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The Uniqueness of Book Presentation of Complete Graph K_{2m}

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Abstract In the present paper we discuss some properties of book presentation of spatial graphs, and prove that the book presentation of minimum sheets of a complete graph K_{2m} with even vertices is unique up to sheet translation and ambient isotopy. We also show this is true for K_7.

Keywords spatial graph; Hamilton path; book presentation.

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1. Introduction

Kobayashi discussed some kinds of standard embeddings into 3-manifolds of spatial graphs in [1] and [2]. He introduced the concepts of locally unknotted spatial graphs and globally unknotted (unlinked) spatial graphs, and defined a standard embedding of spatial graphs so that each spatial graph has such a standard embedding, and the spatial graph has good properties.

The concept of book presentation introduced by Kobayashi, which is also a standard embedding of spatial graphs, is a perfect presentation of spatial graph. In [1] the book presentation of pseudo Hamilton graphs was studied and the book presentation of minimum sheets for K_n was given by Kobayashi. Moreover, he conjectured that the book presentation of minimum sheets of K_n is unique up to the sheet translation and the ambient isotopy. He showed his conjecture is true for K_5 in [2].

In the present paper we discuss some properties of book presentations of spatial graphs, and prove that the book presentation of minimum sheets of a complete graph K_{2m} with even vertices is unique up to sheet translation and ambient isotopy, giving a positive answer to Kobayashi's conjecture when the vertex number of the complete graph is even. We also show this is true for K_7 at the end.

In Section 2, we will preview some definitions and lemmas, and in Section 3 we will give the main results and its proofs.

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2. Preliminary

A graph G is denoted by (V, E), where V is a set. The element in V is called a vertex. Let E be a subset of $V \times V$. Then each of its elements is called an edge.

The set of vertices is denoted by $V(G)$, and the set of edges is denoted by $E(G)$. G is called a finite graph if $V(G)$ and $E(G)$ are both finite sets, otherwise G is called a infinite graph. Only finite graph will be discussed in this paper. The number of the vertices in the graph G is called the rank of the graph. G is called a complete graph if each pair of different vertices are connected by an edge. The complete graph with rank n is denoted by K_n. Obviously a complete graph of rank n contains $C_n^2 = \frac{1}{2}n(n-1)$ edges.

Let G be a finite graph. It is well known that G can be embedded into 3-dimensional Euclidian space in many ways. Let $SE(G)$ be the set of embeddings from G into R^3. An element of $SE(G)$ is called a spatial embedding of the graph or simply a spatial graph.

Definition 2.1 *Let f, $g \in SE(G)$, $f, g : G \to R^3$, and $I = [0, 1]$ be a unit closed interval. The map $\Phi : G \times I \to R^3 \times I$ is called*

1) Level preserving, if for any $t \in I$, there exists a map $\Phi_t : G \to R^3$ so that $\Phi(x, t) = (\Phi_t(x), t)$.

2) Locally flat, if for any point of the image of Φ, there is a neighborhood N s.t. $(N, N \cap \Phi(G \times I))$ is homeomorphic to the standard pairs of disks (D^4, D^2) or $(D^3 \times I, X_n \times I)$, n is non-negative.

3) Between f and g, if there is a real number, so that for all $x \in G, 0 \leq t \leq \varepsilon$, $\Phi(x, t) = (f(x), t)$; and for all $x \in G, 1 - \varepsilon \leq t \leq 1$, $\Phi(x, t) = (g(x), t)$.

Definition 2.2 *Let f, $g \in SE(G)$, $f, g : G \to R^3$, and $I = [0, 1]$ be a unit closed interval. f and g are called*

1) An ambient isotopic, if there is a level preserving and locally flat embedding map $\Phi : G \times I \to R^3 \times I$ between f and g.

2) Cobordism, if there is a locally flat map $\Phi : G \times I \to R^3 \times I$ between f and g.

3) Isotopic, if there is a level preserving map $\Phi : G \times I \to R^3 \times I$ between f and g.

