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Green's Relations on Semigroups of Transformations Preserving Two Equivalence Relations

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Abstract Let $\mathcal{T}_X$ be the full transformation semigroup on a set X. For a non-trivial equivalence F on X, let

$$T_F(X) = \{f \in \mathcal{T}_X : \forall\,(x, y) \in F,\ (f(x), f(y)) \in F\}.$$

Then $T_F(X)$ is a subsemigroup of $\mathcal{T}_X$. Let E be another equivalence on X and $T_{FE}(X) = T_F(X) \cap T_E(X)$. In this paper, under the assumption that the two equivalences F and E are comparable and $E \subseteq F$, we describe the regular elements and characterize Green's relations for the semigroup $T_{FE}(X)$.

Keywords transformation semigroup; equivalence; regular element; Green's relations.

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1. Introduction

Green's relations are five equivalences that have played an important role in the development of semigroup theory[1]. Let X be a set with $|X| \geq 3$ and $\mathcal{T}_X$ be the full transformation semigroup on the set X. In [2], the author observed a kind of transformation semigroup determined by an equivalence F on X, that is,

$$T_F(X) = \{f \in \mathcal{T}_X : \forall\,(x, y) \in F,\ \ (f(x), f(y)) \in F\}.$$

It is easy to see that $T_F(X) = \mathcal{T}_X$ if $F = \{(x, x), x \in X\}$ or $F = X \times X$. Some interesting properties for $T_F(X)$ were studied in some papers. For example, in [3] and [4], the author observed some subsemigroups of $T_F(X)$ which induce certain lattices. In [5] and [6] some special congruences on $T_F(X)$ were investigated, and Green's relations on $T_F(X)$ were described in [7] and so on.

Let E be another equivalence on X. In [2] the author also studied the semigroup

$$T_{FE}(X) = T_F(X) \cap T_E(X),$$

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and determined the suitable lattice of $T_{FE}(X)$.

The regular elements and Green's relation on the semigroups $T(X,\rho,R)$ consisting of transformations preserving an equivalence relation and a cross-section, and on the semigroups $\mathcal{O}_E(X)$ consisting of transformations preserving order and an equivalence relation were considered in [8] and [9], respectively. Clearly, $T_{FE}(X)$ is a subsemigroup of both $T_F(X)$ and $T_E(X)$ and so $f\in T_{FE}(X)$ should preserve two equivalences on X. Naturally, we may ask how to describe the regular elements and Green's relations on $T_{FE}(X)$? However, this is a difficult problem, mainly because we have great difficulty in constructing the desired maps. In this paper, we consider a special case, that is, F and E are comparable. For convenience, we assume, in the remainder, that $T_{FE}(X)$ will denote $T_F(X)\cap T_E(X)$ and that $E\subseteq F$, which is crucial for all that follows. Under the above assumption, each E-class is contained in some F-class, while each F-class is a union of some E-classes.

This paper is organized as follows. In Section 2, we observe the conditions under which an element $f\in T_{FE}(X)$ is regular. In Section 3, Green's relations on $T_{FE}(X)$ are considered and the relations $\mathcal{L},\mathcal{R},\mathcal{H},\mathcal{D}$ and $\mathcal{J}$ are completely characterized for arbitrary elements.

Now we recall some concepts and notations which will be used in the sequel. Denote by X/F the quotient set. The symbol $\pi(f)$ will denote the partition of X induced by $f\in\mathcal{T}_X$, namely,

$$\pi(f)=\{f^{-1}(y):y\in f(X)\}.$$

Also, for a subset $A\subseteq X$, we denote

$$\pi_A(f)=\{M\in\pi(f):M\cap A\neq\emptyset\}.$$

Lemma 1.1[7] *Let $f\in\mathcal{T}_X$. Then $f\in T_F(X)$ if and only if for each $B\in X/F$, there exists some $B'\in X/F$ such that $f(B)\subseteq B'$. Consequently, if $f\in T_F(X)$, then for each $A\in X/F$, the set $f^{-1}(A)$ is a union of some F-classes or $f^{-1}(A)=\emptyset$.*

