For example 2.2, A is not a strongly graded ring of type G and $M \triangleq \tau^{-1}Ae_3 = V_1^{e_3} = k\{\beta^*\}$ is a graded A-module with

$$(\tau^{-1}Ae_3)_{\bar{i}} = \begin{cases} k\{\beta^*\}, & \bar{i} = \bar{1} \\ 0, & \bar{i} \neq \bar{1} \end{cases}.$$

Then it is easy to show that $T = Ae_1 \oplus Ae_2 \oplus M$ is a graded A-module and tilting module and $T_e = k\{e_1\} \oplus k\{e_2\} \oplus 0$. But the number of mutually non-isomorphic indecomposable summands of T_e does not equal to rank$K_0(A_e) = 3$. Hence it is not a tilting A_e-module. □

Example 4.5 Consider the path algebra $A = k\overrightarrow{Q_A}$, where $\overrightarrow{Q_A}$: $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. It is seen from [8] that if we consider the case when $a(\gamma) = b(\gamma) = 1$ for $\forall \gamma \in Q_1$, then the (a,b)-preprojective algebra of the quiver $\overrightarrow{Q_A}$ is turned out to be $A' = (k\overrightarrow{Q'})/I$, where $\overrightarrow{Q'}$ is given by:

$$1 \underset{\alpha^*}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\beta^*}{\overset{\beta}{\rightleftarrows}} 3$$

and the two-sided ideal $I = \langle \{\alpha^*\alpha, \beta^*\beta - \alpha\alpha^*, -\beta\beta^*\} \rangle$. Then the preprojective algebra is turned into a graded k-algebra of type $\mathbb{Z}_3$ by the degrees of pathes. Obviously, it is not strongly graded.

We can prove that $A'e_1 = k\{e_1, \alpha, \beta\alpha\}$, $A'e_2 = k\{e_2, \beta, \beta^*\beta, \alpha^*\}$ and $A'e_3 = k\{e_3, \beta^*, \alpha^*\beta^*\}$ are graded A'-modules respectively with their graded structures as follows:

$$(A'e_1)_{\bar{i}} = \begin{cases} k\{e_1, \alpha, \beta\alpha\}, & \bar{i} = \bar{0} \\ 0, & \bar{i} \neq \bar{0} \end{cases}, \quad (A'e_2)_{\bar{i}} = \begin{cases} k\{e_2, \beta\}, & \bar{i} = \bar{0} \\ k\{\beta^*\beta, \alpha^*\}, & \bar{i} = \bar{1} \\ 0, & \bar{i} = \bar{2} \end{cases},$$

$$(A'e_3)_{\bar{i}} = \begin{cases} k\{e_3\}, & \bar{i} = \bar{0} \\ k\{\ \beta^*\}, & \bar{i} = \bar{1} \\ k\{\ \alpha^*\beta^*\}, & \bar{i} = \bar{2} \end{cases}.$$

It is clear that $A' = (k\overrightarrow{Q'_A})/I = A'e_1 \oplus A'e_2 \oplus A'e_3$ is a tilting A'-module and $(A')_{\bar{0}} = (A'e_1)_{\bar{0}} \oplus (A'e_2)_{\bar{0}} \oplus (A'e_3)_{\bar{0}} = k\{e_1, \alpha, \beta\alpha\} \oplus k\{e_2, \beta\} \oplus k\{e_3\} = Ae_1 \oplus Ae_2 \oplus Ae_3$ is also a tilting $A'_{\bar{0}} = A$-module.

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