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Characterization of the Bounded Symmetric Domain

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Abstract We give a necessary and sufficient condition for a domain to be biholomorphic to a bounded symmetric domain.

Keywords Carathèodory volume element; Eisenmann-kobayashi volume element; bounded symmetric domain.

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1. Introduction

The classification of the bounded domain is an important problem in several complex variables. The best result was got by Cartan, who considered bounded symmetric domains, that is, for any point a, there is a holomorphic automorphism S_a such that $S_a \neq E$ and $S_a^2 = E$. He proved that any bounded symmetric domain must be one of the following domains or their topological product:

(I) $R_1(m;n) = \{z | I - z\overline{z}' > 0,\ z \text{ is an } m \times n \text{ complex matrix}\}$;

(II) $R_{II}(p) = \{z | I - z\overline{z} > 0,\ z \text{ is a } p \times p \text{ complex symmetric matrix}\}$;

(III) $R_{III}(q) = \{z | I + z\overline{z} > 0,\ z \text{ is a } q \times q \text{ complex skew-symmetric matrix}\}$;

(IV) $R_{IV}(N) = \{z \in C^N | 1 + |zz'|^2 - 2z\overline{z}' > 0, 1 - |zz'| > 0\}$;

(V) There are two exceptional domains, and their dimensions are 16 and 27, respectively.

In [1], Wong proved the following famous result:

Let G be a strongly pseudoconvex bounded domain with smooth boundary in C^n. If $\mathrm{Aut}(G)$ is non-compact, then G is biholomorphic to the unit ball B_n. In the course of proof, Carathèodory volume element and Eisenmann-kobayashi volume element played a big part, and their definitions are as follows:

Definition 1.1[2] *Let Ω be a bounded domain in C^n, $z_0 \in \Omega$. The Carathèodory volume element at z_0 on Ω is defined by*

$$C_\Omega(z_0) = \sup\{|\det f'(z_0)| : f : \Omega \to B_n, f(z_0) = 0,\ f \text{ is a holomorphic map}\}. \tag{1}$$

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Definition 1.2[2] *Let Ω be a bounded domain in C^n, $z_0 \in \Omega$. The Eisemann-Kobayashi volume element at z_0 on Ω is defined by*

$$K_\Omega(z_0) = \inf\{\frac{1}{|\mathrm{det} g'(0)|} : g : B_n \to \Omega, g(0) = z_0,\ g \text{ is a holomorphic map}\}. \tag{2}$$

In this paper, we replace B_n with bounded domain.

Definition 1.3 *Let G be a bounded domain containing origin in C^n. For any bounded domain Ω in C^n, suppose that $z_0 \in \Omega$, and the Caratheòdory volume element at z_0 on Ω which is relative to G is defined by*

$$C_\Omega^G(z_0) = \sup\{|\mathrm{det} f'(z_0)| : f : \Omega \to G, f(z_0) = 0,\ f \text{ is a holomorphic map}\}. \tag{4}$$

Definition 1.4 *Let G be a bounded domain containing origin in C^n. For any bounded domain Ω in C^n, suppose that $z_0 \in \Omega$, and the Eisemann-Kobayashi volume element at z_0 on Ω which is relative to G is defined by*

$$K_\Omega^G(z_0) = \inf\{\frac{1}{|\mathrm{det} g'(0)|} : g : G \to \Omega, g(0) = z_0,\ g \text{ is a holomorphic map}\}. \tag{4}$$

In this paper, we proved the following result:

Let Ω be a bounded domain in C^n, and G be a bounded symmetric domain with $0 \in G$. Then Ω is biholomorphic to G if and only if there exists $z_0 \in \Omega$, such that $\frac{C_\Omega^G(z_0)}{K_\Omega^G(z_0)} = 1$.

