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The Crossing Number of the Cartesian Products of W_m with P_n

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Abstract Most results on crossing numbers of graphs focus on some special graphs, such as the Cartesian products of small graphs with path, star and cycle. In this paper, we obtain the crossing number formula of Cartesian products of wheel W_m with path P_n for arbitrary $m \geq 3$ and $n \geq 1$.

Keywords drawing; crossing number; wheel; path; Cartesian product.

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1. Introduction

Let G be a simple graph with vertex set V and edge set E. The crossing number $\mathrm{cr}(G)$ of a graph G is the minimum number of pairs of intersected edges in a drawing of G in the plane. It is well known that the crossing number of a graph is attained only in good drawings of the graph, which are those drawings where no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point except their common vertex. Let D be a good drawing of the graph G, denote by $\mathrm{cr}_D(G)$ the number of crossings in D. If D is a good drawing of G satisfying $\mathrm{cr}_D(G) = \mathrm{cr}(G)$, then D is called an optimal drawing of G.

The suspension of a graph G_1 from a graph G_2 is obtained by adjoining every vertex of G_1 to every vertex of G_2, and is denoted by $G_1 + G_2$[1]. The wheel W_m is obtained by the suspension of K_1 from cycle C_m of length m.

The Cartesian product $G_1 \times G_2$ of graphs G_1 and G_2 has vertex set $V(G_1) \times V(G_2)$ and edge set

$$\begin{aligned} E(G_1 \times G_2) = \{\{(u_i, v_j), (u_h, v_k)\} : u_i = u_h \text{ and } \{v_j, v_k\} \in E(G_2) \\ \text{or} \quad v_j = v_k \text{ and } \{u_i, u_h\} \in E(G_1)\} \end{aligned}$$

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Let G_1 be a graph homeomorphic to G_2, (for the definition[2]), it is readily seen that $\mathrm{cr}(G_1) = \mathrm{cr}(G_2)$. And if G_1 is a subgraph of G_2, then $\mathrm{cr}(G_1) \leq \mathrm{cr}(G_2)$.

The crossing number of a graph is a tantalizingly open problem. Let P_n be the path of length n, S_n be the complete bipartite graph $K_{1,n}$. So far, most researches on crossing number focused on the estimation of the crossing number of special graphs, such as the Cartesian products of small graphs with path P_n, star S_n, and cycle C_n[3−8]. Klešč determined that the crossing number of $W_3 \times P_n$ is $2n$ in [3], and that the crossing number of $W_4 \times P_n$ is $3n-1$ in [7]. Yu and Huang[9] gave the upper bounds of the crossing number of $W_m \times P_n$ for arbitrary m and n, and proved that $\mathrm{cr}(W_m \times P_n) = (n-1)\lfloor \frac{(m-1)^2}{4} \rfloor + (n+1)$ for arbitrary m when $n = 1, 2, 3$. Recently, Bokal[10] determined the crossing number of the Cartesian products of S_m with P_n for an arbitrary star S_m and path P_n using a newly introduced operation. Stimulated by these results, we consider the crossing number of the Cartesian products of wheel W_m with path P_n for arbitrary $m \geq 3$ and $n \geq 1$, and get the main result of this paper:

Theorem 1 For $m \geq 3$ and $n \geq 1$, we have $\mathrm{cr}(W_m \times P_n) = (n-1)(\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1) + 2$.

2. Some lemmas

In [10], Bokal defined that $\hat{G}$ is the graph obtained from G by adding two vertices v_1 and v_2 and the edges $v_i v$ for $i = 1, 2$ and each $v \in V(G)$. If a vertex $v \in V(G)$ is adjacent to all other vertices in G, then v is called a dominating vertex. Let $0, 1, \ldots, n$ be the vertices of the path P_n with 0 and n be its origin and terminus, respectively. With $G \hat{\times} P_n$ we denote the capped Cartesian products of G and P_n, i.e. the graph, obtained from $G \times P_n$ by adding two new vertices v_0 and v_n and connecting v_0 with all the vertices of $G \times \{0\}$ and v_n with all the vertices of $G \times \{n\}$. Bokal gave a relationship between the crossing numbers of $G \hat{\times} P_n$ and $\hat{G}$:

Lemma 2[10] *Let G be a graph with a dominating vertex. Then for $n \geq 0$,* $\mathrm{cr}(G \hat{\times} P_n) = (n+1)\mathrm{cr}(\hat{G})$. □

$W_m \times P_n$ contains a subgraph homeomorphic to $W_m \hat{\times} P_{n-2}$. To obtain the crossing number of $W_m \hat{\times} P_{n-2}$ by Lemma 2, we need to consider the crossing number of $\hat{W_m}$, which is isomorphic to $P_2 + C_m$, see Figure 1.

