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The Crossing Numbers of Cartesian Products of Stars and 5-Vertex Graphs

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Abstract In this paper, the crossing numbers of the Cartesian products of a specific 5-vertex graph with a star are given, and thus the result fills up the crossing number list of Cartesian products of all 5-vertex graphs with stars (presented by Marián Klešč). In addition, we also give an up to date description of Cartesian products of 5-vertex graphs with stars, whose crossing numbers are known.

Keywords graph; drawing; crossing number; star; cartesian product.

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1. Introduction

For graph theory terminology not defined here we refer to [1]. A drawing of an (undirected) graph $G=(V,E)$ is a mapping f that assigns to each vertex in V a distinct point in the plane and to each edge uv in E a continuous arc (i.e., a homeomorphic image of a closed interval) connecting $f(u)$ and $f(v)$, not passing through the image of any other vertex. As for the drawing we need the additional assumptions: (1) No three edges have an interior point in common; (2) If two edges share an interior point p, then they cross at p; (3) Any two edges of a drawing have only a finite number of crossings (common interior points).

The crossing number $\mathrm{cr}(G)$ of a graph G is the minimum number of edge crossings in any drawing of G in the plane.

It is well known that the crossing number of a graph is attained only in these drawing where no edge crosses itself and no two edges cross more than once. Let ϕ be a drawing of graph G. We denote the number of crossings in ϕ by $\mathrm{cr}_\phi(G)$. For more on the theory of crossing numbers, we refer the reader to [2].

Computing the crossing number of graphs is a classical problem[2,3], and yet it is also an elusive one. In fact, Garey and Johnson[4] proved that in general the problem of determining

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the crossing number of a graph is NP-complete (the reader can also refer to two recent results on complexity of the crossing number graphs in [5] and [6], respectively).

The Cartesian product $G_1 \times G_2$ of graphs G_1 and G_2 has vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge set

$$E(G_1 \times G_2) = \{(u_i, v_j)(u_h, v_k) | u_i = u_h \text{ and } u_j v_k \in E(G_2) \text{ or } v_j = v_k \text{ and } u_i u_h \in E(G_1)\}.$$

At present, only few families of graphs with arbitrarily large crossing number for the plane are known. Most of them are Cartesian products of special graphs. Let C_n and P_n be the cycle and the path with n edges, and S_n the star $K_{1,n}$. The crossing numbers of the Cartesian products of all 4-vertex graphs with cycles are determined in [7] and [8] and with paths and stars in [9] and [10]. There are several known exact results on the crossing numbers of Cartesian products of paths, cycles and stars with 5-vertex graphs (in [11], Marian Klešč gave a description of Cartesian products of all 5-vertex graphs with paths, cycles and stars by a table, whose crossing numbers are known). In Section 3, the table with all the 21 connected graphs G of order five together with the up to date results of crossing numbers of $G_i \times S_n$ is given. To fill up a blank of the table we prove in Section 2 that $\mathrm{cr}(G_{12} \times S_n) = n(n-1)$.

2. Crossing number of $G_{12} \times S_n$

Firstly, let us denote by H_n the graph obtained by adding six edges to the graph $K_{5,n}$ (containing n vertices of degree 5 and one vertex of degree $n+1$, one vertex of degree $n+2$, three vertices of degree $n+3$, and $5n+6$ edges (see Figure 2)). Consider now the graph G_{12} in Figure 1. It is easy to see that $H_n = G_{12} \cup K_{5,n}$, where the five vertices of degree n in $K_{5,n}$ and the vertices of G_{12} are the same. Let, for $i = 1, 2, \ldots, n$, T^i denote the subgraph of $K_{5,n}$ which consists of the five edges incident with a vertex of degree five in $K_{5,n}$. Thus, we have

$$H_n = G_{12} \cup K_{5,n} = G_{12} \cup (\bigcup_{i=1}^{n} T^i). \tag{1}$$

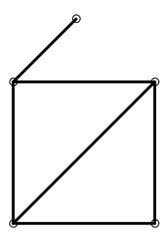

Figure 1 G_{12}

We now explain some notations. Let A and B be two sets of edges of a graph G. We use the sign $\mathrm{cr}_\phi(A, B)$ to denote the number of all crossings whose two crossed edges are respectively in A and in B. Especially, $\mathrm{cr}_\phi(A, A)$ is simply written as $\mathrm{cr}_\phi(A)$. If G has the edge set E, the two signs $\mathrm{cr}_\phi(G)$ and $\mathrm{cr}_\phi(E)$ are essentially the same. The following Lemma 1, which can be shown easily, is usually used in the proofs of our theorem.

