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Positive Solutions of Singular Boundary Value Problems of Fourth-Order Differential Equations

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Abstract The singular boundary value problem

$$\begin{cases} \varphi^{(4)}(x) - h(x)f(\varphi(x)) = 0, \quad 0 < x < 1, \\ \varphi(0) = \varphi(1) = \varphi'(0) = \varphi'(1) = 0 \end{cases}$$

is considered under some conditions concerning the first eigenvalues corresponding to the relevant linear operators, where $h(x)$ is allowed to be singular at both $x = 0$ and $x = 1$. The existence results of positive solutions are obtained by means of the cone theory and the fixed point index.

Keywords singular boundary value problems; positive solution; cone; fixed point index.

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1. Introduction and main results

The existence of positive solutions for nonlinear fourth-order two-point boundary value problems has been studied by many authors using nonlinear alternative of Leray-Schauder, coincidence degree theory and fixed point theorem[1,2]. In this paper, we consider the singular boundary value problem of fourth-order differential equations

$$\begin{cases} \varphi^{(4)}(x) - h(x)f(\varphi(x)) = 0, \quad 0 < x < 1; \\ \varphi(0) = \varphi(1) = \varphi'(0) = \varphi'(1) = 0, \end{cases} \tag{1.1}$$

where $h(x)$ is allowed to be singular at both $x = 0$ and $x = 1$. We obtain the existence results of positive solutions by means of the cone theory and the fixed point index under some conditions concerning the first eigenvalues corresponding to the relevant linear operators.

In the Banach space $C[0,1]$ in which the norm is defined by $\|\varphi\| = \max_{0\le x\le 1} |\varphi(x)|$, we set $P = \{\varphi \in C[0,1] \mid \varphi(x) \ge 0,\ x \in [0,1]\}$. P is a positive cone in $C[0,1]$. Throughout this paper, the partial ordering is always given by P. We denote by $B_r = \{\varphi \in C[0,1] \mid \|\varphi\| < r\}(r > 0)$ the open ball of radius r.

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Now let $G(x,y)$ be the Green's function of the linear problem $\varphi^{(4)}(x)=0$, $x\in(0,1)$ subject to the boundary conditions of problem (1.1), which can be explicitly given by

$$G(x,y)=\frac{1}{6}\begin{cases} x^2(1-y)^2[(y-x)+2(1-x)y], & 0\le x\le y\le 1,\\ y^2(1-x)^2[(x-y)+2(1-y)x], & 0\le y\le x\le 1.\end{cases} \tag{1.2}$$

We make the following assumptions:

(H_1) $h:(0,1)\to[0,+\infty)$ is continuous, $h(x)\not\equiv 0$, and $\int_0^1 x(1-x)h(x)\mathrm{d}x<+\infty$.

(H_2) $f:[0,+\infty)\to[0,+\infty)$ is continuous.

Set

$$(A\varphi)(x)=\int_0^1 G(x,y)h(y)f(\varphi(y))\mathrm{d}y, \tag{1.3}$$

$$(T\varphi)(x)=\int_0^1 G(x,y)h(y)\varphi(y)\mathrm{d}y. \tag{1.4}$$

φ is said to be positive solution of (1.1) if $\varphi\in C[0,1]\cap C^{(4)}(0,1)$, $\varphi(x)>0$, $x\in(0,1)$ and satisfies (1.1).

Theorem 1 *Suppose that the conditions (H_1), (H_2) are satisfied, and*

$$\liminf_{u\to 0^+}\frac{f(u)}{u}>\lambda_1,\quad \limsup_{u\to+\infty}\frac{f(u)}{u}<\lambda_1,$$

where λ_1 is the first eigenvalue of T. Then the singular boundary value problem (1.1) has at least one positive solution.

Theorem 2 *Suppose that the conditions (H_1), (H_2) are satisfied, and*

$$\liminf_{u\to+\infty}\frac{f(u)}{u}>\lambda_1, \tag{1.5}$$

$$\limsup_{u\to 0^+}\frac{f(u)}{u}<\lambda_1, \tag{1.6}$$

where λ_1 is the first eigenvalue of T. Then the singular boundary value problem (1.1) has at least one positive solution.

2. Preliminaries

In this section, we will establish several lemmas for the proof of our main results. Here Lemma 1 is directly obtained by (1.2) and Lemma 2 follows from the Arzera-Ascoli theorem.

Lemma 1 *The function $G(x,y)$ has the following properties:*

(i) $\frac{1}{3}x^2(1-x)^2y^2(1-y)^2\le G(x,y)\le\frac{1}{2}x(1-x)y^2(1-y)^2$, $\forall\, x,y\in[0,1]$;

(ii) $G(x,y)\le\frac{1}{2}x^2(1-x)^2y(1-y)$, $\forall\, x,y\in[0,1]$.