For each $f\in T_F(X)$, let

$$F(f)=\{f^{-1}(A):A\in X/F\ \text{ and } f^{-1}(A)\neq\emptyset\}.$$

Then $F(f)$ is also a partition of X. It is clear that $\pi(f)$ refines $F(f)$ and that $x,\,y\in V\in F(f)$ if and only if $(f(x),f(y))\in F$. Moreover, for each $V\in F(f)$, there exists some $A\in X/F$ such that $f(V)=A\cap f(X)$. We have similar notations for $f\in T_E(X)$. For standard terms and concepts in semigroup theory, one may consult [1].

2. The regular elements of $T_{FE}(X)$

In this section, we observe when an element $f\in T_{FE}(X)$ is regular.

Theorem 2.1 *Let $f\in T_{FE}(X)$. Then f is regular if and only if for each $A\in X/F$, there exists some $B\in X/F$ such that $A\cap f(X)\subseteq f(B)$, while for each E-class $A'\subseteq A$, there exists some E-class $B'\subseteq B$ such that $A'\cap f(X)\subseteq f(B')$.*

Proof Suppose that f is regular in $T_{FE}(X)$. Then there exists $g\in T_{FE}(X)$ such that $f=fgf$.

Let $A \in X/F$. If $A \cap f(X) = \emptyset$, then $A \cap f(X) \subseteq f(B)$ for some F-class B. If $A \cap f(X) \neq \emptyset$, take $y \in A \cap f(X)$ and $x \in X$ so that $y = f(x)$. Let $g(A) \subseteq B \in X/F$. Then

$$y = f(x) = fgf(x) = fg(y) \in fg(A) \subseteq f(B)$$

and it follows that $A \cap f(X) \subseteq f(B)$. Let $A' \in X/E$ with $A' \subseteq A$. If $A' \cap f(X) = \emptyset$, then $A' \cap f(X) \subseteq f(B')$ for some $B' \in X/E$ with $B' \subseteq B$. Now suppose that $A' \cap f(X) \neq \emptyset$. Let $y' \in A' \cap f(X)$. Then there exists some $x' \in X$ such that $y' = f(x')$. Assume $g(A') \subseteq B' \in X/E$. Then

$$y' = f(x') = fgf(x') = fg(y') \in fg(A') \subseteq f(B'),$$

so $A' \cap f(X) \subseteq f(B')$. Noticing that $A' \subseteq A$, $g(A') \subseteq B'$ and $g(A) \subseteq B$, we have

$$g(A') \subseteq g(A) \subseteq B.$$

By the hypothesis $E \subseteq F$, we can deduce that $B' \subseteq B$ and the necessity follows.

Conversely, suppose the condition holds and we need to find some $g \in T_{FE}(X)$ such that $f = fgf$. Let $A \in X/F$ and $A \cap f(X) \subseteq f(B)$ for some $B \in X/F$. Suppose $B = \cup_{i \in I} B_i$ where $B_i \in X/E$. Thus $A \cap f(X) \subseteq f(\cup_{i \in I} B_i)$. If $A \cap f(X) = \emptyset$, then we define $g(x) = x$ for each $x \in A$. If $A \cap f(X) \neq \emptyset$, fix $b \in B$ and $b'_i \in B_i$ for each i. For each $x \in A$, there exists some $A' \in X/E$ such that $x \in A' \subseteq A$. Moreover, by the hypothesis, there exists E-class $B_i \subseteq B$ such that $A' \cap f(X) \subseteq f(B_i)$. We first consider the case that $A' \cap f(X) \neq \emptyset$. If $x \in A' \cap f(X)$, then $x = f(b_i)$ for some $b_i \in B_i$ and define $g(x) = b_i$. If $x \notin A' \cap f(X)$, then define $g(x) = b'_i$. Secondly, if $A' \cap f(X) = \emptyset$, then we define $g(x) = b$ for each $x \in A'$. Thus we have defined g on each $A \in X/F$, consequently, on all of X. One routinely verifies that $g \in T_{FE}(X)$. To see that $f = fgf$, take any $x \in X$ and let $y = f(x) \in A' \cap f(X) \subseteq A \cap f(X)$ where $A \in X/F$ and $A' \in X/E$. By the definition of g, we have $g(y) = b_i$ where $b_i \in B_i \subseteq B$ with $f(b_i) = y$. Thus $f(g(f(x))) = f(g(y)) = f(b_i) = y = f(x)$, which implies $f = fgf$ and f is regular in $T_{FE}(X)$. The proof is completed.