Lemma 1 *Let A, B, C be mutually disjoint subsets of E. Then*

$$\mathrm{cr}_\phi(A \bigcup B) = \mathrm{cr}_\phi(A) + cr_\phi(B) + \mathrm{cr}_\phi(A, B);$$

$$\mathrm{cr}_\phi(A, B \bigcup C) = \mathrm{cr}_\phi(A, B) + cr_\phi(A, C),$$

where ϕ is a good drawing of E. □

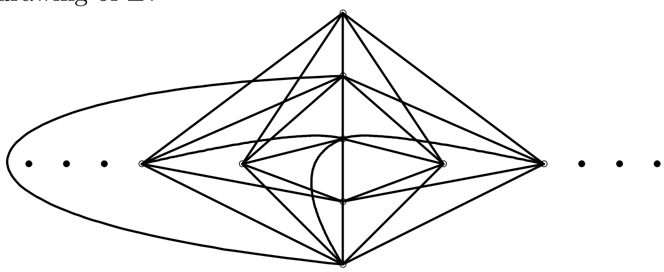

Figure 2 A good drawing of H_n

On the crossing numbers of complete bipartite graphs $K_{m,n}$, Kleitman obtained the following result in [12].

Lemma 2 *If $m \leq 6$, then*

$$\mathrm{cr}(K_{m,n}) = Z(m, n),$$

where $Z(m, n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. □

Lemma 3 *Let ϕ be a good drawing of H_n. If there exist $1 \leq i \neq j \leq n$, such that $\mathrm{cr}_\phi(T^i, T^j) = 0$, then*

$$\mathrm{cr}_\phi(G_{12}, T^i \cup T^j) \geq 2.$$

Proof Let K be the subgraph of H_n induced by the edges of $T^i \cup T^j$. Since $cr_\phi(T^i, T^j) = 0$, and in good drawing two edges incident with the same vertex cannot cross, the subdrawing of $T^i \cup T^j$ induced by ϕ induces the map in the plane without crossing, as shown in Figure 3(1). Let a, b, c, d, e denote the five vertices of the subgraph G_{12} (see Figure 3(2)). Clearly, in figure 3(1) for any $x \in V(G_{12})$, there are exactly two other vertices of G_{12} on the boundary of common region with x. By $d_{G_{12}}(e) = 3$, at the edges incidenting with e, there is at least one crossing with edges of K. Similarly, at the edges incidenting with c, there is at least one crossing with edges of K. If the two crossings are different, this completes the proof. Otherwise, the same crossing only can be found at edge ec. However, by $d_{G_{12}}(b) = 3$, at the edges incidenting with b, there is at least one crossing with edges of K. Clearly, it is different from the crossing in the ec. Therefore, we complete the proof. □

Let H be a graph isomorphic to G_{12}. Consider a graph G_H obtained by joining all vertices of H to five vertices of a connected graph G such that every vertex of H will only be adjacent to exactly one vertex of G. Let G_H^* be the graph obtained from G_H by contracting the edges of H.

Lemma 4 The crossing number of G_H^* is no more than the crossing number of G_H, i.e., $\mathrm{cr}(G_H^*) \leq \mathrm{cr}(G_H)$.

Proof Let ϕ be an optimal drawing of G_H. Since the plane is a normal space, for an edge e

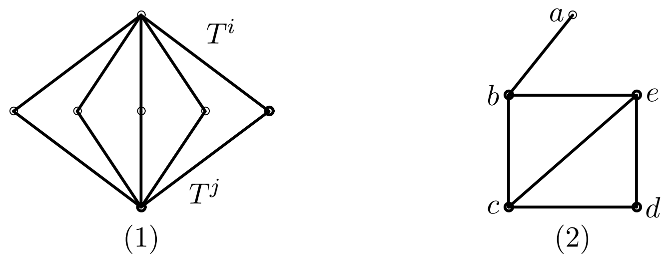

Figure 3 A good drawing of $T^i \cup T^j$ and G_{12}

of the drawing ϕ there is an open set M_e homeomorphic to the open disk such that M_e contains e, together with ends of edges incident with endpoints of e, and open arcs of edges which are crossing e (see Figure 4(a)). All remaining edges of ϕ are disjoint with M_e.

Let x denote the number of crossings of e in ϕ. If we draw in M_e two edges e_1 and e_2 instead of e, these two edges have $2x$ crossings (see Figure 4(b)).

