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An Upper Bound for the Adjacent Vertex-Distinguishing Total Chromatic Number of a Graph

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Abstract Let $G=(V,E)$ be a simple connected graph, and $|V(G)|\geq 2$. Let f be a mapping from $V(G)\cup E(G)$ to $\{1,2,\ldots,k\}$. If $\forall uv\in E(G), f(u)\neq f(v), f(u)\neq f(uv), f(v)\neq f(uv)$; $\forall uv, uw\in E(G) (v\neq w)$, $f(uv)\neq f(uw)$; $\forall uv\in E(G)$ and $u\neq v$, $C(u)\neq C(v)$, where

$$C(u)=\{f(u)\}\cup\{f(uv)|uv\in E(G)\}.$$

Then f is called a k-adjacent-vertex-distinguishing-proper-total coloring of the graph G(k-$AVDTC$ of G for short). The number $\min\{k|k\text{-}AVDTC \text{ of } G\}$ is called the adjacent vertex-distinguishing total chromatic number and denoted by $\chi_{at}(G)$. In this paper we prove that if $\Delta(G)$ is at least a particular constant and $\delta\geq 32\sqrt{\Delta\ln\Delta}$, then $\chi_{at}(G)\leq\Delta(G)+10^{26}+2\sqrt{\Delta\ln\Delta}$.

Keywords total coloring; adjacent vertex distinguishing total coloring; adjacent vertex distinguishing total chromatic number; Lovász local lemma.

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1. Introduction

All graphs $G=(V,E)$ discussed in this paper are finite, undirected, simple and connected. Let $\delta(G)$ ($\Delta(G)$) denote the minimum (maximum) degree of a graph G. A proper edge-coloring of a simple graph G is called vertex-distinguishing[1−3], if for any two distinct vertices u and v in G, the set of colors assigned to the edges incident to u differs from the set of colors assigned to the edges incident to v. A vertex-distinguishing proper edge-coloring is also called a strong edge-coloring. The minimal number of colors required for a strong edge-coloring of G is called the vertex distinguishing edge chromatic number of G (or observability), and denoted by $\chi'_s(G)$ or $\chi'_{vd}(G)$.

Let $n_d=n_d(G)$ denote the number of vertices of degree d in a graph G. It is clear that $\binom{\chi'_s(G)}{d}\geq n_d$ for all d with $\delta(G)\leq d\leq\triangle(G)$. The following conjecture was given in [3].

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Conjecture 1[3] *Let G be a graph and let k be the minimum integer such that $\binom{k}{d} \geq n_d$ for all d such that $\delta(G) \leq d \leq \triangle(G)$. Then $\chi'_s(G) = k$ or $k+1$.*

In [4], the adjacent strong edge-coloring of a graph G was proposed. A proper edge-coloring of a simple graph G is called an adjacent-vertex-distinguishing edge-coloring if for any two adjacent vertices u and v in G, the set of colors assigned to the edges incident to u differs from the set of colors incident to v. An adjacent-vertex-distinguishing proper edge-coloring is also called an adjacent strong edge-coloring. The minimal number of colors required for an adjacent strong edge-coloring of G is called the adjacent vertex distinguishing edge chromatic number of G (or adjacent strong edge chromatic number), and denoted by $\chi'_{as}(G)$. The following conjecture was proposed by Zhang et al. in [4].

Conjecture 2[4] *Let G be a connected graph with $|G| \geq 3$, and $G \neq C_5$ (5-cycle). Then $\triangle(G) \leq \chi'_{as}(G) \leq \triangle(G) + 2$.*

Definition 1[5] *Let G be a simple connected graph, and $|V(G)| \geq 2$. A k-adjacent-vertex-distinguishing-total coloring of a graph G is a mapping f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, k\}$, such that*

1) f is a proper k-total coloring, i.e., $\forall uv \in E(G)$, $f(u) \neq f(v)$, $f(u) \neq f(uv)$, $f(v) \neq f(uv)$;

2) $\forall uv, uw \in E(G) (v \neq w)$, $f(uv) \neq f(uw)$;

3) $\forall uv \in E(G)$, $C(u) \neq C(v)$, where $C(u) = \{f(u)\} \cup \{f(uv) | uv \in E(G)\}$. Then f is called a k-adjacent-vertex-distinguishing-total coloring of the graph G (k-AVDTC of G for short). The number $\min\{k | k\text{-AVDTC of } G\}$ is called the adjacent-vertex-distinguishing total chromatic number and denoted by $\chi_{at}(G)$.