2. Preliminaries

Theorem 2.1[3] *Let D be a bounded domain in C^n and $f(z) = (f_1(z), f_2(z), \ldots, f_n(z))$ be a group of functions defined on D, where $f_l(z)\,(l = 1, \ldots, n)$ is a holomorphic function satisfying $|f_1(z)|^2 + \cdots + |f_n(z)|^2 \le M^2$, and M is a positive constant. Then*

$$\frac{\partial f}{\partial z}\overline{\frac{\partial f}{\partial z}}' \le M^2 T_D(z, \overline{z}), \text{where } T_D(z, \overline{z}) \text{ is the Bergman metric matrix of } D\ .$$

Lemma 2.2 *Let A be a Hermite matrix with $A \ge 0$, and B be a Hermite positive definite matrix with $A \le B$. Then $\mathrm{det}A \le \mathrm{det}B$.*

Proof There exists matrix Q satisfying $\det(Q) \ne 0$, such that $B = Q\overline{Q}'$. It follows from the fact $A \le B$ that $Q^{-1}A\overline{Q^{-1}}' \le I$. There exists unitary matrix U and $\lambda_1 \ge 0, \ldots, \lambda_n \ge 0$, such that $Q^{-1}A\overline{Q^{-1}}' = U[\lambda_1, \ldots, \lambda_n]\overline{U}'$, where $[\lambda_1, \ldots, \lambda_n]$ denotes an $n \times n$ diagonal matrix whose entries on diagonal are $\lambda_1, \ldots, \lambda_n$ respectively. Hence we have $\lambda_j \le 1\,(j = 1, \ldots, n)$. Therefore $\det(Q^{-1}A\overline{Q^{-1}}') = \prod_{j=1}^n \lambda_j \le 1$. It is obvious that $\mathrm{det}A \le \mathrm{det}Q\mathrm{det}\overline{Q'} = \mathrm{det}B$.

Lemma 2.3 Let Ω and G be bounded domains in C^n. Then $0 < C_\Omega^G(z_0) < +\infty$.

Proof It follows from the fact $0 \in G$ that there exists $\delta > 0$, such that $B(0, \delta) \subset G$. There exists $M_1 > 0$, such that for $\forall z \in \Omega, |z - z_0|/M_1 < 1$. Let $\varphi(z) = \delta(z - z_0)/M_1$. Then $\varphi : \Omega \to G, \varphi(z_0) = 0, |\det\varphi'(z_0)| = (\delta/M_1)^n$. According to (3), we have $C_\Omega^G(z_0) > 0$. Next, we will prove $C_\Omega^G(z_0) < +\infty$. Suppose that $f : \Omega \to G, f(z_0) = 0$, and f is a holomorphic mapping, and $f = (f_1, f_2, \ldots, f_n)$. It follows from the fact that G is a bounded domain that there exists $M > 0$, such that for $\forall z \in \Omega$, $|f_1(z)|^2 + \cdots + |f_n(z)|^2 \le M^2$. According to Theorem 2.1, we have

$$\frac{\partial f}{\partial z}|_{z=z_0} \overline{\frac{\partial f}{\partial z}}' |_{z=z_0} \le M^2 T_\Omega(z_0, \overline{z_0}).$$

It follows from Lemma 2.2 that

$$\det\Big[\frac{\partial f}{\partial z}|_{z=z_0} \overline{\frac{\partial f}{\partial z}}' |_{z=z_0}\Big] \le \det[M^2 T_\Omega(z_0, \overline{z_0})].$$

We can easily get that

$$\Big|\det\frac{\partial f}{\partial z}|_{z=z_0}\Big| \le M^n \sqrt{\det T_\Omega(z_0, \overline{z_0})}.$$

Hence $C_\Omega^G(z_0) < +\infty$.

Lemma 2.4 Let Ω and G be bounded domains in C^n. Then $0 < K_\Omega^G(z_0) < +\infty$.