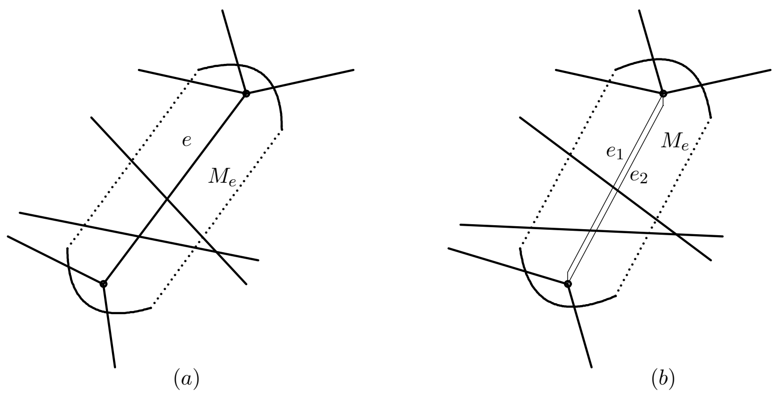

Figure 4 Open disks which contain edge

The subgraph H has six edges and let x_1, x_2, x_3, x_4, x_5 and x_6 denote the numbers of crossings on the edges of H (see Figure 5).

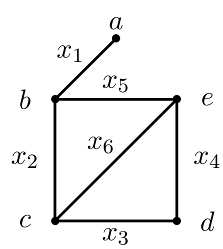

Figure 5 The numbers of crossing on the edge of H

Let $x_2 \leq x_4 + x_5$. Figure 6 shows that H can be contracted to the vertex c without increasing the number of crossings.

Let $x_5 \leq x_2 + x_3$. Figure 7 shows that H can be contracted to the vertex e without increasing the number of crossings. This completes the proof, because for nonnegative integers the system of inequalities

$$x_2 > x_4 + x_5,$$

$$x_5 > x_2 + x_3$$

holds only for $x_3 + x_4 < 0$. This is impossible because x_3 and x_4 are all nonnegative integers.□

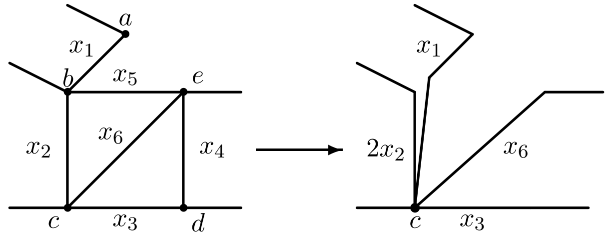

Figure 6 H is contracted to the vertex c

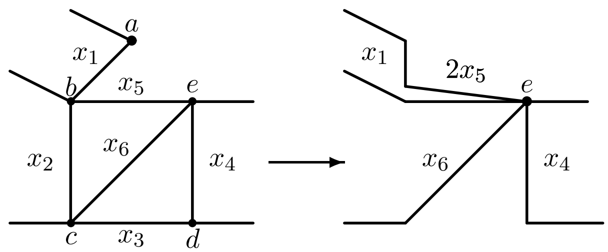

Figure 7 H is contracted to the vertex e

Theorem 1 *For $n \geq 1$, we have* $\mathrm{cr}(H_n) = n(n-1)$.

Proof The drawing in Figure 2 shows that

$$\mathrm{cr}(H_n) \leq \mathrm{cr}(K_{5,n}) + 2\lfloor\frac{n}{2}\rfloor = 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor = n(n-1).$$

Thus, in order to prove theorem, we need only to prove that $\mathrm{cr}_{\phi'}(H_n) \geq n(n-1)$ for any drawing ϕ' of H_n. We prove the reverse inequality by induction on n. The cases $n = 1$ and 2 are trivial. Suppose now that for $n \geq 3$

$$\mathrm{cr}(H_{n-2}) \geq (n-2)(n-3) \tag{2}$$

and consider such a drawing ϕ of H_n that

$$\mathrm{cr}_\phi(H_n) < n(n-1). \tag{3}$$

Our next analysis depends on whether or not there are different subgraphs T^i and T^j that do not cross each other in ϕ.

First, we suppose that every pair of T^i crosses each other. By Lemmas 1 and 2, using (1), we have

$$\mathrm{cr}_\phi(H_n) = \mathrm{cr}_\phi(K_{5,n}) + \mathrm{cr}_\phi(G_{12}) + \mathrm{cr}_\phi(K_{5,n}, G_{12}).$$

This, together with our assumption that $\mathrm{cr}_\phi(H_n) < 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor$, implies that

$$\mathrm{cr}_\phi(G_{12}) + \mathrm{cr}_\phi(K_{5,n}, G_{12}) < 2\lfloor\frac{n}{2}\rfloor$$

and, using (1),

$$\mathrm{cr}_\phi(G_{12}) + \sum_{i=1}^{n} \mathrm{cr}_\phi(G_{12}, T^i) < 2\lfloor\frac{n}{2}\rfloor.$$

We can see that in ϕ there is at least one subgraph T^i which does not cross G_{12}. Let us suppose that $\mathrm{cr}_\phi(G_{12}, T^n) = 0$ and let F be the subgraph $G_{12} \cup T^n$ of the graph H_n.