Definition 2.3 *Let $P_1' = \{(x, y, z) \in R^3 | z = 0, y \geq 0\}$. Set*

$$\begin{aligned}
P_2' =& \{(x_2, y_2, z_2) \in R^3 | x_2 = x, y_2 = y\cos\theta - z\sin\theta, z_2 = y\sin\theta + z\cos\theta, \\
& (x, y, z) \in P_1', \text{ and } \theta = 2\pi/n\} \\
& \cdots \\
P_k' =& \{(x_k, y_k, z_k) \in R^3 | x_k = x, y_k = y\cos\theta - z\sin\theta, z_k = y\sin\theta + z\cos\theta, \\
& (x, y, z) \in P_1', \text{ and } \theta = 2(k-1)\pi/n\} \\
& \cdots \\
P_n' =& \{(x_n, y_n, z_n) \in R^3 | x_n = x, y_n = y\cos\theta - z\sin\theta, z_n = y\sin\theta + z\cos\theta, \\
& (x, y, z) \in P_1', \text{ and } \theta = 2(n-1)\pi/n\}.
\end{aligned}$$

We call $B_n = \bigcup_{i=1}^{n} P'_i$ *a book. Let* $E = \{(x, y, z) \in R^3 | y = z = 0\}$*. We call* E *the binder of* B_n*. Let* $P_i = P'_i - E$*. We call* P_i *the* $i-th$ *sheet of* B_n*.*

Thus $B_n = \bigcup_{i=1}^{n} P'_i = E \cup \bigcup_{i=1}^{n} P_i$ is a book with n-sheets $\{P_i\}$ and the binder E (See Figure 1).

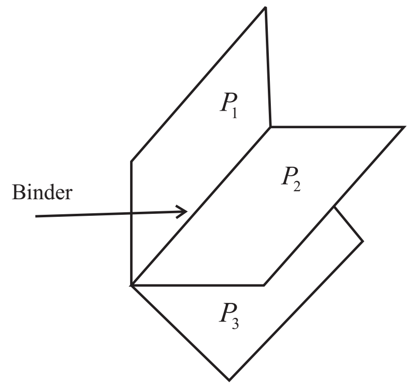

Figure 1 A book with 3-sheets

Definition 2.4 *Let* $\psi : G \longrightarrow B_n$ *be an embedding satisfying that*

(1) $\psi(V(G)) \subset E \subset B_n$,

(2) For any edge $e \in E(G), \psi(e) \subset E$ *or* $\psi(\mathrm{Int}(e)) \subset P_i$ *for some* P_i *and*

(3) For any sheet P_i *there is at least one edge* e *of* G *with* $\psi(\mathrm{Int}(e)) \subset P_i$*.*

Then we call $\widetilde{G} = \psi(G)$ *(or the embedding* ψ*) a book presentation of* G *with* n *sheets. It is clear that* $0 \leq n \leq |E(G)|$*. When* n *is minimum, we call* $\widetilde{G}$ *a book presentation of* G *with minimum sheets.*

Definition 2.5 *For a finite graph* G*, if there is a simple edge path (or simple arc) in* G *containing all vertices of* G*, we call the path a Hamilton path and* G *a pseudo Hamiltonian. Furthermore, if there is a simple closed path on* G *containing all vertices of* G*, we call it a Hamilton cycle. Let* Δ *be a Hamilton path. If a book presentation* $\psi : G \longrightarrow B_n$ *satisfies* $\psi(\Delta) \subset E$*, we call* $\widetilde{G} = \psi(G)$ *a book presentation with respect to the Hamilton path* Δ*, or simply, call* ψ *a B.P.H.*Δ*.*

Some properties of spatial graphs can be found in [3] and [4]. The following three lemmas can be found in [1]:

Lemma 2.6 *Let* $\psi : K_n \longrightarrow B_p$ *be a B.P.H.*Δ *of a complete graph* K_n*. Then every sheet of* B_p *cannot contain* $(n-1)$ *edges.*

Lemma 2.7 *Let* $\psi : K_n \longrightarrow B_p$ *be a B.P.H.*Δ *of a complete graph* K_n*. Then there is at most one sheet containing* $(n-2)$ *edges.*

Lemma 2.8 *Let* K_n *be a complete graph with* n *vertices and* $\psi : K_n \longrightarrow B_p$ *be a B.P.H.*Δ *with* p*-sheets. Then* $[(n+1)/2] \leq p \leq (n-1)(n-2)/2$ *for* $n \geq 4$ *and* $p = 1$ *for* $n = 3$*, where* $[k]$ *is the greatest integer less than or equal to* k*.*

Definition 2.9 *Let $B_p = E \cup \bigcup_{i=1}^{p} P_i$ be a book with p sheets, where P_i is the i-th sheet. If $h : B_p \longrightarrow B_p$ is a homeomorphism satisfying*

(1) $h|E = \mathrm{id}$, and

(2) $h(P_i) = P_{\sigma_{(i)}}$, where σ is a permutation of $\{1, 2, \ldots, p\}$,

then we call h a sheet translation.