Conjecture 3[5] *Let G be a connected graph of order n ($n \geq 2$). Then*

$$\chi_{at}(G) \leq \Delta(G) + 3.$$

Zhang et al. proved that Conjecture 3 is true for cycles, complete graphs, complete bipartite graphs, fans, wheels and trees in [5] and obtained their adjacent vertex distinguishing total chromatic numbers.

It is interesting that $\chi'_s(C_5) = \chi'_{as}(C_5) = 5$ and $\chi_{at}(C_5) = 4$.

It is easy to verify that $2\Delta + 1$ is an upper bound for the adjacent-vertex-distinguishing total chromatic number of G. In this paper we improve this bound to $\chi_{at}(G) \leq \Delta(G) + 10^{26} + 2\sqrt{\Delta \ln \Delta}$. For the other terminology and notations we refer to [6, 7, 8].

2. Main results

Lemma A[9] (Lovász Local Lemma) *Consider a set $\varepsilon = \{A_1, A_2, \ldots, A_n\}$ of (typically bad) events such that each A_i is mutually independent of $\varepsilon - (D_i \cup A_i)$ for some $D_i \subseteq \varepsilon$. If we have reals $x_1, x_2, \ldots, x_n \in [0, 1)$ such that for each $1 \leq i \leq n$, $\Pr(A_i) \leq x_i \prod_{A_j \in D_i}(1 - x_j)$, then the probability that none of the events in ε occurs is at least $\prod_{i=1}^{n}(1 - x_i) > 0$.*

Lemma B[10] *If a simple graph has maximum degree $\Delta(G)$ at least a particular constant, then*

$\chi_t(G) \leq \Delta(G) + 10^{26}$.

Theorem 1 *Let $G(V, E)$ be a graph with maximum degree $\Delta(G)$ at least a particular constant and $\delta \geq 32\sqrt{\Delta \ln \Delta}$. Then*

$$\chi_{at}(G) \leq \Delta(G) + 10^{26} + 2\sqrt{\Delta \ln \Delta}.$$

Proof By Lemma B, it is possible to have a $\Delta + 10^{26}$-total coloring f_0. And then each of edges and vertices in G are recolored randomly and independently with an equal probability $\frac{1}{16\Delta}$ by one of the $2\sqrt{\Delta \ln \Delta}$ new colors and with probability $(1 - \frac{1}{8}\sqrt{\frac{\ln \Delta}{\Delta}})$ preserving its previous color. Naming this total coloring of G as f, we will use the Lovász Local Lemma to show that with a positive probability, f is an adjacent vertex distinguishing total coloring.

We must show that with positive probability the obtained coloring is an adjacent vertex distinguishing total coloring. The following four types of "bad" events are defined in order to satisfy this.

(I) For each pair of adjacent edges $A = \{e_1, e_2\}$, let E_A be the event that both e_1 and e_2 are recolored with the same color.

(II) For each edge $B = \{uv \in E(G)\}$, let E_B be the event that both u and v are recolored with the same color.

(III) For each edge $C_{e(u,v)} = \{e = uv \in E(G)\}$, let $E_{C_{e(u,v)}}$ be the event that e and each end vertex of e are recolored with the same color.

(IV) For each edge $e = uv$ such that $d(u) = d(v) \geq \delta(G)$, let D_e be the set of two vertices u and v, and all edges which are incident with u or v. Then E_{D_e} is the event that two vertices u and v, and the edges which are incident with u and v are colored properly, and $C(u) = C(v)$.