Consider the subdrawings ϕ^* and ϕ^{**} of G_{12} and F, respectively, induced by ϕ. Since $\mathrm{cr}_\phi(G_{12}, T^n) = 0$, the subdrawing ϕ^* divides the plane in such a way that all vertices are on the boundary of one "region". It is easy to verify that all possibilities of the subdrawing ϕ^* are shown in Figure 8. Thus, all possibilities of the subdrawing ϕ^{**} are shown in Figure 9.

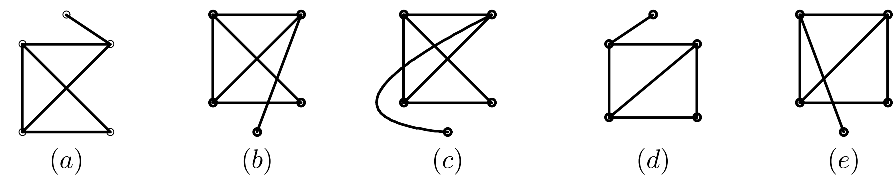

Figure 8 All possibilities of the subdrawing ϕ^*

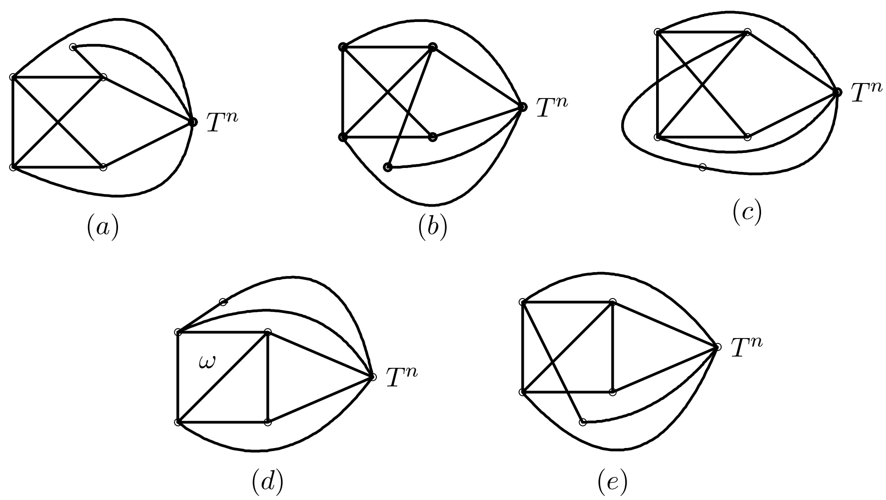

Figure 9 All possibilities of the subdrawing ϕ^{**}

Consider now a subdrawing of $F \cup T^i$ of the drawing ϕ for some $i \in \{1, 2, \ldots, n-1\}$ and let x be the vertex of T^i of degree five. If x is in ϕ^{**} in a region with one or two vertices of G_{12} on its boundary, the edges of T^i at least three times cross the edges of F. If x is in a region with three vertices of G_{12} on its boundary except the region ω in Figure 9 (d), at least one vertex of G_{12} is in a region having no common edge with the region containing x. In this case, the edges of T^i again at least three times cross the edges of F. Only if x is in the region ω (Figure 9 (d)), one can draw the edges of T^i with two crossings, but only with the edges of G_{12}. Thus, using $\mathrm{cr}_\phi(T^i, T^j) \geq 1$, we have again at least three crossings on the edges of T^i. Hence, $\mathrm{cr}_\phi(F, T^i) \geq 3$ for all $i = 1, 2, \ldots, n-1$, and

$$\mathrm{cr}_\phi(K_{5,n-1}, F) \geq 3(n-1).$$

Thus, by Lemma 1 and the fact that $H_n = K_{5,n-1} \cup F$, we have

$$\begin{aligned}\mathrm{cr}_\phi(H_n) &= \mathrm{cr}_\phi(K_{5,n-1}) + \mathrm{cr}_\phi(F) + \mathrm{cr}_\phi(K_{5,n-1}, F)\\ &\geq 4\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor + 3(n-1)\end{aligned}$$

$$\geq n(n-1).$$

This contradicts our assumption of the drawing ϕ.