3. The uniqueness of book presentation of K_{2m}

First we prove several properties for book presentations of K_{2m}.

Theorem 3.1 *If there is a Hamilton cycle γ in K_{2m} so that $\Delta = \gamma - (V_s, V_t)$ is a Hamilton path, then the B.P.H.Δ with minimum sheets has 2^{m-2} possibilities.*

Proof For a Hamilton cycle γ in K_{2m}, assume the vertices in order along γ are $V_1, V_2, \ldots, V_{2m}$. Let $\Delta = \gamma - (V_1, V_{2m})$. Given an edge (V_i, V_j), the length of (V_i, V_j) is denoted by $|j - i|$ and the set of edges with length l are denoted by $E(l)$. By Lemma 3, the $B.P.H.\Delta$ of $K_{2m} - (V_1, V_{2m})$ with minimum sheets contains m sheets, denoted by $P_1, P_2, \ldots, P_m$. There are m edges with length m in $K_{2m} - (V_1, V_{2m})$, namely, $(V_1, V_{m+1}), (V_2, V_{m+2}), \ldots, (V_m, V_{2m})$. And these m edges cannot pairwise lie in one sheet. That is, they must be in m distinguished sheets respectively. Without loss of generality, assume $(V_1, V_{m+1}) \subset P_1, (V_2, V_{m+2}) \subset P_2, \ldots, (V_m, V_{2m}) \subset P_m$.

In the following we will divide it into $m - 2$ steps and construct all possible $B.P.H.\Delta$ of $K_{2m} - (V_1, V_{2m})$ by adding $m + i$ edges and $m - i$ edges in each step.

Firstly, add two edges with length $m + 1$ and $m - 1$. There are $m - 1$ edges with length $m+1$, i.e., $E(m+1) = \{(V_1, V_{m+2}), (V_2, V_{m+3}), \ldots, (V_{m-1}, V_{2m})\}$; and there are $m+1$ edges with length $m-1$, i.e., $E(m-1) = \{(V_1, V_m), (V_2, V_{m+1}), \ldots, (V_{m+1}, V_{2m})\}$. Because of the existence of edges with length m, there are only two possibilities for each edge above, and once one edge is fixed, the other edges position will be fixed. For instance the edge (V_1, V_{m+2}) can only be put into sheet P_1 or P_2.

If (V_1, V_{m+2}) is put into sheet P_1, we have one possibility:

$(V_1, V_{m+2}) \subset P_1$; $(V_{m+1}, V_{2m}) \subset P_m$; $(V_m, V_{2m-1}) \subset P_{m-1}$; $\cdots$; $(V_2, V_{m+1}) \subset P_1$; $(V_1, V_m) \subset P_m$; $(V_{m-1}, V_{2m}) \subset P_{m-1}$; $\cdots$; $(V_2, V_{m+2}) \subset P_2$.

If (V_1, V_{m+2}) is put into sheet P_2, we have the other possibility:

$(V_1, V_{m+2}) \subset P_2$; $(V_2, V_{m+3}) \subset P_3$; $\cdots$; $(V_{m-1}, V_{2m}) \subset P_m$; $(V_1, V_m) \subset P_1$; $(V_2, V_{m+1}) \subset P_2$; $\cdots$; $(V_m, V_{2m-1}) \subset P_m$; $(V_{m+1}, V_{2m}) \subset P_1$.

Then we have the following two case.

Case 1 The location of the edges are: $(V_1, V_{m+2}), (V_2, V_{m+1}) \subset P_1$; $(V_2, V_{m+3}), (V_3, V_{m+2}) \subset P_2$; $\cdots$; $(V_{m-1}, V_{2m}), (V_m, V_{2m-1}) \subset P_{m-1}$; $(V_1, V_m), (V_{m+1}, V_{2m}) \subset P_m$.

Case 2 The location of the edges are: $(V_1, V_m), (V_{m+1}, V_{2m}) \subset P_1$; $(V_1, V_{m+2}), (V_2, V_{m+1}) \subset P_2$; $\cdots$; $(V_{m-2}, V_{2m-1}), (V_{m-1}, V_{2m-2}) \subset P_{m-1}$; $(V_{m-1}, V_{2m}), (V_m, V_{2m-1}) \subset P_m$.

