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Improved Local Wellposedness of Cauchy Problem for Generalized KdV-BO Equation

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Abstract In this paper we prove that the Cauchy problem associated with the generalized KdV–BO equation $u_t + u_{xxx} + \lambda\mathcal{H}(u_{xx}) + u^2u_x = 0$, $x \in R$, $t \geq 0$ is locally wellposed in $\widehat{H}^s_r(R)$ for $\frac{4}{3} < r \leq 2$, $b > \frac{1}{r}$ and $s \geq s(r) = \frac{1}{2} - \frac{1}{2r}$. In particular, for $r = 2$, we reobtain the result in [3].

Keywords KdV-BO equation; Cauchy problem; local wellposedness.

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1. Introduction

We investigate the initial value problem (IVP) of the generalized KdV-BO (Korteweg-de Vries-Benjamin-Ono) equation:

$$u_t + u_{xxx} + \lambda\mathcal{H}(u_{xx}) + u^k u_x = 0,\ x \in R,\ t \geq 0, \tag{1}$$

$$u(0,x) = u_0(x),\ x \in R, \tag{2}$$

where $\lambda > 0$ and $\mathcal{H}$ denotes the usual Hilbert transform. The integro-differential equation (1) models the unidirectional propagation of long waves in a two-fluid system, where the lower fluid with greater density is infinitely deep and the interface is subject to capillarity. It was derived by Benjamin[1] to study gravity-capillary surface waves of solitary type on deep water.

Linares[2] showed that there exists a local and global solution for the IVP (1)–(2) with initial data in L^2 with constant coefficient $\lambda > 0$ if $k = 1$.

Guo and Huo[4] proved that IVP (1)–(2) is locally wellposed with data in $H^s(R)$ for $s > -\frac{1}{8}$, if $k = 1$; $s \geq \frac{1}{4}$, if $k = 2$.

Notice that in equation (1), the dispersive term u_{xxx} plays the most important role and the Benjamin-Ono $\mathcal{H}(u_{xx})$ term can be treated as a nonlinear term. Therefore, we expect the IVP (1)–(2) has the similar wellposeness to that of the modified KdV equation:

$$v_t + v_{xxx} + v^k v_x = 0,\ x \in R,\ t \geq 0, \tag{3}$$

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$$v(0,x)=v_0(x),\ x\in R. \tag{4}$$

Indeed, from this point of view, Huo and Guo[3] proved that IVP (1)–(2) is locally wellposed with data in $H^s(R)$ for $s>-\frac{3}{4}$, if $k=1$; $s\ge\frac{1}{4}$, if $k=2$; $s>-\frac{1}{6}$, if $k=3$. Moreover, the solutions of IVP (1)–(2) converge to the solutions of IVP (3)–(4) if λ tends to zero.

Though for $k=1$, Guo and Huo in [4] improved the well-posedness of IVP (1)–(2) a great deal, it did not improve the wellposedness for the case of $k=2$. As Grünrock did in [5], by proving the wellposedness of IVP (1)–(2) in $\widehat{H}^s_r$, we manage to improve the result of the case $k=2$ to some extent. To this end, let us recall some definitions first.

Definition 1[5] *For $s\in R$, $1\le r\le\infty$, we define the space $\widehat{H^r_s}$ by*

$$\widehat{H^r_s}=\{f\in S'(R):\|f\|_{\widehat{H^r_s}}<\infty\}$$

with

$$\|f\|_{\widehat{H^r_s}}:=\|\langle\xi\rangle^s\hat{f}\|_{L^{r'}_\xi},$$

where $\langle\xi\rangle=(1+|\xi|)$, $\frac{1}{r}+\frac{1}{r'}$.

