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# Reconstruction of Non-Bandlimited Functions by Multidimensional Sampling Theorem of Hermite Type

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**Abstract** In this paper, we prove that under some restricted conditions, the non-bandlimited functions can be reconstructed by the multidimensional sampling theorem of Hermite type in the space of $L_p(\mathbb{R}^n)$, $1 < p < \infty$.

**Keywords** multidimensional sampling theorem; non-bandlimited functions; Hermite cardinal series; entire functions of exponential type.

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## 1. Preliminaries and main result

First we introduce some definitions and notations.

**Definition 1** *Given a positive vector $v = (v_1, \dots, v_n)$, i.e., $v_i > 0$, $i = 1, \dots, n$. Let $g_v(z) = g_{v_1,\dots,v_n}(z_1, \dots, z_n)$ be an entire function on $\mathbb{C}^n$, and assume that for $\varepsilon > 0$, there exists a positive number $A = A_\varepsilon$, such that for all $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $z_k = x_k + \mathrm{i}y_k$, $k = 1, \dots, n$, the inequality*

$$| g_v(z) | \le A \exp(\sum_{j=1}^{n} (v_i + \varepsilon) \mid z_j \mid)$$

*is satisfied. Then the function $g_v(z)$ is called an entire function of exponential type $v$.*

Denote by $E_v$ the set of entire functions of exponential type $v$, and let $B_v(\mathbb{R}^n)$ be the collection of entire functions of exponential type $v$ which are bounded on $\mathbb{R}^n$. Set

$$B_{v,p}(\mathbb{R}^n) := B_v(\mathbb{R}^n) \cap L_p(\mathbb{R}^n), \quad 1 \le p < \infty; \quad B_{v,\infty}(\mathbb{R}^n) := B_v(\mathbb{R}^n),$$

where $L_p(\mathbb{R}^n)$, $1 \le p < \infty$, is the classical $p$th power Lebesgue integrable functions space with the usual norm. Then by the Schwartz theorem[3, P10],

$$B_{v,p} = \{f \in L_p(\mathbb{R}^n) : \mathrm{supp}\hat{f} \subset [-v, v]^n\},$$

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where $\hat{f}$ is Fourier transform of $f$ in the sense of generalized functions. As usual, a function on $\mathbb{R}^n$ is said to be bandlimited if its Fourier transform $\hat{f}$ vanishes off $[-v,v]^n$, otherwise it is said to be non-bandlimited.

The Shannon sampling theorem, especially multivariate sampling theorem, plays an important role in a purely mathematical as well as in a practical engineering sense. Many mathematicians have done their best to generalize the sampling theorem since the last century. Especially, some beautiful results [2],[4]–[12],[14] have been obtained in these directions. In this paper, we continue the previous work, studying reconstruction of non-bandlimited functions by interpolation of cardinal series of Hermite type in the space of $L_p(\mathbb{R}^n)$, $1<p<\infty$. First we recall a uniqueness theorem of interpolation on multivariate Hermite cardinal series.

**Theorem A**[14] *(a) Suppose that* $y=\{y_k\}_{k\in\mathbb{Z}^n}$, $y'_j=\{y'_{jk}\}_{k\in\mathbb{Z}^n}\in\ell_p(\mathbb{Z}^n)$, $1<p<\infty$, $j=1,\dots,n$. *Then there exists a unique* $g\in B_{2v,p}(\mathbb{R}^n)$, $1<p<\infty$, *such that* $g(k\pi/v)=y_k$, $g'_j(k\pi/v)=y'_{jk}$, $k\in\mathbb{Z}^n$, $j=1,\dots,n$ *and*

$$g(x)=\sum_{k\in\mathbb{Z}^n}\{y_k+\sum_{j=1}^{n}y'_{jk}(x_j-k_j\pi/v_j)\}\sin c_n^2(v(x-k\pi/v)),\tag{1}$$

*and the series on the right hand side of (1) converges absolutely and uniformly on* $\mathbb{R}^n$.

