Journal of Mathematical Research & Exposition
Mar., 2009, Vol. 29, No. 2, pp. 367–370
DOI:10.3770/j.issn:1000-341X.2009.02.022
Http://jmre.dlut.edu.cn

The Torsion-Freeness of Partially Ordered K_0-Groups for a Class of Exchange Rings

WU Kuo Hua, *LÜ Xin Min*
(Faculty of Science, Jiangxi University of Science and Technology, Jiangxi 341000, China)
(E-mail: wkhlxy@sohu.com)

Abstract A ring R is called orthogonal if for any two idempotents e and f in R, the condition that e and f are orthogonal in R implies the condition that $[eR]$ and $[fR]$ are orthogonal in $K_0(R)^+$, i.e., $[eR]\wedge[fR]=0$. In this paper, we shall prove that the K_0-group of every orthogonal, IBN_2 exchange ring is always torsion-free, which generalizes the main result in [3].

Keywords IBN_2 ring; Orthogonal ring; K_0-group; Partially ordered Abelian group; ℓ-group.

Document code A
MR(2000) Subject Classification 16D90; 16E20; 06F20; 06F15
Chinese Library Classification O153.3

1. Introduction

Throughout, all rings are associative with identity and all modules are unitary right R-modules. For a ring R, we denote by $\mathrm{FP}(R)$ the class of all finitely generated projective right R-modules. A ring R is said to be directly finite if for $x,y\in R$, $xy=1$ implies $yx=1$. A ring R is said to be stably finite or an IBN_2 ring if all matrix rings $M_n(R)$ over R are directly finite for any positive integers n. According to [2, Chapter 15], there is a natural way to make $K_0(R)$ into a pre-order abelian group, as follows: For any $x,y\in K_0(R)$, $x\le y$ if and only if $y-x\in K_0(R)^+$). We call the pre-order on $K_0(R)$ the natural pre-order or the algebraic pre-order on $K_0(R)$.

A partially ordered Abelian group G is an Abelian group that is also a partially ordered set such that for any $a,b,c\in G$, $c+a+d\le c+b+d$ whenever $a\le b$. G^+ will denote the set $\{a\in G: a\ge 0\}$, and is usually called the positive cone of G. $a,b\in G$ are said to be orthogonal if $a\wedge b$ exists in G and $a\wedge b=0$. A partially ordered Abelian group G is said to be archimedean if for $x,y\in G$, the condition $nx\le y$ for all positive integers n implies $x\le 0$. A partially ordered Abelian group G is said to be an ℓ-group if the underlying order is a lattice. According to [1, Proposition 3.5], every ℓ-group is torsion free. In this paper, we need the following criterion of ℓ-groups: A partially ordered Abelian group G is an ℓ-group if and only if for any $g\in G$, there exist $a,b\in G$ such that $a\wedge b=0$, and $g=a-b$.

Received date: 2007-02-07; **Accepted date**: 2007-07-13
Foundation item: the National Natural Science Foundation of China (No. 10571080); the Natural Science Foundation of Jiangxi Province (No. 0611042); the Science and Technology Projiet Foundation of Jiangxi Province (No. G[2006]194) and the Doctor Foundation of Jiangxi University of Science and Technology.

2. Main results

Definition 1 *A ring R is to be orthogonal if for any idempotents e and f in R, e and f are orthogonal in R implies that $[eR]$ and $[fR]$ are orthogonal in $K_0(R)^+$, i.e., $[eR] \wedge [fR] = 0$.*

Let us first consider some examples of orthogonal rings.

Example 2 Let Q be the field of rational numbers, and $R = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$. According to [3, Example 3.8], we see that R is a generalized Abelian exchange ring with stable range 1. The class of such rings is usually denoted by **GAERS**-1. In view of [3, Proposition 3.7], R is an orthogonal ring.

It should be pointed out that an orthogonal ring need not be in the class **GAERS**-1 introduced and studied in [3]. For example, let $R = \mathbb{Z}$. Clearly, R is an orthogonal ring, but $R \notin$ **GAERS**-1. It follows that the class **GAERS**-1 is, in fact, a proper subclass of the class of orthogonal rings.

Now we shall investigate the structure of the K_0-groups of orthogonal and IBN_2 exchange rings. In order to do this, we need the following lemma.