(Note that for each edge $e = uv$, let $D_e = \{u\} \bigcup \{v\} \bigcup \{ux \in E(G) : x \in V(G)\} \bigcup \{vy \in E(G) : y \in V(G)\}$).

It remains to show that with positive probability none of these events occurs.

We must estimate the probability of a given event first.

Lemma 1 *The following four statements hold:*

1) For each event E_A of Type I, we have $\Pr[E_A] = \frac{2}{16^2\Delta}\sqrt{\frac{\ln \Delta}{\Delta}}$.

2) For each event E_B of Type II, we have $\Pr[E_B] = \frac{2}{16^2\Delta}\sqrt{\frac{\ln \Delta}{\Delta}}$.

3) For each event $E_{C_{e(u,v)}}$ of Type III, we have $\Pr[E_{C_{e(u,v)}}] = \frac{4}{16^2\Delta}\sqrt{\frac{\ln \Delta}{\Delta}}$.

4) For each event E_{D_e} of Type IV, we have $\Pr[E_{D_e}] \leq \frac{3}{2}e^{-\frac{1}{4}\sqrt{\frac{\ln \Delta}{\Delta}} \cdot d}$.

Proof If E_A occurs, then e_1 and e_2 in A are recolored with the same color. So we have

$$\Pr[E_A] = \binom{2\sqrt{\Delta \ln \Delta}}{1}(\frac{1}{16\Delta})^2 = \frac{2}{16^2\Delta}\sqrt{\frac{\ln \Delta}{\Delta}}.$$

If E_B occurs, then u and v in B are recolored with the same color. So we get

$$\Pr[E_B] = \binom{2\sqrt{\Delta \ln \Delta}}{1}(\frac{1}{16\Delta})^2 = \frac{2}{16^2\Delta}\sqrt{\frac{\ln \Delta}{\Delta}}.$$

If $E_{C_{e(u,v)}}$ occurs, then e and u, or e and v are recolored with the same color. So we obtain

$$\Pr[E_{C_{e(u,v)}}]=2\binom{2\sqrt{\Delta\ln\Delta}}{1}(\frac{1}{16\Delta})^2=\frac{4}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}}.$$

Let $e=uv, d(u)=d(v)=d$. Suppose that after recoloring, the edges incident with u and v are colored properly and $C(u)\backslash\{f(e)\}=C(v)\backslash\{f(e)\}=C$. Assume that C is a fixed set which has i members of the new colors and $d-i$ members of the old colors. This event happens with the probability of

$$[i!(1-\tfrac{1}{8}\sqrt{\tfrac{\ln\Delta}{\Delta}})^{d-i}(\tfrac{1}{16\Delta})^i]^2\le[i^ie^{-\frac{1}{8}\sqrt{\frac{\ln\Delta}{\Delta}}(d-i)}(\tfrac{1}{16\Delta})^i]^2\le(\tfrac{i}{16\Delta})^{2i}e^{-\frac{1}{4}\sqrt{\frac{\ln\Delta}{\Delta}}(d-i)}.$$

Hence the edges incident with u and v are colored properly and $C(u)=C(v)$ with a probability of at most