Note that in both Cases 1 and 2, only two edges are added in one sheet and each pair of the two edges are fixed which implies that Cases 1 and 2 are equivalent up to ambient isotopy.

Secondly, we will prove the following conclusions by induction.

(1) Each step induces two different possibility;

(2) Two different edges will be added in each sheet each step;

(3) The two different possibilities are ambient isotopy.

Suppose in the i-th step, the sheets with edges of length $m+i$ and $m-i$ satisfy:

$$
\begin{aligned}
&(V_1, V_{m+i+1}), (V_{i+1}, V_{m+1}) \subset P_{x_1};\\
&(V_2, V_{m+i+2}), (V_{i+2}, V_{m+2}) \subset P_{x_2};\\
&\cdots\\
&(V_{m-i}, V_{2m}), (V_m, V_{2m-i}) \subset P_{x_{m-i}};\\
&(V_1, V_{m-i+1}), (V_{m+1}, V_{2m-i+1}) \subset P_{x_{m-i+1}};\\
&\cdots\\
&(V_i, V_m), (V_{m+i}, V_{2m}) \subset P_{x_m},
\end{aligned}
$$

where $\{x_1, x_2, \ldots, x_m\}$ is a permutation of $\{1, 2, \ldots, m\}$.

By the existence of the edges in $E(m-i), E(m-i+1), \ldots, E(m+i-1), E(m+i)$, each edge in $E(m+i+1)$ or $E(m-i-1)$ can only be put in two ways. There are $m-i-1$ edges in $E(m+i+1)$, and they are: $(V_1, V_{m+i+2}), \ldots, (V_{m-i-1}, V_{2m})$; there are $m+i+1$ edges in $E(m-i-1)$, and they are: $(V_1, V_{m-i}), \ldots, (V_{m+i+1}, V_{2m})$.

(V_1, V_{m+i+2}) can be put in two ways, one is in sheet P_{x_1}, and the other is in sheet P_{x_2}. If (V_1, V_{m+i+2}) is fixed in P_{x_1}, the positions of $E(m+i+1)$ and $E(m-i-1)$ are fixed. Hence we get Case I:

$(V_1, V_{m+i+2}) \subset P_{x_1}$; $(V_{m+i+1}, V_{2m}) \subset P_{x_m}$; $(V_{m+i}, V_{2m-1}) \subset P_{x_{m-1}}$; $\cdots$; $(V_{i+2}, V_{m+1}) \subset P_{x_1}$; $(V_{i+1}, V_m) \subset P_{x_m}$; $\cdots$; $(V_1, V_{m-i}) \subset P_{x_{m-i}}$; $(V_{m-i-1}, V_{2m}) \subset P_{x_{m-i-1}}$; $\cdots$; $(V_2, V_{m+i+3}) \subset P_{x_2}$.

Thus we have:

$(V_1, V_{m+i+2}), (V_{i+2}, V_{m+1}) \subset P_{x_1}$; $(V_2, V_{m+i+3}), (V_{i+3}, V_{m+2}) \subset P_{x_2}$; $\cdots$;
$(V_{m-i-1}, V_{2m}), (V_m, V_{2m-i-1}) \subset P_{x_{m-i-1}}$; $(V_1, V_{m-i}), (V_{m+1}, V_{2m-i}) \subset P_{x_{m-i}}$;
$(V_2, V_{m-i+1}), (V_{m+2}, V_{2m-i+1}) \subset P_{x_{m-i+1}}$; $\cdots$; $(V_{i+1}, V_m), (V_{m+i+1}, V_{2m}) \subset P_{x_m}$.

If (V_1, V_{m+i+2}) is fixed in P_{x_2}, then we can put the edges into the book in the following ways, which we call Case II:

$(V_1, V_{m+i+2}) \subset P_{x_2}$; $(V_2, V_{m+i+3}) \subset P_{x_3}$; $\cdots$; $(V_{m-i-1}, V_{2m}) \subset P_{x_{m-i}}$;
$(V_1, V_{m-i}) \subset P_{x_{m-i+1}}$; $\cdots$; $(V_i, V_{m-1}) \subset P_{x_m}$; $(V_{i+1}, V_m) \subset P_{x_1}$; $\cdots$;
$(V_m, V_{2m-i-1}) \subset P_{x_{m-i}}$; $(V_{m+1}, V_{2m-i}) \subset P_{x_{m-i-1}}$; $\cdots$;
$(V_{m+i}, V_{2m-1}) \subset P_{x_m}$; $(V_{m+i+1}, V_{2m}) \subset P_{x_1}$.