3. Green's relations on $T_{FE}(X)$

In this section, we characterize Green's relations on $T_{FE}(X)$ and begin with the relation $\mathcal{L}$. Recall that, in [7], a map $\phi : Y \to Z$ where $Y, Z \subseteq X$ is said to be F-preserving if F is an equivalence on X and $(\phi(y), \phi(y')) \in F$ for each $(y, y') \in F$ with $y,\ y' \in Y$. If ϕ satisfies that $(\phi(y), \phi(y')) \in F$ if and only if $(y, y') \in F$, then ϕ is said to be F^*-preserving.

Definition 3.1 *If ϕ is both F-preserving and E-preserving, then ϕ is said to be FE-preserving. If ϕ is both F^*-preserving and E^*-preserving, then ϕ is said to be F^*E^*-preserving.*

Remark 1 An element $f \in T_{FE}(X)$ being either E^*-preserving and F-preserving, or F^*-preserving and E-preserving, is not necessarily F^*E^*-preserving. For example, let $X = \{1, 2, \ldots\}$, $X/F = \{A_1, A_2\}$ and $X/E = \{A_1, B_1,\ B_2, B_3, \ldots\}$, where $A_1 = \{1, 2\}$, $A_2 = \{3, 4, \ldots\}$, $B_1 =$

$\{3,4\}$, $B_2=\{5,6\}$, $B_3=\{7,8\},\ldots$. It is clear that $E\subseteq F$. Let

$$g=\begin{pmatrix}1&2&3&4&5&6&\cdots\\3&4&5&6&7&8&\cdots\end{pmatrix},\quad h=\begin{pmatrix}1&2&3&4&5&6&7&8&\cdots\\1&2&5&6&5&6&7&8&\cdots\end{pmatrix}.$$

Then both g and h are FE-preserving. It is not hard to verify that g is E^*-preserving, but not F^*-preserving while h is F^*-preserving, but not E^*-preserving. So both g and h are not F^*E^*-preserving.

Theorem 3.2 *Let $f,g\in T_{FE}(X)$. Then the following statements are equivalent:*

(1) $(f,g)\in\mathcal{L}$;

(2) $\pi(f)=\pi(g)$, $F(f)=F(g)$ and $E(f)=E(g)$;

(3) There exists an F^*E^*-preserving bijection $\phi: f(X)\to g(X)$ such that $g=\phi f$.

Proof (1)$\Longrightarrow$(2). Suppose $(f,g)\in\mathcal{L}$ in $T_{FE}(X)$. Then $(f,g)\in\mathcal{L}$ in both $T_F(X)$ and $T_E(X)$. By Theorem 3.1 of [7], it follows readily that $\pi(f)=\pi(g)$, $F(f)=F(g)$ and $E(f)=E(g)$.

(2)$\Longrightarrow$(3). Define $\phi: f(X)\to g(X)$ by $\phi(x)=g(f^{-1}(x))$ for each $x\in f(X)$. Then ϕ is well-defined (since $\pi(f)=\pi(g)$) and $g=\phi f$. It is routine to show ϕ is F^*E^*-preserving.