$$\begin{aligned}\Pr(E_{D_e})&\le\sum_{i=0}^{2\sqrt{\Delta\ln\Delta}}\binom{d}{i}\binom{2\sqrt{\Delta\ln\Delta}}{i}(\tfrac{i}{16\Delta})^{2i}e^{-\frac{1}{4}\sqrt{\frac{\ln\Delta}{\Delta}}(d-i)}\\&\le\sum_{i=0}^{2\sqrt{\Delta\ln\Delta}}(\tfrac{ed}{i})^i(\tfrac{e\cdot2\sqrt{\Delta\ln\Delta}}{i})^i(\tfrac{i}{16\Delta})^{2i}e^{-\frac{1}{4}\sqrt{\frac{\ln\Delta}{\Delta}}(d-i)}\\&\le\sum_{i=0}^{2\sqrt{\Delta\ln\Delta}}(\tfrac{e^{2+\frac{1}{4}\sqrt{\frac{\ln\Delta}{\Delta}}}\cdot d\cdot2\sqrt{\Delta\ln\Delta}}{16^2\Delta^2})^ie^{-\frac{1}{4}\sqrt{\frac{\ln\Delta}{\Delta}}\cdot d}\\&\le e^{-\frac{1}{4}\sqrt{\frac{\ln\Delta}{\Delta}}\cdot d}\cdot\sum_{i=0}^{2\sqrt{\Delta\ln\Delta}}(\tfrac{2e^{2+\frac{1}{4}\sqrt{\frac{\ln\Delta}{\Delta}}}}{16^2}\sqrt{\tfrac{\ln\Delta}{\Delta}})^i.\end{aligned}$$

For $\Delta\ge32\sqrt{\Delta\ln\Delta}$, we get $\sqrt{\frac{\ln\Delta}{\Delta}}\le\frac{1}{32}$. So we have

$$\sum_{i=0}^{2\sqrt{\Delta\ln\Delta}}(\tfrac{2e^{2+\frac{1}{4}\sqrt{\frac{\ln\Delta}{\Delta}}}}{16^2}\sqrt{\tfrac{\ln\Delta}{\Delta}})^i\le\sum_{i=0}^{2\sqrt{\Delta\ln\Delta}}(\tfrac{2e^{2+\frac{1}{4}\cdot\frac{1}{32}}}{16^2}\cdot\tfrac{1}{32})^i\le\sum_{i=0}^{\infty}(\tfrac{2}{256})^i=\sum_{i=0}^{\infty}(\tfrac{1}{128})^i=\tfrac{128}{127}<\tfrac{3}{2}.$$

(Note that $\frac{1}{32}e^{2+\frac{1}{4\times32}}<\frac{1}{32}\times8=\frac{1}{4}<1$)

Therefore, we have $\Pr[E_{D_e}]\le\frac{3}{2}e^{-\frac{1}{4}\sqrt{\frac{\ln\Delta}{\Delta}}\cdot d}$. □

Then we need to estimate the number of events of each type which are incident to any given event.

Lemma 2 *The following four statements hold.*

1) Each event of Type I is incident to at most 4Δ *events of Type I,* 0 *event of Type II,* 4 *events of Type III, and* 3Δ *events of Type IV.*

2) Each event of Type II is incident to at most 0 *event of Type I,* 2Δ *events of Type II,* 2Δ *events of Type III, and* 2Δ *events of Type IV.*

3) Each event of Type III is incident to at most 2Δ *events of Type I,* Δ *events of Type II,* Δ *events of Type III, and* 2Δ *events of Type IV.*

4) Each event of Type IV is incident to at most $4d\Delta$ *events of Type I,* 2Δ *events of Type II,* 4Δ *events of Type III, and* $2\Delta^2$ *events of Type IV.*

Proof 1) For each event E_A of Type I, for any given edge e, less than 2Δ edges are incident to e. And each event of Type I contains two edges. So each event of Type I is incident to at most $2\times 2\Delta = 4\Delta$ events of Type I. Because each event of Type I and each event of Type II have no common edge or vertex, each event of Type I is incident to at most 0 event of Type II. Because each edge is incident with two vertices u and v, there are two edges in an event of Type I; each event of Type I is incident to at most $2\times 2 = 4$ events of Type III, there are three end vertices for each event of Type I; each vertex has at most Δ adjacent vertices, each event of Type I is incident to at most $3\times\Delta = 3\Delta$ events of Type IV.