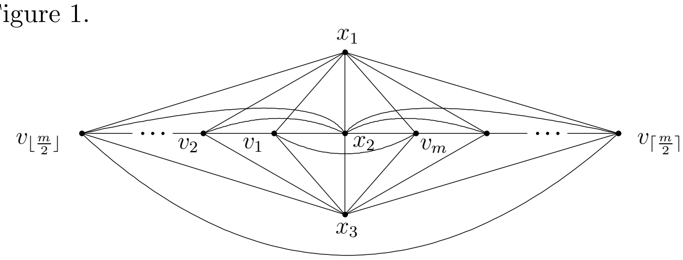

Figure 1 A good drawing of $P_2 + C_m$

Lemma 3 For $m \geq 3$, we have $\mathrm{cr}(P_2 + C_m) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1$.

Proof Figure 1 shows that $cr(P_2+C_m) \leq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor+1$. Now we prove the reverse inequality by assuming that there exists a good drawing D of P_2+C_m with $\mathrm{cr}_D(P_2+C_m) < \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor+1$. Since P_2+C_m contains a subgraph isomorphic to $K_{3,m}$ and $\mathrm{cr}_D(P_2+C_m) \geq \mathrm{cr}(K_{3,m}) = \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$, so $\mathrm{cr}_D(P_2+C_m) = \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$.

We partite the edges of P_2+C_m into three parts: these edges in P_2, these edges in C_m, and these edges in $K_{3,m}$. Obviously, there isn't any crossing on the edges of P_2 or C_m, or else deleting the crossed edges of P_2 or C_m from D, we get a graph containing a subgraph isomorphic to $K_{3,m}$ and a drawing with crossing number less than $\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$, a contradiction! Let x_1, x_2, x_3 be the vertices of P_2, and let $v_1, v_2, \ldots, v_m$ be the vertices of C_m, see Figure 1. Then x_1, x_2, x_3 must lie in the same region of C_m, without loss of generality, we may assume that they lie in the finite region. The edges x_1v_i $(1 \leq i \leq m)$ divide the finite region into m subregions, in one of which x_2 lies. Thus the m edges x_2v_i $(1 \leq i \leq m)$ must cross those edges adjacent to x_1 at least $\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$ times. Similarly, the edges adjacent to x_3 also contribute at least $\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$ crossings. None of these crossings are counted more than once, so there are at least $2\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$ crossings in D, a contradiction to our assumption. $\square$

According to Lemmas 2 and 3, it is easy to get that $\mathrm{cr}(W_m \hat{\times} P_{n-2}) = (n-1)\mathrm{cr}(\hat{W}_m) = (n-1)(\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor+1)$.

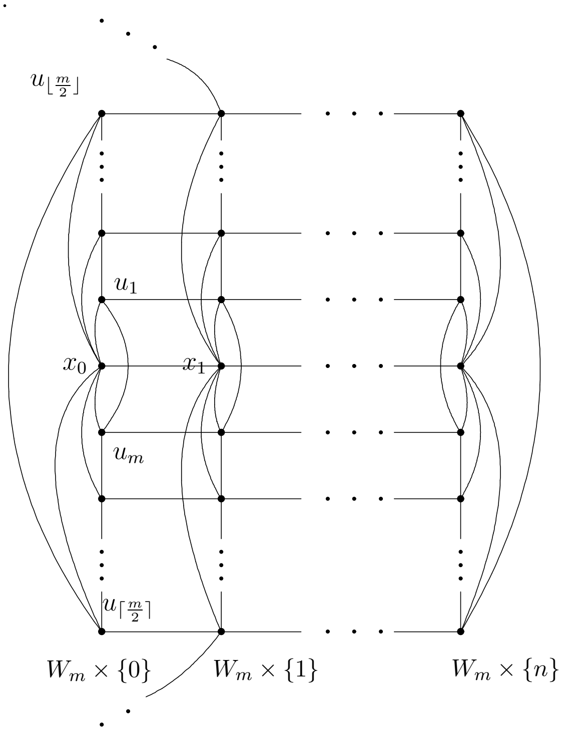

Figure 2 A good drawing of $W_m \times P_n$

3. The proof of Theorem 1

We always assume that $m \geq 4$ and $n \geq 4$ in the proof since the results in [3,7,9] are consistent with Theorem 1. It will be convenient to consider the graph $W_m \times P_n$ in the following way: it has $(m+1)(n+1)$ vertices and edges in the $n+1$ copies $W_m \times \{i\}$, $i = 0, 1, \ldots, n$, and in the $m+1$ paths of length n (see Figure 2). Furthermore, we color the former edges red and the latter ones blue.

Proof Figure 2 shows that $\mathrm{cr}(W_m \times P_n) \leq (n-1)(\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1) + 2$. Now we move to the proof of the reverse inequality.