*(b) Conversely, let* $y=\{y_k\}_{k\in\mathbb{Z}^n}$, $y'_j=\{y'_{jk}\}_{k\in\mathbb{Z}^n}$, $j=1,\dots,n$, *and suppose that there exists a* $g\in B_{2v,p}(\mathbb{R}^n)$, $1<p<\infty$, *such that* $g(k\pi/v)=y_k$, $g'_j(k\pi/v)=y'_{jk}$, $k\in\mathbb{Z}^n$, $j=1,\dots,n$. *Then* $y=\{y_k\}_{k\in\mathbb{Z}^n}$, $y'_j=\{y'_{jk}\}_{k\in\mathbb{Z}^n}\in\ell_p(\mathbb{Z}^n)$, $1<p<\infty$, $j=1,\dots,n$, *where* $\sin cx=\sin x/x$, *if* $x\neq 0$, *and* $0$, *if* $x=0$; $\sin c_n x=\prod_{j=1}^n \sin cx_j$; $(\pi/v)^1=(\pi/v)=\frac{\pi}{v_1}\cdots\frac{\pi}{v_n}$, $f(k\pi/v)=f(k_1\pi/v_1,\dots,k_n\pi/v_n)$; $f'_j=\partial f/\partial x_j$ *and* $f'_j(k\pi/v)=f'_j(k_1\pi/v_1,\dots,k_n\pi/v_n)$.

**Remark 1** Let $f:\mathbb{R}^n\longrightarrow\mathbb{C}$ be a measurable function such that $\sum_{k\in\mathbb{Z}^n}|f(k\pi/v)|^p<\infty$, and $\sum_{k\in\mathbb{Z}^n}|f'_j(k\pi/v)|^p<\infty$, $j=1,\dots,n$. Then by Theorem A, there exists an operator $H_v(f,\cdot)\in B_{2v,p}(\mathbb{R}^n)$, $1<p<\infty$, which interpolates to $f$ at $\{k\pi/v\}_{k\in\mathbb{Z}^n}$, satisfying the following conditions

$$H_v(f,k\pi/v)=f(k\pi/v);\quad H'_{v,j}(f,k\pi/v)=f'_j(k\pi/v),\quad j=1,\dots,n.$$

For convenience, we write also $H_v(f)$ for $H_v(f,\cdot)$.

Assume that $\Re(\mathbb{R}^n)$ is the set of Riemann integrable functions on any bounded fields in $\mathbb{R}^n$, and let

$$L_p^{\ell}(\mathbb{R}^n)=\{f:\ \ f^{(s)}\in L_p(\mathbb{R}^n),\ |s|\le\ell,\ \ \ell\in\mathbb{N}\},$$

where $f^{(s)}(x)=\frac{\partial^{|s|}}{\partial x_1^{s_1}\cdots\partial x_n^{s_n}}f(x_1,\dots,x_n)$, $s=(s_1,\dots,s_n)\in\mathbb{N}^n$, $|s|=s_1+\cdots+s_n$, $0\le s_i\le 1$, $i=1,\dots,n$.

**Definition 2**[12] *Let* $f:\mathbb{R}^n\longrightarrow\mathbb{C}$ *be a measurable function, and let* $h(x)\in L_p(\mathbb{R}^n)$ *be nonnegative, even and non-increasing on* $[0,\infty)$ *with respect to each* $x_i$, $(i=1,\dots,n)$. *We say* $f\in\Lambda_p(\mathbb{R}^n)$, $1<p<\infty$, *if there exists a constant* $C_0$ *independent of* $x$ *such that* $|f(x)|\le C_0h(x)$.

It is clear that if $f\in\Lambda_p(\mathbb{R}^n)$, then $\sum_{k\in\mathbb{Z}^n}|f(k\pi/v)|^p<+\infty$, for all $v\in\mathbb{R}_+^n$. Now we are ready to state our main result.