That is,

$(V_{i+1}, V_m), (V_{m+i+1}, V_{2m}) \subset P_{x_1}$; $(V_1, V_{m+i+2}), (V_{m+i+1}, V_{2m}) \subset P_{x_2}$; $\cdots$;
$(V_{m-i-1}, V_{2m}), (V_m, V_{2m-i-1}) \subset P_{x_{m-i}}$; $(V_1, V_{m-i}), (V_{m+1}, V_{2m-i}) \subset P_{x_{m-i+1}}$; $\cdots$;
$(V_{i-1}, V_{m-1}), (V_{m+i}, V_{2m-1}) \subset P_{x_m}$.

In the above two cases, each sheet contains one pair of edges. For Case I, by an isotopy as

$$P_{x_1} \xrightarrow{(V_1,V_{m+i+2}),(V_{i+2},V_{m+1})} P_{x_2} \xrightarrow{(V_2,V_{m+i+3}),(V_{i+3},V_{m+2})} P_{x_3} \rightarrow \cdots$$
$$\rightarrow P_m \xrightarrow{(V_{i+1},V_m),(V_{m+i+1},V_{2m})} P_{x_1},$$

we get Case II.

The construction will end in $m-2$ steps, and each step is done in two different ways. Therefore the $B.P.H.\Delta$ of $K_{2m} - (V_s, V_t)$ with minimum sheets has 2^{m-2} possibilities. □

Theorem 3.2 *The 2^{m-2} $B.P.H.\Delta$ in Theorem 3.1 are pairwise ambient isotopic.*

Proof By the proof of Theorem 1, there are $m-2$ steps and each step has two possibilities. Without loss of generality, denote the character string by $\{x_1, \ldots, x_{m-2} | x_i = 0 \text{ or } 1\}$. Choose two $B.P.H.\Delta$, which are different only on the $(i+1)-$th step, denoted by $\{a_1, \ldots, a_{i+1}, \ldots, a_{m-2}\}$, $\{b_1, \ldots, b_{i+1}, \ldots, b_{m-2}\}$, $a_j = b_j$, $j \neq i+1$; $a_{i+1} = 0$, $b_{i+1} = 1$.

Consider the $(i+1)$-st step in the construction in Theorem 1. The edges in sheet P_{x_j} whose lengths are greater than or equal to $m+i+1$, and less than or equal to $m-i-1$ are denoted by:

$$S_j = (E(m+i+1) \cup E(m-i-1) \cup \cdots \cup E(2m-2) \cup E(2)) \cap P_{x_j}.$$

Change $\{a_1, \ldots, a_{i+1}, \ldots, a_{m-2}\}$ by isotopy as $P_{x_1} \xrightarrow{S_1} P_{x_i} \xrightarrow{S_2} P_{x_3} \longrightarrow \cdots \longrightarrow P_m \xrightarrow{S_m} P_{x_1}$, We will get $\{b_1, \ldots, b_{i+1}, \ldots, b_{m-2}\}$.

For any two $B.P.H.\Delta$, if they are different in n steps in the construction, then they are equivalent to each other by n ambient isotopies.

This completes the proof. □

By using Theorems 3.1 and 3.2, we can get

Theorem 3.3 *Let Δ be a Hamilton path in K_{2m}. Then the $B.P.H.\Delta$ of K_{2m} with minimum sheets is unique up to sheet translation and ambient isotopy.*

Proof By Theorems 3.1 and 3.2, the $B.P.H.\Delta$ of $K_{2m} - (V_1, V_{2m})$ with minimum sheets is unique up to sheet translation and ambient isotopy. By Lemma 2.8, the $B.P.H.\Delta$ of K_{2m} with minimum sheets needs m sheets, and (V_1, V_{2m}) can be ambient isotopy to any sheet. Therefore the $B.P.H.\Delta$. of K_{2m} with minimum sheets is unique up to sheet translation and ambient isotopy.

The cases of a complete graph K_{2m+1} with vertices $2m+1$ $(m \geq 2)$ are much more complicated. So far only the book presentation of K_5 is known to be unique up to sheet translation and ambient isotopy[2]. In the following we show the same happens for K_7.

