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Any Long Cycles Covering Specified Independent Vertices

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Abstract An invariant $\sigma_2(G)$ of a graph is defined as follows: $\sigma_2(G) := \min\{d(u)+d(v)|u, v \in V(G), uv \notin E(G), u \neq v\}$ is the minimum degree sum of nonadjacent vertices (when G is a complete graph, we define $\sigma_2(G) = \infty$). Let k, s be integers with $k \geq 2$ and $s \geq 4$, G be a graph of order n sufficiently large compared with s and k. We show that if $\sigma_2(G) \geq n+k-1$, then for any set of k independent vertices $v_1, \ldots, v_k$, G has k vertex-disjoint cycles $C_1, \ldots, C_k$ such that $|C_i| \leq s$ and $v_i \in V(C_i)$ for all $1 \leq i \leq k$.

The condition of degree sum $\sigma_2(G) \geq n+k-1$ is sharp.

Keywords vertex-disjoint cycle; degree sum condition; independent vertices.

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1. Introduction

An independent set is a set of pairwise non-adjacent vertices of a graph. In this paper, we only consider finite undirected graphs without loops or multiple edges. We will follow standard terminology and notation from [1] except as indicated. Let $G = (V(G), E(G))$ be a graph, the minimum degree of G will be denoted by $\delta(G)$. An invariant $\sigma_2(G)$ of a graph is defined as follows: $\sigma_2(G) := \min\{d(u)+d(v)|u, v \in V(G), uv \notin E(G), u \neq v\}$ is the minimum degree sum of nonadjacent vertices (when G is a complete graph, we define $\sigma_2(G) = \infty$). For $v \in V(G)$ and $U, W \subset V(G)$. We let $N_G(v, U)$ (or simply $N(v, U)$) denote the neighborhood of v in U, i.e., $N(v, U) := \{u \in U|uv \in E(G)\}$. Let $d_G(v, U)$ (or simply $d(v, U)$) denote the degree of v. Thus $d(v, U) = |N(v, U)|$. When $U = V(G)$, we simply write $N(v) = N(v, V(G))$ and $d(v) = d(v, V(G))$. For any $v \in V(G)$, if there is a cycle passing through v, we say G has v-cycle.

Egawa et al.[2] presented the following theorem.

Theorem 1[2] *Let k be an integer with $k \geq 1$ and G be a graph of order $n \geq 4k-1$ satisfying the condition that $\sigma_2(G) \geq n+2k-2$. Then for any k independent edges $e_1, \ldots, e_k$ of G, G has k vertex-disjoint cycles $C_1, \ldots, C_k$ of order at most four such that $e_i \in E(C_i)$ for each $i \in \{1, \ldots, k\}$.*

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For vertex-disjoint cycles passing through given vertices, The following conjecture from [3],[6] and the result is true.

Theorem 2[3],[6] *Let G be a graph with order $n \geq ck^2$, where c is a large enough absolute constant, with minimum degree at least $\lfloor \sqrt{n + \frac{9}{4}k^2 - 4k + 1} + \frac{3}{2}k - 1 \rfloor$. Then for any k distinct vertices in G there exist k vertex-disjoint cycles $C_1, \ldots, C_k$ of length at most six each of which contains exactly one of the k specified vertices.*

Ishigami and Jiang[3] conjectured the conjecture for $n \geq ck^2$ where c is a constant. Further they showed that under the same condition the cycles can be chosen so that each has length at most six.

Theorem 3[3] *Let δ, k be positive integers with $\delta \geq ck$, where c is a large enough constant. Let G be a graph with order $n \leq \delta^2 + (-3k + 4)\delta - 2k + 3$, and minimum degree δ. Then for any k distinct vertices in G there exist k vertex-disjoint cycles $C_1, \ldots, C_k$ of length at most six each of which contains exactly one of the k specified vertices.*

Ishigami[4] discussed the minimum degree condition of G containing k vertex-disjoint cycles of length at most four each of which contains one of the k prescribed vertices, and proved the following theorem:

Theorem 4[4] *Let $k \geq 1$ be an integer and G a graph of order $n \geq 3k$ with $\delta(G) \geq \lfloor \sqrt{n + k^2 - 3k + 1} \rfloor + 2k - 1$. Then for any k distinct vertices $\{x_1, x_2, \ldots, x_k\}$, there exist k vertex-disjoint cycles $C_1, \ldots, C_k$ of order at most four with $x_i \in V(C_i)$ for $i \in \{1, \ldots, k\}$.*

For a bipartite graph with partite sets V_1 and V_2, we define $\sigma_{1,1}(G) = \min\{d_G(x) + d_G(y) | x \in V_1, y \in V_2, xy \notin E(G)\}$. Matsumura[5]proved the maximum number of 4-cycle passing through given edges in a graph.