Lemma 2 *Suppose (H_1) and (H_2) are satisfied. Then $A:P\to P$ is a completely continuous operator.*

It is not difficult to verify that the nonzero fixed points of the operator A are the positive solutions of singular boundary value problem (1.1). In addition, we have from (H_1) that $T: C[0,1] \to C[0,1]$ is a completely continuous linear operator and $T(P) \subset P$.

Similar to Lemma 3 in [3], we have

Lemma 3 *Suppose that* (H_1) *is satisfied. Then for the operator* T *defined by (1.4), the spectral radius* $r(T) \neq 0$ *and* T *has a positive eigenfunction corresponding to its first eigenvalue* $\lambda_1 = r(T)^{-1}$.

Lemma 4 *Suppose that the conditions* (H_1) *and* (H_2) *are satisfied. If* $\varphi^* \in P$ *is the positive eigenfunction of* T *corresponding to its first eigenvalue* λ_1, *then*

(i) there exist $\delta_1, \delta_2 > 0$ *such that* $\delta_1 G(x,y) \leq \varphi^*(y) \leq \delta_2 y^2(1-y)^2, 0 \leq x, y \leq 1$.

(ii) for $\psi^*(x) = \varphi^*(x)h(x)$, *we have*

$$\int_0^1 \psi^*(x)\mathrm{d}x < +\infty, \quad \psi^*(x) = \lambda_1 \int_0^1 G(y,x)h(x)\psi^*(y)\mathrm{d}y, \quad x \in [0,1].$$

Proof (i) Since $\varphi^* \in P$ is the positive eigenfunction of T, it follows from Lemma 1 that $\varphi^*(y) \geq \frac{\lambda_1}{3} y^2(1-y)^2 \int_0^1 x^2(1-x)^2 h(x)\varphi^*(x)\mathrm{d}x$ and $\varphi^*(y) \leq \frac{\lambda_1}{2} y(1-y) \int_0^1 x^2(1-x)^2 h(x)\varphi^*(x)\mathrm{d}x$, therefore $\int_0^1 x^2(1-x)^2 h(x)\varphi^*(x)\mathrm{d}x > 0$. It follows from (ii) of Lemma 1 that $\varphi^*(y) \leq \frac{\lambda_1}{2} y^2(1-y)^2 \int_0^1 x(1-x)h(x)\varphi^*(x)\mathrm{d}x$. Set

$$\delta_1 = \frac{8\lambda_1}{3} \int_0^1 x^2(1-x)^2 h(x)\varphi^*(x)\mathrm{d}x, \quad \delta_2 = \frac{\lambda_1}{2} \int_0^1 x(1-x)h(x)\varphi^*(x)\mathrm{d}x.$$

Then we have

$$\delta_1 G(x,y) \leq \varphi^*(y) \leq \delta_2 y^2(1-y)^2, \quad 0 \leq x, y \leq 1.$$

(ii) Suppose $\psi^*(x) = \varphi^*(x)h(x)$. We have from (i) and ($\mathrm{H}_1$) that $\int_0^1 \psi^*(x)\mathrm{d}x \leq \delta_2 \int_0^1 x^2(1-x)^2 h(x)\mathrm{d}x < +\infty$. In addition, it follows from $G(x,y) = G(y,x)$ that

$$\begin{aligned} \lambda_1 \int_0^1 G(y,x)h(x)\psi^*(y)\mathrm{d}y &= \lambda_1 h(x) \int_0^1 G(y,x)h(y)\varphi^*(y)\mathrm{d}y \\ &= \lambda_1 h(x) \int_0^1 G(x,y)h(y)\varphi^*(y)\mathrm{d}y = \varphi^*(x)h(x) = \psi^*(x). \end{aligned}$$

□

3. Proof of main theorems

The proof of Theorem 1 is analogous to that in [3, Theorem 1], so we omit it. Now we only need to prove Theorem 2.

Proof of Theorem 2 We introduce

$$P_1 = \Big\{\varphi \in P \mid \int_0^1 \psi^*(x)\varphi(x)\mathrm{d}x \geq \lambda_1^{-1}\delta_1\|\varphi\|\Big\}, \tag{3.1}$$

where ψ^* and δ_1 are defined by Lemma 4. It is easy to check that P_1 is a cone in $C[0,1]$ and

$P_1 \subset P$. It follows from Lemma 4 that, for every $\varphi \in P$

$$\int_0^1 \psi^*(x)(T\varphi)(x)\mathrm{d}x = \lambda_1^{-1}\int_0^1 \psi^*(y)\varphi(y)\mathrm{d}y \geq \lambda_1^{-1}\delta_1 \int_0^1 G(x,y)h(y)\varphi(y)\mathrm{d}y = \lambda_1^{-1}\delta_1(T\varphi)(x).$$

Then $\int_0^1 \psi^*(x)(T\varphi)(x)\mathrm{d}x \geq \lambda_1^{-1}\delta_1\|T\varphi\|$, i.e., $T(P) \subset P_1$.