Theorem 3.4 *Let Δ be a Hamilton path in K_7. Then the $B.P.H.\Delta$ of K_7 with minimum sheets is unique up to sheet translation and ambient isotopy.*

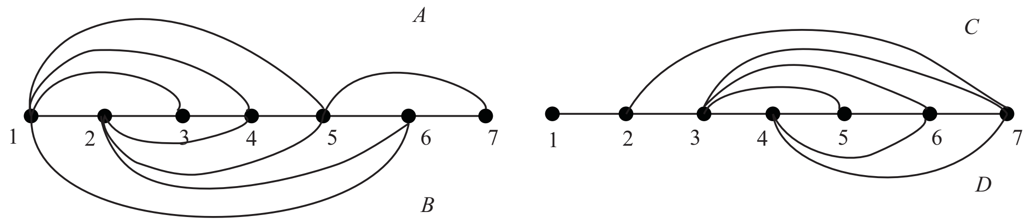

Figure 2 The standard graph of K_7

Proof One $B.P.H.\Delta$ of K_7, which is called standard graph, is given in Figure 2. Without loss of generality, denote the four sheets by A, B, C, D so that $(1,5) \subset A, (2,6) \subset B, (3,7) \subset C$. For any given embedding of K_7, (1,5), (2,6) and (3,7) lie in three different sheets, which are denoted by A', B', C', reapectively, and the remainder one is denoted by D'. Next we only need to prove that A, B, C, D and A', B', C', D' are equivalent by sheet translation and ambient isotopy.

At first if (5,7) is not in sheet A', move (5,7) to sheet A'. Because (1,5) is fixed, only edge (1,6) may be affected by the change. Thus if it is, move (1,6) to B'. Then only (2,7) may be affected. If it is, change (2,7) to sheet C'. Then (1,3) may be affected, and if it is, move it to sheet A'. Note that both two edges (2,4) and (2,5) may be affected. We discuss it in the followings:

1) If only (2,4) is affected, move it to sheet B'. In this case, (3,5) and (3,6) may be affected.

1.1) If only (3,5) is affected, move it to sheet C'. In this case (4,6) and (4,7) may be affected.

1.1.1) If only (4,6) is affected, move it to sheet D'. Then (2,5) may be affected by this change, if it is, move it to B'. Then no other edge will be affected. In this case (4,7) must lie in sheet D', and the remainings (1,4), (3,6) can be changed to the position in the standard graph without affecting the others.

1.1.2) If only (4,7) is affected, (4,7) must lie on sheet C' and (3,6) must lie on sheet D'. In this case, change (3,7) from sheet C' to sheet D', which will not affect any other edge. In this way, the mark of sheet C' and D' are exchanged. Then change (3,5) to the new sheet C' and (4,6) to sheet D' if it is affected. At last move the remains to the position in the standard graph.

1.1.3) If both (4,6) and (4,7) are affected, move both two to sheet D'. In this change, (3,6), (2,5) may be affected. As above we can put all edges to the position as in the standard graph.

1.2) If only (3,6) is affected, we discuss the sheet where (4,7) lies as follows:

1.2.1) If (4,7) is in sheet D', move (3,4) to sheet C' which will not affect the other edges. Then we only need to move the remaining edges to the position as in the standard graph in order by ambient isotopy.

1.2.2) If (4,7) is in sheet C', move (3,6) to sheet D'. In this change (1,4) and (2,5) may be affected, while they both cannot be put in one sheet. If (1,4) is affected, move it to sheet A', then (3,5) may be affected. If it is, move it to sheet D'. And in this change (4,6) may be affected. If it is, move it to sheet C' and it will not affect any other edge again. To be the standard graph, the last step is to move (3,7) and (2,7) to sheet D' and exchange the mark of C' and D'; if (2,5) is affected, move it to sheet B', then (4,6) may be affected. If it is, move it to sheet C'. Then

(1,4) and (3,5) must lie on sheetA' and D' respectively. And move (3,7) and (2,7) to sheet D', as above, exchange the mark of C' and D', it will be the standard graph.

1.3) If both (3,5) and (3,6) are affected, move them to sheet C'. In this changing (4,6) and (4,7) may be affected, which is similar to the conditions in Case 1.1. By the similar discussions we will get the standard graph.

2) If only (2,5) is affected, move it to sheet B'. In the change, only (3,6) can be affected. By the similar discussions to Case 1.2, we again get the standard graph.

3) If both (2,4) and (2,5) are affected, move it to sheet B'. In the change, (3,5) and (3,6) may be affected. It is the same as Case 1 except for that the edge (2,5) is in the standard position. We can discuss similarly to Case 1 to get the standard graph.

This completes the proof. $\square$

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