Proof There exists $\delta > 0$, such that $B(z_0, \delta) \subset \Omega$. There exists $M_2 > 0$, such that for $\forall z \in G, |z| < M_2$. Let $\varphi(z) = \delta z/M_2 + z_0$. Then $\varphi(z) : G \to \Omega, \varphi(0) = z_0$, $\det\varphi'(0) = (\delta/M_2)^n$. According to (4), we have $K_\Omega^G(z_0) < +\infty$. Next we will prove $0 < K_\Omega^G(z_0)$. Suppose that $g : G \to \Omega, g(0) = z_0$, and g is a holomorphic mapping, $g = (g_1, g_2, \ldots, g_n)$. It follows from the fact that Ω is a bounded domain that there exists $M > 0$, such that for $\forall z \in G$, $|g_1(z)|^2+\cdots+|g_n(z)|^2 \le M^2$. According to Theorem 2.1, we have

$$\frac{\partial g}{\partial z}|_{z=0} \overline{\frac{\partial g}{\partial z}}' |_{z=0} \le M^2 T_G(0,0).$$

It follows from Lemma 2.2 that

$$\det\Big[\frac{\partial g}{\partial z}|_{z=0} \overline{\frac{\partial g}{\partial z}}' |_{z=0}\Big] \le \det[M^2 T_G(0,0)].$$

We can easily get

$$\Big|\det\frac{\partial g}{\partial z}|_{z=0}\Big| \le M^n \sqrt{\det T_G(0,0)}.$$

According to formula (4), we have

$$K_\Omega^G(z_0) \ge \frac{1}{M^n \sqrt{\det T_G(0,0)}}.$$

Lemma 2.5 *Let $F : \Omega_1 \to \Omega_2$ be a biholomorphic mapping. Then*

$$C_{\Omega_1}^G(z_0) = C_{\Omega_2}^G(F(z_0))|\det F'(z_0)|,$$
$$K_{\Omega_1}^G(z_0) = K_{\Omega_2}^G(F(z_0))|\det F'(z_0)|.$$

Proof We can easily get the result by the definitions.

Remark $C_\Omega^G(z_0)/K_\Omega^G(z_0)$ is an analytic invariant.

Theorem 2.6[3] *If S is a holomorphic map, and S maps bounded domain D into D, and $S(a) = a$, then $\det S'(a) \le 1$, and $\det S'(a) = 1$ if and only if $S \in \mathrm{Aut}(D)$.*

Lemma 2.7 Let Ω and G be bounded domains in C^n. Then $C_{\Omega}^{G}(z_0) \le K_{\Omega}^{G}(z_0)$.

Proof Let $f : \Omega \to G, f(z_0) = 0$, be a holomorphic mapping, and let $g : G \to \Omega, g(0) = z_0$, be a holomorphic mapping. Suppose that $\varphi(z) = f(g(z))$. Then $\varphi : G \to G, \varphi(0) = 0$. According to Theorem 2.6, we have $|\det\varphi'(0)| \le 1$. Hence $|\det f'(z_0)||\det g'(0)| \le 1$, and $C_{\Omega}^{G}(z_0) \le K_{\Omega}^{G}(z_0)$.

Theorem 2.8 *Let G be a bounded homogeneous domain containing origin in C^n. Then for any $z \in G$, $C_G^G(z)/K_G^G(z) = 1$.*

Theorem 2.9[4] *Let Ω be a domain in C^n, and $F_k : \Omega \to C^n$ be a holomorphic mapping sequence satisfying the following two conditions:*

(i) F_k converges uniformly on any compact subsets of Ω to holomorphic mapping F;

(ii) There exists $a \in \Omega$, such that $\det F'(a) \neq 0$. Then,

(a) There exists a neighbourhood U of $a\,(U \subset \Omega)$ and positive integer k_0 which is large enough, such that, F_k is biholomorphic on U when $k \ge k_0$;

(b) There exists a neighbourhood V of $F(a)$, such that, $V \subset F_k(U)$ when $k \ge k_0$.