Let D be an optimal drawing of $W_m \times P_n$. We denote the $m-$cycles of $W_m \times \{0\}$ and $W_m \times \{n\}$ by C_m^0 and C_m^n, respectively. The following two cases are discussed:

Case 1 Suppose that there is not any crossing on the edges of C_m^0 or C_m^n in D. We may assume, without loss of generality, that there is not any crossing on the edges of C_m^0. We denote the dominating vertex of $W_m \times \{0\}$ by x_0, and the vertices of C_m^0 by $u_1, u_2, \ldots, u_m$.

Since there is not any crossing on the edges of C_m^0, x_0 and all of the other vertices of $W_m \times \{1\}$, $W_m \times \{2\}, \ldots, W_m \times \{n\}$ must lie in the same region of C_m^0. Without loss of generality, we may assume that they lie in the interior region of C_m^0. Let $|A_i|$ denote the number of crossings on the edge x_0u_i $(i = 1, 2, \ldots, m)$. Without loss of generality, we assume that $|A_1| = \min\limits_{1 \leq i \leq m} \{A_i\}$.

Now we move to obtain a good drawing D' of a new graph which is homeomorphic to $W_m \hat{\times} P_{n-2}$: firstly, deleting the edges x_0u_i $(i = 2, 3, \ldots, m)$ from D; secondly, drawing curves connecting u_1u_i along the exterior region of C_m^0 $(i = 3, 4, \ldots, m-1)$ such that there is not any crossing on the curves u_1u_i $(i = 3, 4, \ldots, m-1)$; thirdly, deleting u_iu_{i+1} $(i = 2, 3, \ldots, m-1)$ from D (see Figure 3); and lastly, deleting the edges of C_m^n from $W_m \times \{n\}$. We distinguish two subcases:

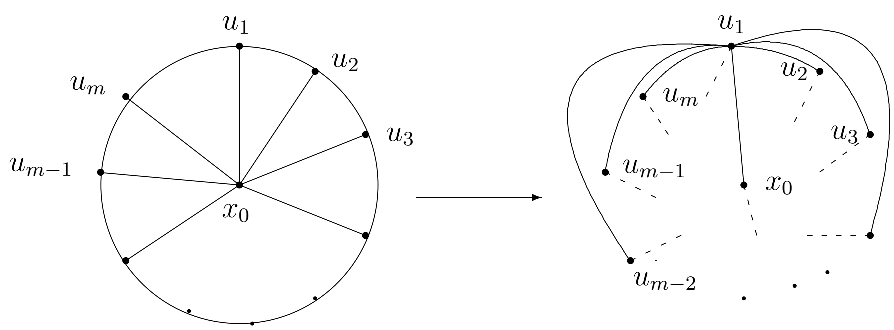

Figure 3 A good drawing D' produced from D

Subcase 1.1 Suppose that $|A_1| = 0$. Let x_1 denote the dominating vertex of $W_m \times \{1\}$. If all of the vertices of $W_m \times \{1\}$ except x_1 lie in the same region, without loss of generality, we may assume that they lie in the region $x_0u_2u_3$, then there will be at least $m-2$ crossings on the edges of x_0u_2 and x_0u_3 made by the blue edges adjacent to u_i $(i \neq 2, 3)$. If all of the vertices of $W_m \times \{1\}$ except x_1 lie in different regions of C_m^0, then there will be at least 2 crossings on the edges x_0u_i $(i \neq 1)$ made by the $m-$cycle of $W_m \times \{1\}$. Thus the first step decreases at least 2 crossings.

Subcase 1.2 Suppose that $|A_1| \neq 0$, that is $|A_1| \geq 1$. It is easy to see that the first step decreases at least $(m-1)|A_1|$ crossings.

Therefore, the first step decreases at least 2 crossings no matter $|A_1| = 0$ or not. The latter three steps don't increase the number of crossings, so we have

$$\mathrm{cr}_D(W_m \times P_n) \geq \mathrm{cr}_{D'}(W_m \hat{\times} P_{n-2}) + 2 \geq (n-1)(\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1) + 2.$$

Case 2 Suppose that both C_m^0 and C_m^n are crossed in D. If the crossings are made between the edges of C_m^0 and C_m^n, then they must cross at least twice; if not, the edges of C_m^0 and C_m^n are crossed at least once respectively. Deleting the edges of C_m^0 and C_m^n from D, we obtain a graph homeomorphic to $W_m \hat{\times} P_{n-2}$ with at least $(n-1)(\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1)$ crossings. So we have

$$\mathrm{cr}_D(W_m \times P_n) \geq (n-1)(\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1) + 2.$$

In all, for an optimal drawing D of $W_m \times P_n$, we have $\mathrm{cr}_D(W_m \times P_n) \geq (n-1)(\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 1) + 2$, which completes the proof of Theorem 1. □

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