Definition 2[5] *For s, $b\in R$, $1\le r\le\infty$, we define the space $X^r_{s,b}$ to be the completion of the Schwartz function space on R^2 with respect to the norm*

$$\|u\|_{X^r_{s,b}}=\|\langle\xi\rangle^s\langle\tau-\xi^3\rangle^b\hat{u}\|_{L^{r'}_{\xi\tau}}=\left(\int \mathrm{d}\xi\mathrm{d}\tau\langle\xi\rangle^{sr'}\langle\tau\rangle^{b'r'}|\mathcal{F}(U(-\cdot)f)(\tau,\xi)|^{r'}\right)^{\frac{1}{r'}},$$

where $\langle\xi\rangle=(1+|\xi|)$, $\frac{1}{r}+\frac{1}{r'}$. $\mathcal{F}u=\hat{u}(\tau,\xi)$ denotes the Fourier transform in variables t and x of u. Denote by $\mathcal{F}_{(\cdot)}u$ the Fourier transform in the $(\cdot)$ variable. $U(t)=\mathcal{F}_x^{-1}e^{it\xi^3}\mathcal{F}_x$ is the unitary operator associated with the Airy equation.

Deduced directly from the definition, the embedding relation

$$\|u\|_{X^r_{s_1,b_1}}\le\|u\|_{X^r_{s_2,b_2}}$$

holds if $s_1\le s_2$, $b_1\le b_2$; for $r=2$, we reobtain Bourgain space $X_{s,b}$.[1]

For $b>\frac{1}{r}$, we have

$$X^r_{s,b}\subset C(R,\widehat{H^r_s}).$$

Let $\psi\in C_0^\infty(R)$ with $\psi\equiv1$ on $[-\frac{1}{2},\frac{1}{2}]$ and supp$\psi\subseteq(-1,1)$. Denote $\psi_\delta(t)=\psi(\frac{t}{\delta})$. Then the time restricted space is defined by

$$X^r_{s,b}(\delta):=\{f=\tilde{f}|_{[-\delta,\delta]\times R}=\psi_\delta(t)\tilde{f}:\tilde{f}\in X^r_{s,b}\},$$

endowed with the norm

$$\|f\|_{X^r_{s,b}}(\delta):=\inf\{\|\tilde{f}\|_{X^r_{s,b}}:\tilde{f}|_{[-\delta,\delta]\times R}=f\}.$$

Our main result reads as follows (for $k=2$):

Theorem *Let $\frac{4}{3}<r\le2$. Then for any $b>\frac{1}{r}$ and $s\ge s(r)=\frac{1}{2}-\frac{1}{2r}$, there exist $T=$*

[1] We call $X_{s,b}$ Bourgain space because it was first introduced by Bourgain in [7]. The norm of $X_{s,b}$ is given by $\|u\|_{X_{s,b}}=\|\langle\xi\rangle^s\langle\tau-\xi^3\rangle^b\hat{u}(\tau,\xi)\|_{L^2_{\xi\tau}}$.

$T(\|u_0\|_{\widehat{H^r_s}}) > 0$ and a unique solution $u \in C([0,T],\widehat{H^r_s}) \cap X^r_{s,b}$ of IVP (1)–(2) with $u_0(x) \in \widehat{H^r_s}$. Moreover, for any given $t \in (0,T)$, the mapping $u_0(x) \to u(t,x)$ is Lipschitz continuous from $\widehat{H^r_s}$ to $C([0,T],\widehat{H^r_s})$.

In the sequel, C will denote a constant which may differ at each appearance, possibly depending on the dimension or other parameters.

2. Estimate of the Benjamin-Ono term

In this section, we concentrate on the estimate of Benjamin-Ono term $\mathcal{H}(u_{xx})$.

Lemma 1 *Let $s \in R$, $b'+1 > b > 0 > b' > -\frac{1}{r'}$, and $b - b' > \frac{2}{3}$, $1 < r < \infty$. We have*

$$\|\mathcal{H}(u_{xx})\|_{X^r_{s,b'}} \le C\|u\|_{X^r_{s,b}}. \tag{5}$$

To prove this lemma, we prove the following preliminary lemma in advance. We agree on that the eigenfunction $\chi_{|\tau|\sim|\xi|^3}$ equals 1 if $|\tau| \sim |\xi|^3$; equals 0, otherwise. Define the operator $\mathcal{P}$ by $\mathcal{P}f(t,x) = \mathcal{F}^{-1}\chi_{|\tau|\sim|\xi|^3}\mathcal{F}f(\tau,\xi)$.