The proofs of 2), 3) and 4) are similar to 1). □

Next we must determine the real constants x_i. Let $\frac{4}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}}$, $\frac{4}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}}$, $\frac{8}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}}$ and $\frac{1}{8\Delta^2}$ be the constants associated with events of Type I, Type II, Type III and Type IV, respectively. We conclude that with positive probability none of the events of Type I, II, III or IV occurs, provided that

$$\frac{2}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}} \le \frac{4}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}}\,(1-\frac{4}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}})^{4\Delta}\,(1-\frac{8}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}})^{4}\,(1-\frac{1}{8\Delta^2})^{3\Delta} \tag{1}$$

$$\frac{2}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}} \le \frac{4}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}}\,(1-\frac{4}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}})^{2\Delta}\,(1-\frac{8}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}})^{2\Delta}\,(1-\frac{1}{8\Delta^2})^{2\Delta} \tag{2}$$

$$\frac{4}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}} \le \frac{8}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}}\,(1-\frac{4}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}})^{2\Delta}\,(1-\frac{4}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}})^{\Delta}\,(1-\frac{8}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}})^{\Delta}\,(1-\frac{1}{8\Delta^2})^{2\Delta} \tag{3}$$

$$\frac{3}{2}e^{-\frac{1}{4}\sqrt{\frac{\ln\Delta}{\Delta}}\cdot d} \le \frac{1}{8\Delta^2}(1-\frac{4}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}})^{4d\Delta}(1-\frac{4}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}})^{2\Delta}(1-\frac{8}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}})^{4\Delta}(1-\frac{1}{8\Delta^2})^{2\Delta^2} \tag{4}$$

For (1), we have

$$\begin{aligned}&(1-\frac{4}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}})^{4\Delta}(1-\frac{8}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}})^{4}(1-\frac{1}{8\Delta^2})^{3\Delta}\\ &\ge (\tfrac{1}{4})^{\frac{1}{16}\sqrt{\frac{\ln\Delta}{\Delta}}+\frac{1}{8\Delta}\sqrt{\frac{\ln\Delta}{\Delta}}+\frac{3}{8\Delta}} = (\tfrac{1}{2})^{\frac{1}{8}\sqrt{\frac{\ln\Delta}{\Delta}}+\frac{1}{4\Delta}\sqrt{\frac{\ln\Delta}{\Delta}}+\frac{3}{4\Delta}}\\ &\ge (\tfrac{1}{2})^{\frac{1}{8\times 32}+\frac{1}{32\times 4\Delta}+\frac{3}{4\Delta}} = (\tfrac{1}{2})^{\frac{1}{32\times 8}+\frac{97}{128\Delta}}.\end{aligned}$$

So in order to prove (1), we only need to prove $\frac{1}{2}\le(\frac{1}{2})^{\frac{1}{32\times 8}+\frac{97}{128\Delta}}$.
This inequality is obviously true when $\Delta>2$. So inequality (1) holds.

For (2), we have

$$\begin{aligned}&(1-\frac{4}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}})^{2\Delta}\,(1-\frac{8}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}})^{2\Delta}\,(1-\frac{1}{8\Delta^2})^{2\Delta}\\ &\ge (\tfrac{1}{4})^{\frac{1}{32}\sqrt{\frac{\ln\Delta}{\Delta}}+\frac{1}{16}\sqrt{\frac{\ln\Delta}{\Delta}}+\frac{1}{4\Delta}} = (\tfrac{1}{2})^{\frac{1}{16}\sqrt{\frac{\ln\Delta}{\Delta}}+\frac{1}{8}\sqrt{\frac{\ln\Delta}{\Delta}}+\frac{1}{2\Delta}}\\ &\ge (\tfrac{1}{2})^{\frac{1}{16\times 32}+\frac{1}{8\times 32}+\frac{1}{2\Delta}} = (\tfrac{1}{2})^{\frac{3}{512}+\frac{1}{2\Delta}}.\end{aligned}$$

So in order to prove (2), we only need to prove $\frac{1}{2}\le(\frac{1}{2})^{\frac{3}{512}+\frac{1}{2\Delta}}$. This inequality is obviously true when $\Delta>1$. So inequality (2) holds.