**Theorem 1** *Let* $f \in L_p^\ell(\mathbb{R}^n)$, $\ell \geq n$, $f_j' \in \Lambda_p(\mathbb{R}^n) \cap \Re(\mathbb{R}^n)$, $1 < p < \infty$, $j = 1, \dots, n$. *Then*

$$\| f - H_v(f) \|_{p(\mathbb{R}^n)} \longrightarrow 0, \quad v \to \infty,$$

*where* $H_v(f) := \sum_{k\in\mathbb{Z}^n}\{f(k\pi/v) + \sum_{j=1}^n f_j'(k\pi/v)(x_j - k_j\pi/v_j)\}\sin c_n{}^2(v(x - k\pi/v))$, *i.e, under the condition of the theorem, the non-band limited functions can be approximately reconstructed by the multivariate Hermite type cardinal series.*

## 2. Proof of main result

In order to prove the theorem, we need also the several lemmas, where $C_p$, $C_n, \dots$ denote the positive constants depending only on $p$, $n, \dots$ , respectively.

**Lemma 1**[14] *Let* $f \in L_p^\ell(\mathbb{R}^n)$, $\ell \geq 1$, $\{f(k\pi/v)\}_{k\in\mathbb{Z}^n}$, $\{f_j'(k\pi/v)\}_{k\in\mathbb{Z}^n} \in \ell_p(\mathbb{Z}^n)$, $1 < p < \infty$, $j = 1, \dots, n$, *and let* $g \in B_{2v,p}(\mathbb{R}^n)$. *Then there exists a constant* $C_p$, *such that*

$$\begin{aligned}\|f - H_v(f)\|_{p(\mathbb{R}^n)} \leq & C_p((\pi/v)\sum_{k\in\mathbb{Z}^n}|f(k\pi/v) - g(k\pi/v)|^p)^{1/p} + \\ & C_p\sum_{j=1}^n 1/v_j((\pi/v)\sum_{k\in\mathbb{Z}^n}|f_j'(k\pi/v) - g_j'(k\pi/v)|^p)^{1/p} + \\ & \| f - g \|_{p(\mathbb{R}^n)} .\end{aligned}$$

Let $K_r(t) = A_r(\sin c_n(t/2r))^{2r}$, $r \in \mathbb{N}$, $t \in \mathbb{R}^n$, where the constant $A_r$ is chosen to satisfy the condition $\| K_r(t) \|_{1(\mathbb{R}^n)} = 1$, and let

$$K_{r,v}(t) = A_r(v)(\sin c_n(vt/2r))^{2r}, \quad v = (v_1, \dots, v_n) \in \mathbb{R}_+^n, \quad (v) = v_1 \cdots v_n.$$

Then $K_{r,v} \in B_{v,1}(\mathbb{R}^n)$, and $\| K_{r,v}(t) \|_{1(\mathbb{R}^n)} = 1$.

**Lemma 2**[12] *Let* $h \in L_p(\mathbb{R}^n)$, $1 < p < \infty$ *be nonnegative, even and non-increasing on* $[0, \infty)$ *with respect to every* $x_i$, $i = 1, \dots, n$, *and let* $g(x) = \int_{\mathbb{R}^n} h(x + t)K_{2,v}(t)\mathrm{d}t$, $v > 1$ *i.e.,* $v_i > 1$, $i = 1, \dots, n$. *Then* $g \in B_{v,p}(\mathbb{R}^n) \cap \Lambda_p(\mathbb{R}^n)$.

**Lemma 3**[12] *If* $f \in L_p^\ell(\mathbb{R}^n)$, $\ell \geq n$, $1 \leq p < \infty$, *then*

$$\begin{aligned}\Big((\pi/v)\sum_{k\in\mathbb{Z}^n}|f(k\pi/v)|^p\Big)^{1/p} \leq & \|f\|_{p(\mathbb{R}^n)} + \sum_{1\leq i\leq n}\frac{\pi}{v_i}\|f_i'\|_{p(\mathbb{R}^n)} + \\ & \sum_{1\leq i<j\leq n}\frac{\pi}{v_i}\frac{\pi}{v_j}\Big\|\frac{\partial^2 f}{\partial x_i \partial x_j}\Big\|_{p(\mathbb{R}^n)} + \cdots + (\frac{\pi}{v})\Big\|\frac{\partial^n}{\partial x_1 \cdots \partial x_n}f\Big\|_{p(\mathbb{R}^n)}.\end{aligned}$$