Lemma 2 *Let $s \in R$, $b'+1 > b > 0 > b' > -\frac{1}{r'}$, and $b - b' > \frac{2}{3}$, $1 < r < \infty$. Assume that the Fourier transform $\mathcal{F}f = \hat{f}(\tau,\xi)$ of f is supported in $\{(\tau,\xi) : |\xi| > 1\}$. Then there holds that*

$$\|\mathcal{P}f\|_{X^r_{s,b'}} \le C\|f\|_{X^r_{s-2,b}}.$$

Proof We have

$$\begin{aligned}
\|\mathcal{P}f\|_{X^r_{s,b'}} &= \Big(\int \mathrm{d}\xi\mathrm{d}\tau \langle\xi\rangle^{sr'}\langle\tau\rangle^{b'r'}|\mathcal{F}[U(-\cdot)\mathcal{P}f](\tau,\xi)|^{r'}\Big)^{\frac{1}{r'}}\\
&= \Big(\int \mathrm{d}\xi\mathrm{d}\tau \frac{\langle\xi\rangle^{2r'}}{\langle\tau\rangle^{(b-b')r'}}\langle\xi\rangle^{(s-2)r'}\langle\tau\rangle^{br'}|\mathcal{F}[U(-\cdot)\mathcal{P}f](\tau,\xi)|^{r'}\Big)^{\frac{1}{r'}}\\
&\le \Big(\int \mathrm{d}\xi\mathrm{d}\tau \frac{\langle\xi\rangle^{2r'}\chi_{|\tau|\sim|\xi|^3}}{\langle\tau\rangle^{(b-b')r'}}\langle\xi\rangle^{(s-2)r'}\langle\tau\rangle^{br'}|\mathcal{F}[U(-\cdot)\mathcal{P}f](\tau,\xi)|^{r'}\Big)^{\frac{1}{r'}}\\
&\le \Big(\int \mathrm{d}\xi\mathrm{d}\tau \langle\xi\rangle^{(s-2)r'}\langle\tau\rangle^{br'}|\mathcal{F}[U(-\cdot)\mathcal{P}f](\tau,\xi)|^{r'}\Big)^{\frac{1}{r'}}\\
&= C\|f\|_{X^r_{s-2,b}}.
\end{aligned}$$

This completes the proof. □

Proof of Lemma 1

Case 1 If the Fourier transform $\mathcal{F}\mathcal{H}(\partial_{xx}u)$ is supported in $\{(\tau,\xi) : |\xi| \le 1\}$, we obtain

$$\begin{aligned}
\|\mathcal{H}(u_{xx})\|_{X^r_{s,b'}} &= \Big\|\frac{\langle\xi\rangle^s}{\langle\tau-\xi^3\rangle^{-b'}}\mathcal{F}\mathcal{H}(u_{xx})(\tau,\xi)\Big\|_{L^{r'}_{\tau,\xi}}\\
&= \Big\|\frac{\xi|\xi|}{\langle\tau-\xi^3\rangle^{b-b'}}\langle\xi\rangle^s\langle\tau-\xi^3\rangle^b\hat{u}(\tau,\xi)\Big\|_{L^{r'}_{\tau,\xi}}\\
&\le C\|u\|_{X^r_{s,b}}.
\end{aligned}$$

Case 2 If the Fourier transform $\mathcal{F}\mathcal{H}(u_{xx})$ is supported in $\{(\tau,\xi) : |\xi| \ge 1\}$, we rewrite $\mathcal{H}(u_{xx})$ as $\mathcal{H}(u_{xx}) = \mathcal{P}\mathcal{H}(u_{xx}) + (1-\mathcal{P})\mathcal{H}(u_{xx})$ and estimate the two terms respectively.