Theorem 2.10 *Let D be a domain in C^n, and f_k be a holomorphic function sequence and $f_k \neq 0$ at every point of D. If f_k converges uniformly on any compact subset of D to f, then $f \equiv 0$ on D, or $f \neq 0$ at every point of D.*

Theorem 2.11[5] *Every bounded symmetric domain is biholomorphic to a bounded symmetric and circled domain containing origin.*

Theorem 2.12[5] *A circled bounded symmetric domain containing origin is convex.*

Theorem 2.13 *Let G be a bounded domain in C^n, and D be a bounded convex domain containing origin in C^n, and $g : G \to C^n$ be a holomorphic mapping. If $g(G) \subset \overline{D}$ and there exists $z_0 \in G$, such that $g(z_0) \in D$, then $g(G) \subset D$ ($\overline{D}$ is the closure of D).*

Proof Suppose that there exists $a \in G$, such that $g(a) \in \partial D$ and let $g(a) = (x_1^0 + \sqrt{-1}y_1^0, x_2^0 + \sqrt{-1}y_2^0, \ldots, x_n^0 + \sqrt{-1}y_n^0), x_0 = (x_1^0, y_1^0, x_2^0, y_2^0, \ldots, x_n^0, y_n^0)$. Define

$$D^* = \{x = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n) | (x_1 + \sqrt{-1}y_1, x_2 + \sqrt{-1}y_2, \ldots, x_n + \sqrt{-1}y_n) \in D\}.$$

It is obvious that D^* is a bounded convex domain in R^{2n} containing the origin and $x_0 \in \partial D^*$. Define the the Minkowski functional $p(x)$ on R^{2n} to be

$$P(x) = \inf\{\lambda > 0 | \frac{x}{\lambda} \in D^*\}, x \in R^{2n}.$$

According to the knowledge of the functional analysis, we know that $P(x)$ is a real continuous function and a sublinear functional. Define $X_0 = \{\lambda x_0 | \lambda \in R\}, f_0(\lambda x_0) = \lambda$. It is obvious that f_0 is a linear functional defined on X_0, $P(x_0) = 1, P(x) \le 1, x \in \overline{D^*}; P(x) < 1, x \in D^*$. $f_0(\lambda x_0) = \lambda \le P(\lambda x_0)$. It follows from the Hahn–Banach Theorem that there exists linear functional $f(x)$ on R^{2n} satisfying the following conditions:

$$f(x) \le P(x), x \in R^{2n}; \quad f(x) = f_0(x), x \in X_0.$$

Let $f(x) = a_1x_1 + b_1y_1 + a_2x_2 + b_2y_2 + \cdots + a_nx_n + b_ny_n$. Because for $\forall x \in D^*$, $P(x) < 1$, $f_0(x_0) = 1$, we have $\forall x \in D^*, f(x) < 1$; $f(x_0) = 1$. It is obvious that $\forall x \in \overline{D^*}$,

$$a_1(x_1 - x_1^0) + b_1(y_1 - y_1^0) + a_2(x_2 - x_2^0) + b_2(y_2 - y_2^0) + \cdots + a_n(x_n - x_n^0) + b_n(y_n - y_n^0) \leq 0.$$

Let

$$\begin{aligned} F(z) =&(a_1 - \sqrt{-1}b_1)(z_1) + (a_2 - \sqrt{-1}b_2)(z_2) + \cdots + (a_n - \sqrt{-1}b_n)(z_n) - \\ &(a_1 - \sqrt{-1}b_1)(x_1^0 + \sqrt{-1}y_1^0) + (a_2 - \sqrt{-1}b_2)(x_2^0 + \sqrt{-1}y_2^0) + \cdots + \\ &(a_n - \sqrt{-1}b_n)(x_n^0 + \sqrt{-1}y_n^0), \end{aligned}$$

where $z_j = x_j + \sqrt{-1}y_j$. For $\forall z \in D, \mathrm{Re}F(z) < 0; \forall z \in \overline{D}, \mathrm{Re}F(z) \leq 0$. We consider the holomorphic mapping $F \circ g$. It is obvious that $F(g(a)) = 0, \mathrm{Re}F(g(z_0)) < 0$. Therefore, $F \circ g$ is not a constant function. According to the open mapping theorem, $F(g(G))$ is an open set in C. There exists $\delta > 0$, such that $B(0, \delta) \subset F(g(G))$. Therefore there exists $b \in G$, such that $F(g(b)) = \delta/2$. This contradicts the fact that $\mathrm{Re}F(g(b)) \leq 0$.