(3)$\Longrightarrow$(1). Suppose that (3) holds. We need to find some h, $k\in T_{FE}(X)$ such that $g=hf$ and $f=kg$. For $A\in X/F$, assume $A=\cup_{i\in I}B_i$ where $B_i\in X/E$. Denote $A'=A\cap f(X)$. If $A'=\emptyset$, then define $h(x)=x$ for each $x\in A$. Now assume $A'\neq\emptyset$. Since ϕ is F^*-preserving, there exists $D\in X/F$ such that $\phi(A')\subseteq D\cap g(X)$. Fix $d\in D$. Notice that ϕ is also E^*-preserving. For each $i\in I$ with $B_i\cap f(X)\neq\emptyset$, there exists some $C_i\in X/E$ such that $\phi(B_i\cap f(X))\subseteq C_i\subseteq D$. Fix $c_i\in C_i$ for each $i\in I$ with $B_i\cap f(X)\neq\emptyset$ and define

$$h(x)=\begin{cases}\phi(x), & x\in A',\\ c_i, & x\in A-A',\ x\in B_i\in X/E \text{ and } B_i\cap f(X)\neq\emptyset,\\ d, & x\in A-A',\ x\in B_i\in X/E \text{ and } B_i\cap f(X)=\emptyset.\end{cases}$$

In this way, we have defined the map h on each F-class A and, consequently, on all of X. It is not difficult to check that $h\in T_F(X)$ and $h\in T_E(X)$, namely, $h\in T_{FE}(X)$. Finally, we verify that $g=hf$. Let $x\in X$ and assume $f(x)\in A\cap f(X)$ for $A\in X/F$. Then $hf(x)=\phi(f(x))=g(x)$ and $g=hf$. Similarly, one may find some $k\in T_{FE}(X)$ such that $f=kg$. So $(f,g)\in\mathcal{L}$.

In what follows we investigate the relation $\mathcal{R}$. We need some preparations before stating the conclusion. Let f, $g\in T_F(X)$. Recall that a map $\psi:\pi(f)\to\pi(g)$ is said to be F-admissible, if for each $A\in X/F$, there exists some $B\in X/F$ such that $B\cap\psi(P)\neq\emptyset$ for each $P\in\pi_A(f)$. If ψ is bijective and both ψ and ψ^{-1} are F-admissible, then ψ is said to be F^*-admissible. This concept was useful in describing the relation $\mathcal{R}$ on $T_F(X)$ in [7]. To describe the relation $\mathcal{R}$ on $T_{FE}(X)$, we need the following terminology.

Definition 3.3 *Let $\psi:\pi(f)\to\pi(g)$ be a map with f, $g\in T_{FE}(X)$. Suppose for each $A\in X/F$, there exists $B\in X/F$ such that $B\cap\psi(P)\neq\emptyset$ for each $P\in\pi_A(f)$, while for each $A'\in X/E$ with $A'\subseteq A$, there exists $B'\in X/E$ with $B'\subseteq B$ such that $B'\cap\psi(P')\neq\emptyset$ for each $P'\in\pi_{A'}(f)$. Then ψ is said to be FE-admissible. If ψ is bijective and both ψ and ψ^{-1} are FE-admissible,* *then ψ is said to be F^*E^*-admissible.*

Remark 2 If $\psi:\pi(f)\to\pi(g)$ is FE-admissible, then ψ is both F-admissible and E-admissible. However, the converse is not, in general, true. For example, let $X=\{1,2,\ldots\}, X/F=\{A_1,A_2\}$ and $X/E=\{B_1, B_2, B_3,\ldots\}$, where $A_1=\{1,2,3,4\}$, $A_2=\{5,6,\ldots\}$, $B_1=\{1,2\}$, $B_2=\{3,4\}$, $B_3=\{5,6\},\ldots$. It is clear that $E\subseteq F$. Let

$$f=\begin{pmatrix}1&2&3&4&5&6&\cdots\\5&6&7&8&9&10&\cdots\end{pmatrix},\quad g=\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&\cdots\\5&6&7&7&5&5&6&6&7&8&9&10&\cdots\end{pmatrix}.$$