Let $h = (h_1, \dots, h_n) \in \mathbb{R}^n$, $|h| = \sum_{i=1}^n h_i = 1$, and $\omega_h^k(f, \delta)_{p(\mathbb{R}^n)}$ denote the $k$-th modulus of smoothness of $f$ on $L_p(\mathbb{R}^n)$ along with the direction $h$[3, p140–149]. i.e., $\omega_h^k(f, \delta)_{p(\mathbb{R}^n)} = \sup_{|t|\leq\delta} \| \Delta_{th}^k f \|_{p(\mathbb{R}^n)}$ . Assume $f \in L_p^\ell(\mathbb{R}^n)$ and let

$$\Omega_{\mathbb{R}^n}^k(f^{(\ell)}, \delta)_{p(\mathbb{R}^n)} = \sup_{h\in\mathbb{R}^n}\omega_h^k(f_h^{(\ell)}, \delta)_{p(\mathbb{R}^n)}, \quad f_h^{(\ell)} := \sum_{|s|=l} f^{(s)}h^s$$

be the $k$-th continuity modulus of the $\ell$-th derivative of $f(x)$.

**Lemma 4**[12] *Let* $f \in L_p^\ell(\mathbb{R}^n)$, $\ell \geq n$, $1 \leq p < \infty$, $k \in \mathbb{N}$. *Then there exists a* $g \in B_{v,p}(\mathbb{R}^n)$, $v = (v_1, \ldots, v_n) \in \mathbb{R}_+^n$, *and a constant* $C$, *such that*

$$\| f - g \|_{p(\mathbb{R}^n)} \leq C\Omega^k_{\mathbb{R}^n}(f, 1/\delta)_{p(\mathbb{R}^n)}, \quad \delta = \min_{1\leq i\leq n}\{v_i\},$$

$$\Big\|\frac{\partial f}{\partial x_i} - \frac{\partial g}{\partial x_i}\Big\|_{p(\mathbb{R}^n)} \leq C\Omega^k_{\mathbb{R}^n}(f^{(\ell)}, 1/\delta)_{p(\mathbb{R}^n)}, \quad i = 1, \ldots, n,$$

$$\cdots$$

$$\Big\|\frac{\partial^{\ell_1+\cdots+\ell_n}}{\partial x_1^{\ell_1}\cdots\partial x_n^{\ell_n}}(f-g)\Big\|_{p(\mathbb{R}^n)} \leq C\Omega^k_{\mathbb{R}^n}(f^{(\ell_1+\cdots+\ell_n)}, 1/\delta)_{p(\mathbb{R}^n)}, \quad 0 \leq \ell_i \leq 1, \quad i = 1, \ldots, n.$$

**Proof of Theorem 1** For notational convenience, we only prove the case $n = 2$. Let $f \in L_p^\ell(\mathbb{R}^2)$, $\ell \geq 2$, $f_1'$, $f_2' \in \Lambda(\mathbb{R}^2) \cap \Re(\mathbb{R}^2)$. Then from Lemma 2 and the definition of $\Lambda(\mathbb{R}^2)$, $\{f(k\pi/v)\}_{k\in\mathbb{Z}^2} \in \ell_p(\mathbb{Z}^2)$, $\{f_j'(k\pi/v)\}_{k\in\mathbb{Z}^2} \in \ell_p(\mathbb{Z}^2)$, $j = 1, 2$. Let

$$g(x) := \int_{\mathbb{R}^2} f(x+t)K_{2,2v}(t)\mathrm{d}t.$$

Then $g \in B_{2v,p}(\mathbb{R}^2)$. By the definition of $\Lambda(\mathbb{R}^2)$, there exist $h_i(x) \in L_p(\mathbb{R}^2)$, $i = 1, 2$, satisfying the following conditions:

$$|f_i'(x)| \leq C_{0i}|h_i(x)|, \quad i = 1, 2, \tag{2}$$

where $C_{0i}$, $i = 1, 2$, are two positive constants. In view of the Lebesgue dominated convergence theorem

$$g_i'(x) := \int_{\mathbb{R}^2} f_i'(x+t)K_{2,2v}(t)\mathrm{d}t, \quad i = 1, 2, \tag{3}$$

consequently, $g_i'(x) \in B_{2v,p}(\mathbb{R}^2) \cap \Lambda(\mathbb{R}^2)$, $i = 1, 2$. It follows from (2) and (3),

$$|g_i'(x)| \leq C_{0i}\int_{\mathbb{R}^2} h_i(x+t)K_{2,2v}(t)\mathrm{d}t, \quad i = 1, 2.$$