For $\mathcal{PH}(u_{xx})$, by Lemma 2, we have

$$\|\mathcal{PH}(u_{xx})\|_{X^r_{s,b'}} \le C\|\mathcal{H}(u_{xx})\|_{X^r_{s-2,b}} \le C\|u\|_{X^r_{s,b}}.$$

For $(1-\mathcal{P})\mathcal{H}(u_{xx})$, it is clear that $\mathcal{F}(1-\mathcal{P})\mathcal{H}(u_{xx})$ is supported in

$$\{(\tau,\xi) : |\tau| \ll |\xi|^3 \text{ or } |\tau| \gg |\xi|^3\},$$

correspondingly, we have either $|\tau-\xi^3| \sim |\xi|^3$ or $|\tau-\xi^3| \gg |\xi|^3$. Then we have

$$\begin{aligned}
&\|(1-\mathcal{P})\mathcal{H}(u_{xx})\|_{X^r_{s,b'}}\\
&= \Big\|\frac{\xi|\xi|}{\langle\tau-\xi^3\rangle^{b-b'}}\langle\xi\rangle^s\langle\tau-\xi^3\rangle^b\chi_{|\tau-\xi^3|\sim|\xi|^3 \text{ or } |\tau-\xi^3|\gg|\xi|^3}\hat{u}(\tau,\xi)\Big\|_{L^{r'}_{\tau\xi}}\\
&\le C\|u\|_{X^r_{s,b}}.
\end{aligned}$$

Thus, collecting all the estimates together, we obtain

$$\|\mathcal{H}(u_{xx})\|_{X^r_{s,b'}} \le \|\mathcal{PH}(u_{xx})\|_{X^r_{s,b'}} + \|(1-\mathcal{P})\mathcal{H}(u_{xx})\|_{X^r_{s,b'}} \le C\|u\|_{X^r_{s,b}}.$$

3. Proof of the main results

In order to smooth the proof of the main result, let us recall some facts as lemmas first.

Lemma 3[5] *1) For $\phi \in \widehat{H^r_s}$, we have*

$$\|\psi(t)S(t)\phi\|_{X^r_{s,b}} \le C_\psi\|\phi\|_{\widehat{H^r_s}}.$$

2) Assume $1 < r < \infty$ and $b'+1 \ge b \ge 0 \ge b' > -\frac{1}{r'}$. Then

$$\Big\|\psi_\delta(t)\int_0^t S(t-t')f(t')dt'\Big\|_{X^r_{s,b}} \le C\delta^{1+b'-b}\|f\|_{X^r_{s,b'}}.$$

Lemma 4[5] *Let $\frac{4}{3} < r \le 2$ and $s \ge s(r) = \frac{1}{2} - \frac{1}{2r}$. Then for all $b' < \frac{1}{2r} - \frac{5}{8}$ and $b > \frac{1}{r}$ the estimate*

$$\|(\Pi_{i=1}^3 u_i)_x\|_{X^r_{s,b'}} \le C\Pi_{i=1}^3\|u_i\|_{X^r_{s,b}}$$

holds true.

Lemma 5 *Let $\frac{4}{3} < r \le 2$ and $s \ge s(r) = \frac{1}{2} - \frac{1}{2r}$. Then for all $b' < \frac{1}{2r} - \frac{5}{8}$ and $b > \frac{1}{r}$ the estimates*

$$\|u^2u_x\|_{X^r_{s,b'}} \le C\|u\|^3_{X^r_{s,b}} \tag{6}$$

and

$$\|u^2u_x - v^2v_x\|_{X^r_{s,b'}} \le C[\|u\|^2_{X^r_{s,b}} + \|v\|^2_{X^r_{s,b}}]\|u-v\|_{X^r_{s,b}} \tag{7}$$

hold true.