Clearly, $f,g\in T_{FE}(X)$, while $\pi(f)=\{\{1\},\{2\},\ldots\}$ and $\pi(g)=\{\{1,5,6\},\ \{2,7,8\},\ \{3,4,9\}, \{10\},\ \{11\},\ldots\}$. Define $\psi:\pi(f)\to\pi(g)$ as follows:

$$\psi(\{1\})=\{1,5,6\},\ \psi(\{2\})=\{2,7,8\},\ \psi(\{3\})=\{3,4,9\},$$
$$\psi(\{4\})=\{10\},\ \psi(\{5\})=\{11\},\ \psi(\{6\}=\{12\},\ldots.$$

It is not hard to verify that ψ is both F-admissible and E-admissible, but not FE-admissible. In fact, for E-classes B_1 and B_2 which are contained in the F-class A_1, there exist E-classes B_1 and B_5 such that $\psi(\pi_{B_1}(f))\subseteq\pi_{B_1}(g)$ and $\psi(\pi_{B_2}(f))\subseteq\pi_{B_5}(g)$. While there is no E-class $B\neq B_5$ such that $\psi(\pi_{B_2}(f))\subseteq\pi_B(g)$. Note that B_1 and B_5 are contained in the different F-classes. By Definition 3.3, ψ is not FE-admissible.

For each $h\in\mathcal{T}_X$, let h_* denote the map from $\pi(h)$ into $h(X)$ defined by $h_*(P)=h(P)$ for $P\in\pi(h)$.

Theorem 3.4 *Let $f,g\in T_{FE}(X)$. Then the following statements are equivalent:*

(1) $(f,g)\in\mathcal{R}$;

(2) For each $A\in X/F$, there exist $B,\ C\in X/F$ such that $f(A)\subseteq g(B)$, $g(A)\subseteq f(C)$ and for each $A'\in X/E$ with $A'\subseteq A$, there exist $B',\ C'\in X/E$ with $B'\subseteq B$, $C'\subseteq C$ such that $f(A')\subseteq g(B')$ and $g(A')\subseteq f(C')$;

(3) There exists an F^*E^*-admissible bijection $\psi:\pi(f)\to\pi(g)$ such that $f_*=g_*\psi$.

Proof (1)$\Longrightarrow$ (2). It is clear.

(2)$\Longrightarrow$(3). By the hypothesis, we have $f(X)=g(X)$. Define $\psi:\pi(f)\to\pi(g)$ by $\psi(P)=g^{-1}(f_*(P))$ for each $P\in\pi(f)$. Obviously, ψ is well-defined and $f_*=g_*\psi$ and, by Theorem 3.2 of [7], ψ is F-admissible. What remains for us is to show that ψ is E-admissible. Now for each $A'\in X/E$ with $A'\subseteq A\in X/F$, by the hypothesis, there exists $B'\in X/E$ with $B'\subseteq B\in X/F$ such that $f(A')\subseteq g(B')$. Let $\pi_{A'}(f)=\{P_i:i\in I\}$ and $\{x_i'\}=f_*(P_i)(i\in I)$. Then $x_i'\in f(A')\subseteq g(B')$, so $B'\cap g^{-1}(x_i')\neq\emptyset$. Consequently,

$$B'\cap\psi(P_i)=B'\cap g^{-1}(f_*(P_i))=B'\cap g^{-1}(x_i')\neq\emptyset$$

for each $P_i\in\pi_{A'}(f)$ which means that ψ is FE-admissible. Similarly, one may show that ψ^{-1} is also FE-admissible. And $\psi:\pi(f)\to\pi(g)$ is F^*E^*-admissible, as required.