Theorem 5[5] *Suppose $\sigma_{1,1}(G) \geq \max\{\lceil \frac{4n+2s-1}{3} \rceil, \lceil \frac{2n-1}{3} \rceil + 2k\}$, and $k \geq 1$, $1 \leq s \leq k$, $n \geq 2k$. Then for any k independent edges $e_1, \ldots, e_k$ of G, G contains k vertex-disjoint cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$, $|C_i| \leq 6$, and there are at least s 4-cycle in $\{C_1, \ldots, C_k\}$.*

Ishigami and Jiang [3], Ishigami[4], Matsumura [5] considered short cycles passing through specified vertices or edges, they limited the length of cycle at most four or at most six. In the following, we consider any length of cycle covering specified vertices, that is the following Theorem 6:

Theorem 6 *Let k, s be integers with $k \geq 2$ and $s \geq 4$. Let G be a graph of order $n \geq s(k-1)+5$. If $\sigma_2(G) \geq n+k-1$, then for any set of k independent vertices $v_1, \ldots, v_k$, G has k vertex-disjoint cycles $C_1, \ldots, C_k$ such that $|C_i| \leq s$ and $v_i \in V(C_i)$ for all $1 \leq i \leq k$.*

To demonstrate the sharpness of the condition of degree sum $\sigma_2(G) \geq n + k - 1$ in Theorem 6, we construct the following example.

Example Suppose $n \geq 3k$. Consider three vertex disjoint graphs G_1, G_2 and G_3. Let $G_1 = \{x\}$ be a vertex, G_2 be independent vertex set of order k , G_3 be a complete graph of order

$n-k-1$. Join x completely to G_2, and join G_2 completely to G_3. Thus we get graph G. Then $\min\{d_G(x)+d_G(y)|xy\notin E(G), x\in V(G_1), y\in V(G_3)\}=k+k+n-k-2=n+k-2$. Clearly, every cycle passing through x must contain at least two vertices in G_2. Therefore, for k independent vertices in G_2, G has no k cycles satisfying the property of Theorem 6.

2. Proof of Theorem 6

We choose G to be a maximal counterexample, that is, if x and y are nonadjacent vertices in G, then $G+xy$ contains k vertex disjoint cycles $C_1,\dots,C_k$ such that $|C_i|\le s$ and $v_i\in V(C_i)$ for all $1\le i\le k$. We may assume that $xy\in E(C_k)$. Then $C_1,\dots,C_{k-1}$ are vertex disjoint cycles such that $|C_i|\le s$ and $v_i\in V(C_i)$ for all $1\le i\le k-1$, $v_k\notin\bigcup_{i=1}^{k-1}V(C_i)$, and $\sum_{i=1}^{k-1}|V(C_i)|\le n-3$. Among all possible choices of a set of $k-1$ vertex disjoint cycles such that $|C_i|\le s$, $v_i\in V(C_i)$ for all $1\le i\le k-1$, $v_k\notin\bigcup_{i=1}^{k-1}V(C_i)$ and $\sum_{i=1}^{k-1}|V(C_i)|\le n-3$, select one collection satisfying

$$\sum_{i=1}^{k-1}|V(C_i)| \text{ is minimum.} \tag{1}$$

Subject to (1), we may further choose $C_1,C_2,\dots,C_{k-1}$ such that

$$\sum_{i=1}^{k-1}d(v_k,C_i) \text{ is as small as possible.} \tag{2}$$

We denote $k-1$ cycles satisfying the above (1), (2) as follows:

$C_1,\dots,C_{m_1},C_{m_1+1},\dots,C_{m_1+m_2},C_{m_1+m_2+1},\dots,C_{m_1+m_2+m_3},C_{m_1+m_2+m_3+1},\dots,$
$C_{m_1+m_2+m_3+m_4},\dots,C_{m_1+m_2+m_3+\cdots+m_{s-3}},C_{m_1+m_2+m_3+\cdots+m_{s-3}+1},\dots,C_{k-1}$.
$|C_i|=3$, $i\le m_1$, $|C_i|=4$, $m_1+1\le i\le m_1+m_2$,
$|C_i|=5$, $m_1+m_2+1\le i\le m_1+m_2+m_3$,
$|C_i|=s-1$, $m_1+m_2+\cdots+m_{s-3}+1\le i\le m_1+m_2+m_3+\cdots+m_{s-3}+m_{s-2}$,
$|C_i|=s$, $m_1+m_2+\cdots+m_{s-2}+1\le i\le k-1$.

Let $L=G[\bigcup_{i=1}^{k-1}V(C_i)]$, $H=G-L$.

We also assume that in this selection any permutation of the vertices $\{v_1,v_2,\dots,v_k\}$ can be used.

Claim 1 We claim $xv_k\in E(G)$ for all $x\in V(H)$.