By the definition of $\Lambda(\mathbb{R}^2)$, there exist functions $\psi_i$, $i = 1, 2$, and two positive constants $C_{p,\psi_i}$, $i = 1, 2$, such that

$$|g_i'(x)| \leq C_{0i}C_{p,\psi_i}|\psi_i(x)|, \quad \forall x \in \mathbb{R}^2, \quad i = 1, 2.$$

In view of Lemma 1,

$$\begin{aligned}\|f - H_v(f)\|_{p(\mathbb{R}^2)} \leq & C_p((\pi/v)\sum_{k\in\mathbb{Z}^2}|f(k\pi/v) - g(k\pi/v)|^p)^{1/p} + \\ & C_p\sum_{j=1}^{2} 1/v_j((\pi/v)\sum_{k\in\mathbb{Z}^2}|f_j'(k\pi/v) - g_j'(k\pi/v)|^p)^{1/p} + \\ & \| f - g \|_{p(\mathbb{R}^2)}\ .\end{aligned} \tag{4}$$

We first estimate the second term on the right hand side of the above inequality. Since $h_i(x)$, $\psi_i(x) \in L_p(\mathbb{R}^2)$, $i = 1, 2$, for given $\varepsilon > 0$ there exists $r_0 > 0$, such that for $r > r_0$

$$C_{0i}C_p\Big(\int_{\mathbb{R}^2\setminus Q(r)}|h_i(x)|^p\mathrm{d}x\Big)^{1/p} \leq \varepsilon/16,$$

$$C_{0i}C_{p\psi_i}\Big(\int_{\mathbb{R}^2\setminus Q(r)}|\psi_i(x)|^p\mathrm{d}x\Big)^{1/p} \leq \varepsilon/16, \quad i = 1, 2,$$

where $Q(r) := \{x = (x_1, x_2) : |x_i| < r, i = 1, 2\}$.

Set $\alpha(v_i) = \left[\frac{v_i r_0}{\pi}\right] + 1,\ i = 1, 2$, where $[a]$ denotes the integer part of $a$. Then from the relation between integral and series, for $v = (v_1, v_2) > 1$, we get

$$\begin{aligned} &C_p\Big((\pi/v)\big(\sum_{k\in\mathbb{Z}^2} - \sum_{|k_i|<\alpha(v_i)}\big)|f_i'(k\pi/v)|^p\Big)^{1/p} \\ &\leq C_0 C_p\Big(\int_{\mathbb{R}^2\setminus Q(r)} |h_i(x)|^p \mathrm{d}x\Big)^{1/p} \leq \varepsilon/16, \end{aligned} \tag{5}$$

similarly

$$C_p\Big((\pi/v)\big(\sum_{k\in\mathbb{Z}^2} - \sum_{|k_i|<\alpha(v_i)}\big)|g_i'(k\pi/v)|^p\Big)^{1/p} \leq \varepsilon/16, \quad i = 1, 2. \tag{6}$$

On the other hand, in view of the fact that $f_i' \in \Re(\mathbb{R}^2)$, $g_i' \in B_{2v,p}(\mathbb{R}^2)$, $i = 1, 2$, then there exists $v_0 = (v_1^0, v_2^0)$, such that for $v \geq v_0$,

$$\begin{aligned} \Big((\pi/v)\sum_{|k_i|<\alpha(v_i)} |f_i'(k\pi/v) - g_i'(k\pi/v)|^p\Big)^{1/p} &\leq \|f_i' - g_i'\|_{p(Q)} + \varepsilon/16 \\ &\leq \|f_i' - g_i'\|_{p(\mathbb{R}^2)} + \varepsilon/16, \quad i = 1, 2. \end{aligned} \tag{7}$$