(3)$\Longrightarrow$(1). Suppose that (3) holds. We need to find $h,\ k\in T_{FE}(X)$ such that $f=gh$ and $g=fk$. Since ψ is F-admissible, for each $A\in X/F$, there exists $B\in X/F$ such that $B\cap\psi(P)\neq\emptyset$

for each $P \in \pi_A(f)$. Assume $A = \cup_{i\in I} A_i$ where $A_i \in X/E$ and let $P_x = f^{-1}(f(x))$ for every $x \in A_i$. Then $x \in P_x \in \pi_{A_i}(f)$. Therefore there exists some $B_i \in X/E$ with $B_i \subseteq B$ such that $B_i \cap \psi(P_x) \neq \emptyset$ for each $P_x \in \pi_{A_i}(f)$. Choose $y \in B_i \cap \psi(P_x)$ and define $h(x) = y$. Then $gh(x) = g(y) = g_*(\psi(P_x))$ and $\psi(P_x) = g^{-1}(gh(x))$. Now we have defined the map h on each F-class A, consequently, on all of X. It is clear that $h \in T_{FE}(X)$ and $f = gh$. Similarly, one can find some $k \in T_{FE}(X)$ such that $g = fk$. Consequently, $(f, g) \in \mathcal{R}$.

Using Theorems 3.2 and 3.4, we can establish the next result.

Theorem 3.5 *Let $f, g \in T_{FE}(X)$. Then the following statements are equivalent:*

(1) $(f, g) \in \mathcal{H}$;

(2) $\pi(f) = \pi(g)$, $F(f) = F(g)$, $E(f) = E(g)$. For each $A \in X/F$, there exist $B,\ C \in X/F$ such that $f(A) \subseteq g(B)$ and $g(A) \subseteq f(C)$, while for each $A' \in X/E$ with $A' \subseteq A$, there exist $B',\ C' \in X/E$ with $B' \subseteq B,\ C' \subseteq C$ such that $f(A') \subseteq g(B'),\ g(A') \subseteq f(C')$;

(3) There exist an F^*E^*-preserving bijection $\phi : f(X) \to g(X)$ and an F^*E^*-admissible bijection $\psi : \pi(f) \to \pi(g)$ such that $g = \phi f$ and $f_* = g_*\psi$.

Next we consider the relation $\mathcal{D}$.

Theorem 3.6 *Let $f, g \in T_{FE}(X)$. Then the following statements are equivalent:*

(1) $(f, g) \in \mathcal{D}$;

(2) There exist an F^*E^*-admissible bijection $\psi : \pi(f) \to \pi(g)$ and an F^*E^*-preserving bijection $\phi : f(X) \to g(X)$ such that $\phi f_* = g_*\psi$.

The proof is similar to that of Theorem 3.4 of [7] and it is omitted.

Now we discuss the last relation $\mathcal{J}$. Recall that, in a semigroup S, $J_a \leq J_b$ means that $S^1aS^1 \subseteq S^1bS^1$ where J_x denotes the $\mathcal{J}$-class containing $x \in S$.

Lemma 3.7 *Let $f, g \in T_{FE}(X)$. Then $J_f \leq J_g$ if and only if there exists an FE-preserving surjection $\phi : g(X) \to f(X)$ such that for each $A \in X/F$, there exists $B \in X/F$ such that $f(A) \subseteq \phi(g(B))$, while for each $C \in X/E$ with $C \subseteq A$, there exists $D \in X/E$ with $D \subseteq B$ such that $f(C) \subseteq \phi(g(D))$.*

Proof Suppose $J_f \leq J_g$. Then there exist $h, k \in T_{FE}(X)$ such that $f = hgk$. Take $A \in X/F$ with $A \cap g(X) \neq \emptyset$. Assume $A = \cup_{i\in I} B_i$, where $B_i \in X/E$. Denote $A' = A \cap g(X)$, $A'' = A \cap gk(X)$, $B_i' = B_i \cap g(X)$ and $B_i'' = B_i \cap gk(X)$. Fix $a \in h(A'') \subseteq f(X)$ and $x_i \in B_i''$ for each i with $B_i'' \neq \emptyset$. Define

$$\phi(x) = \begin{cases} h(x), & x \in A'', \\ h(x_i), & x \in A' - A'',\ x \in B_i' \text{ and } B_i'' \neq \emptyset, \\ a, & x \in A' - A'',\ x \in B_i' \text{ and } B_i'' = \emptyset. \end{cases}$$