It follows from (1.5) that there exists $\varepsilon > 0$ such that $f(u) \geq (\lambda_1 + \varepsilon)u$ when u is sufficiently large. We know by (H_2) that there exists $b \geq 0$ such that $f(u) \geq (\lambda_1 + \varepsilon)u - b$. Take $R > (\varepsilon\delta_1)^{-1}b\lambda_1\int_0^1 \psi^*(x)\mathrm{d}x$. In the following, we prove

$$\varphi - A\varphi \neq \mu\varphi^*, \ \forall\ \varphi \in \partial B_R \cap P,\ \mu \geq 0, \tag{3.2}$$

where $\varphi^* \in P$ is the positive eigenfunction of T corresponding to its first eigenvalue λ_1. If otherwise, then there exist $\varphi_1 \in \partial B_R \cap P$ and $\mu_0 \geq 0$ such that

$$\varphi_1 - A\varphi_1 = \mu_0\varphi^*. \tag{3.3}$$

Since $T(P) \subset P_1$ and f is continuous, we have $A(P) \subset P_1$. It follows from (3.3) that $\varphi_1 \in P_1$. Therefore by Lemma 4 and (3.1), we have

$$\begin{aligned}
&\int_0^1 \psi^*(x)(A\varphi_1)(x)\mathrm{d}x - \int_0^1 \psi^*(x)\varphi_1(x)\mathrm{d}x \\
&\geq (\lambda_1+\varepsilon)\lambda_1^{-1}\int_0^1 \psi^*(x)\varphi_1(x)\mathrm{d}x - b\lambda_1^{-1}\int_0^1 \psi^*(x)\mathrm{d}x - \int_0^1 \psi^*(x)\varphi_1(x)\mathrm{d}x \\
&\geq \varepsilon\lambda_1^{-1}\lambda_1^{-1}\delta_1\|\varphi_1\| - b\lambda_1^{-1}\int_0^1 \psi^*(x)\mathrm{d}x = \lambda_1^{-1}\Big[\varepsilon\lambda_1^{-1}\delta_1 R - b\int_0^1 \psi^*(x)\mathrm{d}x\Big] > 0.
\end{aligned}$$

On the other hand, we have from (3.3) that

$$\int_0^1 \psi^*(x)\varphi_1(x)\mathrm{d}x - \int_0^1 \psi^*(x)(A\varphi_1)(x)\mathrm{d}x = \mu_0\int_0^1 \psi^*(x)\varphi^*(x)\mathrm{d}x \geq 0.$$

It is a contradiction. So from Corollary 2.3.1 in [4], we have

$$i(A,\ B_R \cap P,\ P) = 0. \tag{3.4}$$

It follows from (1.6) that there exists $0 < r < R$ such that $f(u) \leq \lambda_1 u$, $\forall 0 \leq u \leq r$. If there are $\varphi_2 \in \partial B_r \cap P$ and $\mu_1 \geq 1$ such that $A\varphi_2 = \mu_1\varphi_2$, we may suppose that $\mu_1 > 1$, otherwise we are done. Thus

$$\mu_1\varphi_2(x) = \int_0^1 G(x,y)h(y)f(\varphi_2(y))\mathrm{d}y \leq \lambda_1\int_0^1 G(x,y)h(y)\varphi_2(y)\mathrm{d}y. \tag{3.5}$$

After multiplying the both sides in (3.5) by ψ^* and integrating them, we get from Lemma 4(ii) that

$$\begin{aligned}
\mu_1\int_0^1 \psi^*(x)\varphi_2(x)\mathrm{d}x &\leq \lambda_1\int_0^1 \psi^*(x)\mathrm{d}x\int_0^1 G(x,y)h(y)\varphi_2(y)\mathrm{d}y \\
&= \lambda_1\int_0^1 \varphi_2(x)\mathrm{d}x\int_0^1 G(y,x)h(x)\psi^*(y)\mathrm{d}y = \int_0^1 \psi^*(x)\varphi_2(x)\mathrm{d}x. \qquad (3.6)
\end{aligned}$$

Since $A(P) \subset P_1$ and $\varphi_2 = \frac{1}{\mu_1}A\varphi_2$, $\varphi_2 \in P$, we have $\varphi_2 \in P_1$ which implies $\int_0^1 \psi^*(x)\varphi_2(x)\mathrm{d}x > 0$.

Then (3.6) implies $\mu_1 \leq 1$, a contradiction. So from Lemma 2.3.1 in [4], we have

$$i(A,\ B_r \cap P,\ P) = 1. \tag{3.7}$$

By (3.4) and (3.7), we have

$$i(A,\ (B_R \cap P) \setminus (\overline{B_r} \cap P),\ P) = i(A,\ B_R \cap P,\ P) - i(A,\ B_r \cap P,\ P) = -1.$$

Then A has at least one fixed point on $(B_R \cap P) \setminus (\overline{B_r} \cap P)$. This means that the singular boundary value problem (1.1) has at least one positive solution.

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