Hence, there are at least two different subgraphs T^i and T^j that do not cross each other in ϕ. Without loss of generality, we may assume that $\mathrm{cr}_\phi(T^{n-1}, T^n) = 0$. By Lemma 3, $\mathrm{cr}_\phi(G_{12}, T^{n-1} \cup T^n) \geq 2$. As $\mathrm{cr}(K_{3,5}) = 4$, for all $i = 1, 2, \ldots, n-2$, $\mathrm{cr}_\phi(T^i, T^{n-1} \cup T^n) \geq 4$. This implies that

$$\mathrm{cr}_\phi(H_{n-2}, T^{n-1} \cup T^n) \geq 4(n-2) + 2 = 4n - 6. \tag{4}$$

Since $H_n = H_{n-2} \cup (T^{n-1} \cup T^n)$, using (1),(2) and (4), we have

$$\begin{aligned}\mathrm{cr}_\phi(H_n) &= \mathrm{cr}_\phi(H_{n-2}) + \mathrm{cr}_\phi(T^{n-1} \cup T^n) + \mathrm{cr}_\phi(H_{n-2}, T^{n-1} \cup T^n)\\ &\geq (n-2)(n-3) + 4n - 6 = n(n-1).\end{aligned}$$

This contradiction to (3) completes the proof. □

Consider now the graph $\mathrm{cr}(G_{12} \times S_n)$. For $n \geq 1$ it has $5(n+1)$ vertices and edges that are the edges in $n+1$ copies G_{12}^i for $i = 0, 1, \ldots, n$, and in the five stars S_n (see Figure 10), where the vertices of G_{12}^0 are the central vertices of the stars S_n.

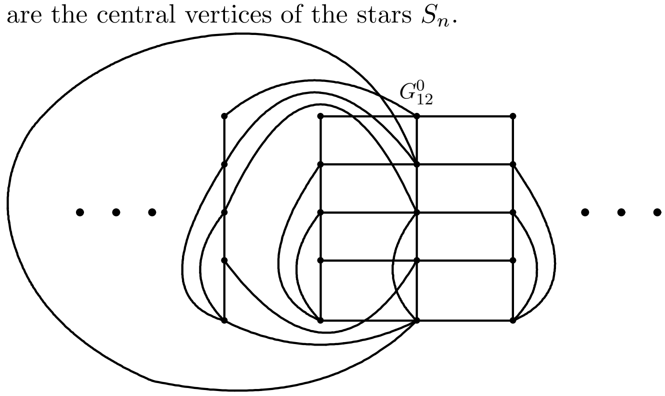

Figure 10 A optimal drawing of $G_{12} \times S_n$

G_i	G_1	G_2	G_3	G_4	G_5	G_6	G_7
$\mathrm{cr}(G_i) \times S_n$	$Z(5,n)$ [13]	$n(n-1)$ [14]					
G_i	G_8	G_9	G_{10}	G_{11}	G_{12}	G_{13}	G_{14}
$\mathrm{cr}(G_i) \times S_n$				$n(n-1)$ [16]	$n(n-1)$	$Z(5,n) + \lfloor \frac{n}{2} \rfloor$ [15]	$n(n-1)$ [16]
G_i	G_{15}	G_{16}	G_{17}	G_{18}	G_{19}	G_{20}	G_{21}
$\mathrm{cr}(G_i) \times S_n$	$Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor$ [17]		$Z(5,n) + 2n$ [11]		$Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor$ [18]		$Z(5,n) + 5n$ $\lfloor \frac{n}{2} \rfloor + 1$ [19]

Table 1 The up to date results of the Cartesian products of 5-vertex graphs with stars

Theorem 2 *For $n \geq 1$, we have $\mathrm{cr}(G_{12} \times S_n) = n(n-1)$.*

Proof The drawing in Figure 10 shows that $\mathrm{cr}(G_{12} \times S_n) \le \mathrm{cr}(K_{5,n}) + 2\lfloor \frac{n}{2} \rfloor = n(n-1)$. To complete the proof, assume that there is an optimal drawing ϕ of $G_{12} \times S_n$ with fewer than $n(n-1)$ crossings. Contracting the edges of G_{12}^i for all $i = 1, 2, \ldots, n$ in ϕ results in a graph isomorphic to H_n. In accordance with Lemma 4, we have: $\mathrm{cr}(H_n) \le n(n-1)$. This is impossible because in Theorem 1 it is shown that $\mathrm{cr}(H_n) = n(n-1)$. Therefore, the graph $G_{12} \times S_n$ has crossing number $n(n-1)$. □

3. Summary of results

To conclude, we present Table 1 to give a picture of the up to date results of the Cartesian products of 5-vertex graphs with stars.

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