Proof Inequality (6) is a direct result of Lemma 4. Inequality (7) follows from Lemma 4 and

$$\begin{aligned}
3(u^2u_x - v^2v_x) &= (u^3-v^3)_x = [(u-v)(u^2+uv+v^2)]_x\\
&= [(u-v)u^2]_x + [(u-v)uv]_x + [(u-v)v^2]_x.
\end{aligned}$$

Sketch of proof of the Theorem Take $b = \frac{1}{r} + \varepsilon$, $b' = -\frac{1}{r'} + 2\varepsilon$, for $0 < \varepsilon < \frac{1}{2r'}$. Then $b' + 1 > b > 0 > b' > -\frac{1}{r'}$, and $b - b' > \frac{2}{3}$, $b > \frac{1}{r}$.

For $u \in X^r_{s,b}(\delta)$ with extension $\widetilde{u} \in X^r_{s,b}$, define operator

$$\mathcal{T}(u) = \psi(t)U(t)u_0 - i\psi_\delta(t)\int_0^t U(t-\tau)F(u(\tau))\mathrm{d}\tau,$$

then extension of $\mathcal{T}(u)$ is given by

$$\widetilde{\mathcal{T}}(u) = \psi(t)U(t)u_0 - i\psi_\delta(t)\int_0^t U(t-\tau)F(\widetilde{u}(\tau)\mathrm{d}\tau,$$

where $F(\widetilde{u}(\tau)) = (\widetilde{u}^3)_x + \lambda\mathcal{H}(\widetilde{u}_{xx})$.

By Lemma 3 and the first inequality of Lemma 5, we obtain

$$\begin{aligned}\|\mathcal{T}(u)\|_{X^r_{s,b}(\delta)} &\le \|\psi(t)U(t)u_0\|_{X^r_{s,b}} + \|\psi_\delta(t)\int_0^t U(t-\tau)F(\widetilde{u}(\tau))\mathrm{d}\tau\|_{X^r_{s,b}} \\ &\le C\|u_0\|_{\widehat{H^r_s}} + C\delta^{1-b+b'}\|\widetilde{u}\|^3_{X^r_{s,b}} + C\delta^{1-b+b'}\|\widetilde{u}\|_{X^r_{s,b}}.\end{aligned}$$

This holds for all extension $\widetilde{u} \in X^r_{s,b}$ of $u \in X^r_{s,b}(\delta)$. Hence

$$\|\mathcal{T}(u)\|_{X^r_{s,b}(\delta)} \le C\|u_0\|_{\widehat{H^r_s}} + C\delta^{1-b+b'}\|u\|^3_{X^r_{s,b}(\delta)} + C\delta^{1-b+b'}\|u\|_{X^r_{s,b}(\delta)}. \tag{8}$$

Similarly, by Lemma 3 and the second inequality of Lemma 5, for given $u \in X^r_{s,b}(\delta)$, $v \in X^r_{s,b}(\delta)$, we obtain

$$\|\mathcal{T}(u) - \mathcal{T}(v)\|_{X^r_{s,b}(\delta)} \le C\delta^{1-b+b'}(\|u\|^2_{X^r_{s,b}(\delta)} + \|v\|^2_{X^r_{s,b}(\delta)})\|u-v\|_{X^r_{s,b}(\delta)} + C\delta^{1-b+b'}\|u-v\|_{X^r_{s,b}(\delta)}. \tag{9}$$

Inequalitis (8) and (9) show that for $R = 2C\|u_0\|_{\widehat{H^r_s}}$ and $\delta^{1-b+b'} < \min\{\frac{1}{4CR^2}, \frac{1}{2}\}$, the mapping $\mathcal{T}$ is a contract mapping of the closed ball of radius R in $X^r_{s,b}(\delta)$ into itself. The Banach fixed point theorem now guarantees the existence of a solution of $\mathcal{T}(u) = u$. Because of $b > \frac{1}{r}$, any solution $u \in X^r_{s,b}(\delta)$ also belongs to $C([0,\delta], \widehat{H^r_s})$. Finally, the statement about continuous dependence can be shown in a straightforward manner using the same estimates as the above.

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