By the definitions of $g(x)$ and $K_{r,v}(t)$, we have

$$\begin{aligned} f(x) - g(x) &= \int_{\mathbb{R}^2} (f(x) - f(x+t))K_{2,2v}(t)\mathrm{d}t, \\ f_i'(x) - g_i'(x) &= \int_{\mathbb{R}^2} (f_i'(x) - f_i'(x+t))K_{2,2v}(t)\mathrm{d}t, \quad i = 1, 2. \end{aligned}$$

Accordingly, there exists $v_{00} = (v_1^{00}, v_2^{00}) > 0$, for $v > v_{00}$, such that

$$\begin{aligned} C_p\|f - g\|_{p(\mathbb{R}^2)} &\leq C_p \int_{\mathbb{R}^2} \|(f(x) - f(x+t))\|_{p(\mathbb{R}^2)} K_{2,2v}(t)\mathrm{d}t \\ &\leq C_p \Omega_{\mathbb{R}^2}(f, \frac{1}{\delta})_{p(\mathbb{R}^2)} \int_{\mathbb{R}^2} (1 + \delta \mid t \mid)K_{2,2v}(t)\mathrm{d}t \\ &\leq C^* C_p \Omega_{\mathbb{R}^2}(f, \frac{1}{\delta})_{p(\mathbb{R}^2)} \leq \varepsilon/4, \quad \delta = \min\{v_1, v_2\} \end{aligned} \tag{8}$$

and

$$\begin{aligned} C_p\|f_i' - g_i'\|_{p(\mathbb{R}^2)} &\leq C_p \int_{\mathbb{R}^2} \|(f_i'(x) - f_i'(x+t))\|_{p(\mathbb{R}^2)} K_{2,2v}(t)\mathrm{d}t \\ &\leq C_p \Omega_{\mathbb{R}^2}(f_i', \frac{1}{\delta})_{p(\mathbb{R}^2)} \int_{\mathbb{R}^2} (1 + \delta \mid t \mid)K_{2,2v}(t)\mathrm{d}t \\ &\leq C^* C_p \Omega_{\mathbb{R}^2}(f_i', \frac{1}{\delta})_{p(\mathbb{R}^2)} \leq \varepsilon/16, \quad i = 1, 2. \end{aligned} \tag{9}$$

Now we turn to estimate the first term on the right hand side of (4). By Lemmas 3 and 4, we get

$$\begin{aligned} &C_p((\pi/v)\sum_{k\in\mathbb{Z}^2} |f(k\pi/v) - g(k\pi/v)|^p)^{1/p} \\ &\leq C_p\big(\|f - g\|_{p(\mathbb{R}^2)} + \frac{\pi}{v_1}\|f_1' - g_1'\|_{p(\mathbb{R}^2)} + \frac{\pi}{v_2}\|f_2' - g_2'\|_{p(\mathbb{R}^2)} + \frac{\pi}{v_1}\frac{\pi}{v_2}\|f_{12}'' - g_{12}''\|_{p(\mathbb{R}^2)}\big) \end{aligned}$$

$$\le C_p\big(\Omega^3_{\mathbb{R}^2}(f,1/\delta)+\frac{1}{\delta}\Omega^3_{\mathbb{R}^2}(f_1',1/\delta)+\frac{1}{\delta}\Omega^3_{\mathbb{R}^2}(f_2',1/\delta)+\frac{1}{\delta^2}\Omega^3_{\mathbb{R}^2}(f_{12}'',1/\delta)\big)$$
$$\le C_p\frac{1}{\delta^2}\Omega_{\mathbb{R}^2}(f'',1/\delta)\le\varepsilon/4. \tag{10}$$

Combining (10) with (4)–(9) gives

$$\|f-H_v(f)\|_{p(\mathbb{R}^2)}\le\frac{\varepsilon}{4}+4\times\frac{\varepsilon}{16}+2\times\frac{\varepsilon}{16}+2\times\frac{\varepsilon}{16}+\frac{\varepsilon}{4}=\varepsilon,$$

where $v>\max\{1,v_0,v_{00}\}$. The proof of Theorem 1 is completed. □

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