For (3), we have

$$(1-\frac{4}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}})^{2\Delta}(1-\frac{4}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}})^{\Delta}(1-\frac{8}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}})^{\Delta}(1-\frac{1}{8\Delta^2})^{2\Delta}$$

$$\geq (\tfrac{1}{4})^{\frac{3}{64}\sqrt{\frac{\ln\Delta}{\Delta}}+\frac{1}{16}\sqrt{\frac{\ln\Delta}{\Delta}}+\frac{1}{4\Delta}} = (\tfrac{1}{2})^{\frac{7}{32}\sqrt{\frac{\ln\Delta}{\Delta}}+\frac{1}{2\Delta}}$$
$$\geq (\tfrac{1}{2})^{\frac{7}{32\times 32}+\frac{1}{2\Delta}} = (\tfrac{1}{2})^{\frac{7}{1024}+\frac{1}{2\Delta}}.$$

So, in order to prove (3), we only need to prove $\frac{1}{2} \leq (\frac{1}{2})^{\frac{7}{1024}+\frac{1}{2\Delta}}$. This inequality is obviously true when $\Delta > 1$. So inequality (3) holds.

For (4), we have

$$\frac{1}{8\Delta^2}(1-\frac{4}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}})^{4d\Delta}\ (1-\frac{4}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}})^{2\Delta}\ (1-\frac{8}{16^2\Delta}\sqrt{\frac{\ln\Delta}{\Delta}})^{4\Delta}\ (1-\frac{1}{8\Delta^2})^{2\Delta^2}$$
$$\geq e^{-\ln(8\Delta^2)}\cdot e^{-\frac{1}{8}\sqrt{\frac{\ln\Delta}{\Delta}}d}\cdot e^{-\frac{1}{16}\sqrt{\frac{\ln\Delta}{\Delta}}}\cdot e^{-\frac{1}{4}\sqrt{\frac{\ln\Delta}{\Delta}}}\cdot e^{-\frac{1}{2}}$$
$$= e^{-2\ln\Delta-\frac{1}{8}\sqrt{\frac{\ln\Delta}{\Delta}}d-\frac{5}{16}\sqrt{\frac{\ln\Delta}{\Delta}}-\ln 8-0.5}.$$

So in order to prove (4), it is sufficient to prove the following

$$\frac{3}{2}e^{-\frac{1}{4}\sqrt{\frac{\ln\Delta}{\Delta}}\cdot d} \leq e^{-2\ln\Delta-\frac{1}{8}\sqrt{\frac{\ln\Delta}{\Delta}}d-\frac{5}{16}\sqrt{\frac{\ln\Delta}{\Delta}}-\ln 8-0.5}$$
$$-\frac{1}{4}\sqrt{\frac{\ln\Delta}{\Delta}}\cdot d+\ln\frac{3}{2} \leq -2\ln\Delta-\frac{1}{8}\sqrt{\frac{\ln\Delta}{\Delta}}d-\frac{5}{16}\sqrt{\frac{\ln\Delta}{\Delta}}-\ln 8-0.5$$
$$\frac{1}{8}\sqrt{\frac{\ln\Delta}{\Delta}}d-\frac{5}{16}\sqrt{\frac{\ln\Delta}{\Delta}}-2\ln\Delta-\ln 12-0.5 \geq 0.$$

Since $\delta \geq 32\sqrt{\Delta\ln\Delta}$, $\frac{1}{8}\sqrt{\frac{\ln\Delta}{\Delta}}d \geq 4\ln\Delta$, it suffices to show $2\ln\Delta-\frac{5}{16}\times\frac{1}{32}-\ln 12-0.5 \geq 0$. This inequality is obviously true when $\Delta \geq 5$. So inequality (4) holds.

So, by the Lovász Local Lemma, G has a $\Delta(G)+10^{26}+2\sqrt{\Delta\ln\Delta}$-adjacent vertex-distinguishing total coloring. □

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