In this way, we can define the map ϕ on $g(X)$. To see $\phi(g(X)) \subseteq f(X)$, for each $x \in g(X)$, if $x \in gk(X)$, then $\phi(x) = h(x) \in hgk(X) = f(X)$; if $x \in g(X) - gk(X)$, $x \in B_i'$ and $B_i'' \neq \emptyset$ for some i, then $\phi(x) = h(x_i) \in hgk(X) = f(X)$, too. So ϕ indeed maps $g(X)$ into $f(X)$. One routinely verifies that ϕ is FE-preserving. For each $A \in X/F$, let $k(A) \subseteq B$ for some $B \in X/F$.

Thus

$$f(A) = hgk(A) = \phi(gk(A)) \subseteq \phi(g(B)),$$

which implies that ϕ is surjective. Similarly, For each $C \in X/E$ with $C \subseteq A$, there exists some $D \in X/E$ such that $k(C) \subseteq D$ and $f(C) \subseteq \phi(g(D))$. By the hypothesis $E \subseteq F$ and $C \subseteq A$, it follows that $k(C) \subseteq k(A)$ and $D \subseteq B$.

Conversely, suppose there exists such a map ϕ. We shall construct some $h, k \in T_{FE}(X)$ such that $f = hgk$. Let $A \in X/F$ and assume $A = \cup_{i \in I} B_i$ where $B_i \in X/E$. Denote $A' = A \cap g(X)$. If $A' = \emptyset$, then define $h(x) = x$ for each $x \in A$. If $A' \neq \emptyset$, let

$$\mathcal{B} = \{B_i : B_i \cap g(X) \neq \emptyset\}.$$

Fix $x_i \in B_i \cap g(X)$ for each $B_i \in \mathcal{B}$. Since ϕ is FE- preserving, there exists some $D \in X/F$ such that $\phi(A') \subseteq D$. Fix $b \in D$ and define

$$h(x) = \begin{cases} \phi(x), & x \in A', \\ \phi(x_i), & x \in A - A' \text{ and } x \in B_i \in \mathcal{B}, \\ b, & x \in A - A' \text{ and } x \in B_i \notin \mathcal{B}. \end{cases}$$

It is not hard to verify that $h \in T_{FE}(X)$.

Now we construct k. By the hypothesis, for each $A \in X/F$ there exists $B \in X/F$ such that $f(A) \subseteq \phi(g(B))$, while for each $C \in X/E$ with $C \subseteq A$, there exists $D \in X/E$ with $D \subseteq B$ such that $f(C) \subseteq \phi(g(D))$. Thus, for each $x \in C \subseteq A$, there exists some $y \in D \subseteq B$ such that $f(x) = \phi(g(y))$. Define $k(x) = y$. Clearly, $k \in T_{FE}(X)$. One may routinely verify that $f = hgk$. This completes the proof.

As an immediate consequence of Lemma 3.7, we have the following

Theorem 3.8 *Let $f, g \in T_{FE}(X)$. Then $(f, g) \in \mathcal{J}$ if and only if there exist FE-preserving surjections $\phi : g(X) \to f(X)$ and $\psi : f(X) \to g(X)$ such that for each $A \in X/F$, there exists B, $B' \in X/F$ such that $f(A) \subseteq \phi(g(B))$, $g(A) \subseteq \psi(f(B'))$, while for each $C \in X/E$ with $C \subseteq A$, there exists D, $D' \in X/E$ with $D \subseteq B$ and $D' \subseteq B'$ such that $f(C) \subseteq \phi(g(D))$, $g(C) \subseteq \psi(f(D'))$.*