Suppose to the contrary, $xv_k\notin E(G)$ for any $x\in V(H)$, $x\ne v_k$. As $G[H]$ has no v_k cycle, and $G[H]$ has no cycle of length four passing through v_k, $d(v_k,H)+d(x,H)\le|V(H)|-2+1=|V(H)|-1$. Thus $d(v_k,L)+d(x,L)\ge n+k-1-(|V(H)|-1)=|L|+k=\sum_{i=1}^{k-1}(|C_i|+1)+1$. This implies there exists C_i in L such that $d(v_k,C_i)+d(x,C_i)\ge|C_i|+2$. If $|C_i|=3$, let $C_i=x_iy_iv_ix_i$, then $d(x,C_i)=3, d(v_k,C_i)=2$ ($v_kv_i\notin E(G)$). Say $C_i'=xy_iv_ix$ and $d(v_k,C_i')=1$, a contradiction to (2). So $|C_i|\ge4$. $d(v_k,C_i)+d(x,C_i)\ge|C_i|+2$. By (1) and $v_kv_i\notin E(G)$, it is easy to check that $d(x,C_i)\le3$, and if $d(x,C_i)=3$, then x is only adjacent to the consecutive three vertices of $C_i-\{v_i\}$ (For otherwise, there is a cycle C_i' containing x and v_i and $|C_i'|<|C_i|$), a contradiction to (1). So $d(v_k,C_i)=|C_i|-1$, that is, v_k is adjacent to every vertex of $V(C_i)-\{v_i\}$, hence we may get a v_k-cycle C_k such that $|C_k|<|C_i|$, a contradiction to (1). As claimed.

As $|L| = 3m_1 + 4m_2 + \cdots + sm_{s-2}$, $m_1 + m_2 + \cdots + m_{s-2} = k - 1$, $|H| = n - (3m_1 + 4m_2 + \cdots + sm_{s-2}) = n - [3m_1 + 4m_2 + \cdots + (s-1)m_{s-3} + s(k - 1 - m_1 - m_2 - \cdots - m_{s-3})] = n + (s-3)m_1 + (s-4)m_2 + \cdots + m_{s-3} - s(k-1) \geq s(k-1) + 5 - s(k-1) = 5$.

By Claim 1, $V(H) - \{v_k\}$ are independent. For any $u_1, u_2, u_3, u_4 \in V(H)$ ($|H| \geq 5$), $d(u_1, L) + d(u_2, L) + d(u_3, L) + d(u_4, L) \geq 2n + 2k - 2 - 4 = n - 5 + n - 5 + 2k + 4 \geq |L| + |L| + 2(k-1) + 6 = 2(\sum_{i=1}^{k-1}(|C_i|+1)) + 6$. This implies there exists C_i in L such that $d(u_1, C_i) + d(u_2, C_i) + d(u_3, C_i) + d(u_4, C_i) \geq 2(|C_i| + 1) + 1 = 2|C_i| + 3$.

If $|C_i| = 3$, then $d(u_1, C_i) + d(u_2, C_i) + d(u_3, C_i) + d(u_4, C_i) \geq 9$. Say $C_i = x_i y_i v_i x_i$. Without loss of generality we may assume $d(u_1, C_i) = 3$, $d(u_2, C_i) \geq 2$. Then $d(u_3, C_i) + d(u_4, C_i) \geq 3$. Without loss of generality we may assume $d(u_3, C_i) \geq 2$.

We claim $u_2 x_i \notin E(G)$.

Suppose to the contrary $u_2 x_i \in E(G)$. If $u_3 x_i \in E(G)$, say $C_k = v_k u_2 x_i u_3 v_k$, $C_i' = v_i u_1 y_i v_i$, we get k desired cycles, a contradiction. So $u_3 v_i \in E(G)$, $u_3 y_i \in E(G)$. Say $C_i' = v_i u_3 y_i v_i$, $C_k = v_k u_1 x_i u_2 v_k$, we get k desired cycles, a contradiction. So $u_2 x_i \notin E(G)$. As claimed.

Hence $u_2 v_i \in E(G)$, $u_2 y_i \in E(G)$. If $u_3 x_i \in E(G)$, then we say $C_i' = v_i u_2 y_i v_i$, $C_k = v_k u_1 x_i u_3 v_k$, we get k desired cycles, a contradiction. So $u_3 v_i \in E(G)$, $u_3 y_i \in E(G)$. Say $C_i' = v_i u_1 x_i v_i$, $C_k = v_k u_2 y_i u_3 v_k$, we get k desired cycles, a contradiction.

So $|C_i| \neq 3$.

If $|C_i| = 4$, then $d(u_1, C_i) + d(u_2, C_i) + d(u_3, C_i) + d(u_4, C_i) \geq 11$. Let $C_i = v_i x_i y_i z_i v_i$. By (1), we may assume $d(u_i, C_i) = 3$, $i \in \{1, 2, 3\}$, and $u_i v_i \notin E(G)$. Say $C_i' = v_i x_i u_1 z_i v_i$, $C_k = v_k u_2 y_i u_3 v_k$, we get k desired cycles, a contradiction.

Therefore $|C_i| \geq 5$. By (1), $d(u_i, C_i) \leq 3$, thus $d(u_1, C_i) + d(u_2, C_i) + d(u_3, C_i) + d(u_4, C_i) \leq 12$. Contradicting to $d(u_1, C_i) + d(u_2, C_i) + d(u_3, C_i) + d(u_4, C_i) \geq 2|C_i| + 3 \geq 13$. This completes the proof of Theorem 6.

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