Lemma 3 *Let R be an orthogonal and IBN_2 ring. Then for any two orthogonal idempotents e and f in R, and any two positive integers m and n, we have*

$$m[eR] \wedge n[fR] = 0, \quad m[eR] \vee n[fR] = m[eR] + n[fR]$$

Proof First, we should notice that if for any $[eR]$, $[fR] \in K_0(R)$, $[eR] \wedge [fR]$ exists in $K_0(R)$, then for any positive integer s, $s([eR] \wedge [fR])$ must exist in $K_0(R)$, and

$$s([gR] \wedge [hR]) = s[gR] \wedge \{(s-1)[gR] + [hR]\} \wedge \cdots \wedge \{[gR] + (s-1)[hR]\} \wedge s[hR].$$

Now, take $s = 2k$, where $k = \max\{m, n\}$. Then we have

$$0 \leq m[eR] \wedge n[fR] \leq k[gR] \wedge k[hR] \leq s([gR] \wedge [hR]) = 0.$$

It follows that $m[eR] \wedge n[fR]$ exists in $K_0(R)$, and $m[eR] \wedge n[fR] = 0$. Since $K_0(R)$ is a partially ordered Abelian group, and $m[eR] \wedge n[fR]$ exists in $K_0(R)$, we have that $m[eR] \vee n[fR]$ exists in $K_0(R)$, and

$$m[eR] \vee n[fR] = m[eR] + n[fR] - m[eR] \wedge n[fR] = m[eR] + n[fR].$$

For any a given ring R and any $x \in K_0(R)$, $x = [A] - [B]$ for suitable $A, B \in FP(R)$. Now we shall construct a special subset $k_0(R)$ of $K_0(R)$ satisfied the following conditions:

(1) For any $x \in k_0(R)$, there exist pairwise orthogonal idempotents $e_1, \ldots, e_k$ in R and positive integers $n_1, \ldots, n_k$ such that

$$x = n_1[e_1R] + \cdots + n_s[e_sR] - n_{s+1}[e_{s+1}R] - \cdots - n_k[e_kR].$$

(2) For any $x, y \in k_0(R)$, there exist pairwise orthogonal idempotents $e_1, \ldots, e_k$ in R, and integers $m_1, \ldots, m_k$ and $n_1, \ldots, n_k$ such that

$$x = m_1[e_1R] + \cdots + m_k[e_kR], \quad y = n_1[e_1R] + \cdots + n_k[e_kR].$$

Clearly, $k_0(R)$ is an Abelian subgroup of $K_0(R)$. In particular, if R is an IBN_2 ring, then it is also a partially ordered Abelian subgroup of $K_0(R)$.

We now prove the main result of this paper.

Theorem 4 *If R is an orthogonal and IBN_2 ring, then $k_0(R)$ is Archimedean ℓ-subgroup of $K_0(R)$, i.e., $k_0(R)$ itself is an Archimeden ℓ-group.*

Proof First, we show that $k_0(R)$ is an ℓ-group. Notice that $k_0(R)$ is a partially ordered Abelian group. So, according to [1, Proposition 3.5], it suffices to show that for any $x \in k_0(R)$, there exist $a, b \in K_0(R)$ such that $x = a - b$, and $a \wedge b = 0$.

Now, for any $x \in k_0(R)$, by assumption, there exist pairwise orthogonal idempotents $e_1, \dots, e_s, e_{s+1}, \dots, e_k$ in R and nonnegative integers $n_1, \dots, n_s, p_{s+1}, \dots, p_k$ such that

$$x = n_1[e_1R] + \cdots + n_s[e_sR] - p_{s+1}[e_{s+1}R] + \cdots - p_k[e_kR].$$

Let

$$a = n_1[e_1R] + \cdots + n_s[e_sR], \quad b = p_{s+1}[e_{s+1}R] + \cdots + p_k[e_kR].$$

Then we have that $x = a - b$. Since $[e_1R], [e_2R], \dots, [e_kR]$ are pairwise orthogonal, we get

$$\begin{aligned} a \wedge b &= (n_1[e_1R] + \cdots + n_s[e_sR]) \wedge (p_{s+1}[e_{s+1}R] + \cdots + p_k[e_kR]) \\ &= (n_1[e_1R] \vee \cdots \vee n_s[e_sR]) \wedge (p_{s+1}[e_{s+1}R] \vee \cdots \vee p_k[e_kR]) \\ &= (n_1[e_1R] \wedge p_{s+1}[e_{s+1}R]) \vee \cdots \vee (n_s[e_sR] \wedge p_k[e_kR]) \\ &= 0. \end{aligned}$$

It follows that $k_0(R)$ is an ℓ-group.