Remark 3 In general, $\mathcal{J} \neq \mathcal{D}$ in the semigroup $T_{FE}(X)$. For example, let $X = \{0, 1, 2, 3, \ldots\}$ and $X/F = \{A_1, B_1, A_2, B_2, A_3, B_3, \ldots\}$, $X/E = \{C_1, C_2, B_1, A_2, B_2, A_3, B_3, \ldots\}$, where $A_1 = \{0, 2, 4, 6\}$, $B_1 = \{1, 3, 5\}$, $A_2 = \{8, 10\}$, $B_2 = \{7, 9\}$, $A_3 = \{12, 14\}$, $B_3 = \{11, 13\}, \ldots$, $C_1 = \{0, 2\}$, $C_2 = \{4, 6\}$. Then $E \subseteq F$. Let f, $g \in \mathcal{T}_X$ be such that

$$f(C_1) = f(C_2) = C_1,\ f(B_1) = C_2,\ f(A_2) = A_2,\ f(B_2) = A_3,$$

$$f(A_3) = A_4,\ f(B_3) = A_5, \ldots$$

and

$$g(C_1) = g(C_2) = \{1, 3\},\ g(B_1) = \{5\},\ g(A_2) = B_2,$$

$$g(B_2) = B_3,\ g(A_3) = B_4,\ g(B_3) = B_5, \ldots .$$

Clearly, $f,\, g \in T_{FE}(X)$. Now we define $\psi : f(X) \to g(X)$ and $\phi : g(X) \to f(X)$, respectively, as follows:

$$\psi(C_1) = \{1,3\},\, \psi(C_2) = \{5\},\, \psi(A_2) = B_2,\, \psi(A_3) = B_3, \ldots$$

and

$$\phi(\{1,3\}) = C_1,\, \phi(\{5\}) = \{2\},\, \phi(B_2) = C_2,\, \phi(B_3) = A_2,\, \phi(B_4) = A_3, \ldots .$$

Then ψ and ϕ satisfy the conditions in Theorem 3.8. Therefore, $(f,g) \in \mathcal{J}$. However, since $f(X) = \cup\{A_i : i = 1,2,\ldots\}$ and $g(X) = \cup\{B_i : i = 1,2,\ldots\}$, there is no F^*E^*-preserving bijection from $f(X)$ onto $g(X)$. In fact, suppose there exists such one, say ρ, then $\rho(A_1) = B_i$ for some i. Note that $|A_1| = 4$ and $|B_i| \le 3$ for each i. So ρ is impossible to be bijective. Thus, by Theorem 3.6, $(f,g) \notin \mathcal{D}$.

References

[1] HOWIE J M. *Fundamentals of Semigroup Theory* [M]. Oxford University Press, New York, 1995.

[2] PEI Huisheng. *Equivalences, α-semigroups and α-congruences* [J]. Semigroup Forum, 1994, **49**(1): 49–58.

[3] PEI Huisheng. *A regular α-semigroup inducing a certain lattice* [J]. Semigroup Forum, 1996, **53**(1): 98–113.

[4] PEI Huisheng. *Some α-semigroups inducing certain lattices* [J]. Semigroup Forum, 1998, **57**(1): 48–59.

[5] PEI Huisheng, GUO Yufang. *Some congruences on $S(X)$* [J]. Southeast Asian Bull. Math., 2000, **24**(1): 73–83.

[6] PEI Huisheng. *A unique atom in $[C(\Omega), C_\alpha(\Omega)]$* [J]. East-West J. Math., 1999, **1**(2): 197–205.

[7] PEI Huisheng. *Regularity and Green's relations for semigroups of transformations that preserve an equivalence* [J]. Comm. Algebra, 2005, **33**(1): 109–118.

[8] ARAÚJO J, KONIEZNY J. *Semigroups of transformations preserving an equivalence relation and a cross-section* [J]. Comm. Algebra, 2004, **32**(5): 1917–1935.

[9] PEI Huisheng, ZOU Dingyu. *Green's equivalences on semigroups of transformations preserving order and an equivalence relation* [J]. Semigroup Forum, 2005, **71**(2): 241–251.