Secondly, we shall show that $k_0(R)$ has the Archimedean property. Given any $x, y \in G$ with the condition $nx \leq y$ for all positive integers n, by assumption, there exist pairwise orthogonal idempotents $e_1, \dots, e_s, e_{s+1}, \dots, e_k$ in R and integers $\{p_i\}_{i=1}^k$ and $\{q_i\}_{i=1}^k$ such that

$$x = p_1[e_1R] + \cdots + p_k[e_kR], \quad y = q_1[e_1R] + \cdots + q_k[e_kR].$$

In order to prove $x \leq 0$, we need show that for each $p_i \leq 0$, where $i = 1, \dots, k$.

Suppose by way of contradiction that there exist some i such that $p_i < 0$. Without loss of generality, we may further assume that $p_1 > 0$. According to the inequality $nx \leq y$, we have

$$n(p_1[e_1R] + \cdots + p_k[e_kR]) \leq q_1[e_1R] + \cdots + q_k[e_kR].$$

Then

$$(np_1 - q_1)[e_1R] + (np_2 - q_2)[e_2R] + \cdots + (np_k - q_k)[e_kR]) \leq 0.$$

Since $p_1 > 0$, and n is any a positive integer, we can choose a positive integer n_0 such that $n_0p_1 - q_1 > 0$, while $n_0p_i - q_i > 0$ for $i = 2, \dots, s$ $(2 \leq s \leq k)$, and $n_0p_j - q_j < 0$ for $j = s+1, \dots, k$. For convenience, let $r_i = n_0p_i - q_i$ for $i = 1, 2, \dots, s$, and $t_j = -n_0p_j + q_j > 0$ for $j = s+1, \dots, k$. So we have

$$r_1[e_1R] + r_2[e_2R] + \cdots + r_s[e_sR] \leq t_{s+1}[e_{s+1}R] + \cdots + t_k[e_kR]).$$

Similarly, since $[e_1R],[e_2R],\ldots,[e_kR]$) are pairwise orthogonal, we have

$$\begin{aligned}[e_1R] &= [e_1R]\wedge r_1[e_1R]\\ &= [e_1R]\wedge(r_1[e_1R]\vee r_2[e_2R]\vee\cdots\vee r_s[e_sR])\\ &= [e_1R]\wedge(r_1[e_1R]+r_2[e_2R]+\cdots+r_s[e_sR])\\ &\le [e_1R]\wedge(t_{s+1}[e_{s+1}R]+\cdots+t_k[e_kR])\\ &= [e_1R]\wedge(t_{s+1}[e_{s+1}R]\vee\cdots\vee t_k[e_kR])\\ &= 0.\end{aligned}$$

This is a contradiction. So, each $p_i \le 0$ for $i = 1,\ldots,k$. Hence $k_0(R)$ is an Archimedean ℓ-group. □

Following [4], we say that a ring R is an exchange ring if for every R-module A_R and any decompositions

$$A = B \oplus C = (\bigoplus_{i\in I} A_i) \text{ with } B \cong R_R$$

as right R-modules, there exist submodules $A_i' \subseteq A_i$ for each $i \in I$ such that $A = B \oplus (\bigoplus_{i\in I} A_i')$. In view of [5, Corollary 2.2], for any an exchange ring R, $k_0(R) = K_0(R)$. So, as a corollary of Theorem 4, we have

Corollary 5 *Let R be an orthogonal and* IBN_2 *ring. If R is an exchange ring, then $K_0(R)$ is an Archimedean ℓ-group. So, in this case, $K_0(R)$ is always torsion free.*

References

[1] DARNEL M R. *Theory of Lattice-Ordered Groups* [M]. Marcel Dekker, Inc., New York, 1995.

[2] GOODEARL K R. *Von Neumann Regular Rings* [M]. Second Edition. Robert E. Krieger Publishing Co., Inc., Malabar, FL, 1991.

[3] LU Xinmin, QIN Hourong. *Boolean algebras, generalized abelian rings, and Grothendieck groups* [J]. Comm. Algebra, 2006, **34**(2): 641–659.

[4] WARFIELD R B JR. *Exchange rings and decompositions of modules* [J]. Math. Ann., 1972, **199**: 31–36.

[5] WU Tongsuo, TONG Wenting. *Finitely generated projective modules over exchange rings* [J]. Manuscripta Math., 1995, **86**(2): 149–157.