3. Characterization of bounded symmetric domain by Carathèodory volume element and Eisenmann-kobayashi volume element

Theorem 3.1 *Let Ω be a bounded domain in C^n, and G be a bounded symmetric domain containing origin in C^n. Then Ω is biholomorphic to G if and only if there exists $z_0 \in \Omega$, such that $\frac{C_\Omega^G(z_0)}{K_\Omega^G(z_0)} = 1$.*

Proof (i) Necessity. Let $F : \Omega \to G$ be a biholomorphic mapping. According to Lemma 2.5 and Theorem 2.8, we get that for $\forall z \in \Omega$

$$\frac{C_\Omega^G(z)}{K_\Omega^G(z)} = \frac{C_G^G(F(z))|\mathrm{det}F'(z)|}{K_G^G(F(z))|\mathrm{det}F'(z)|} = 1.$$

(ii) Sufficiency.

We first suppose that G is a bounded symmetric circled domain containing origin in C^n.

(a) We first show that there is a holomorphic mapping $f : \Omega \to G$, $f(z_0) = 0$, such that $C_\Omega^G(z_0) = |\mathrm{det}f'(z_0)|$. It follows from the definotion of $C_\Omega^G(z_0)$ that for $\forall$ natural number k, there exists holomorphic mapping $f_k : \Omega \to G$ satisfying $f_k(z_0) = 0$ and

$$C_\Omega^G(z_0) - (1/k) < |\mathrm{det}f_k'(z_0)| \leq C_\Omega^G(z_0). \tag{5}$$

There exists a subsequence $\{k_j\}$ of $\{k\}$, such that f_{k_j} converges uniformly on any compact subset of Ω to holomorphic mapping f by Montel's normal family theory. It is obvious that $f(z_0) = 0$. According to formula (5), we have $|\mathrm{det}f'(z_0)| = C_\Omega^G(z_0)$. Next, we will prove that $f(\Omega) \subset G$. We can get $f(z) \in \overline{G}$ because $\lim_{j\to+\infty} f_{k_j}(z) = f(z)$. It is obvious that $f(\Omega) \subset G$ by Theorems 2.12 and 2.13.

(b) Next, we will prove that there exists holomorphic mapping $g : G \to \Omega$, satisfying $g(0) = z_0$, such that $K_\Omega^G(z_0) = \frac{1}{|\mathrm{det}g'(0)|}$. According to the definition of $K_\Omega^G(z_0)$, one can easily get that for $\forall$ natural number k, there exists holomorphic mapping $g_k : G \to \Omega$, satisfying

$g_k(0) = z_0$ and

$$K_\Omega^G(z_0) \le \frac{1}{|\mathrm{det} g_k'(0)|} \le K_\Omega^G(z_0) + \frac{1}{k}. \tag{6}$$

Consider $H_k = f \circ g_k$. It is obvious that $H_k : G \to G, H_k(0) = 0$. There exists a subsequence $\{k_j\}$ of $\{k\}$, such that H_{k_j} converges uniformly on any compact subset of G to holomorphic mapping H by Montel's normal family theory. It is obvious that $H(0) = 0$. According to formula (6), we have $|\mathrm{det} H'(0)| = \lim_{j\to+\infty} |\mathrm{det} f'(z_0)||\mathrm{det} g_{k_j}'(0)| = 1$. One can get $H(G) \subset G$ by Theorems 2.12 and 2.13. It follows from Theorem 2.6 that $H \in \mathrm{Aut}(G))$. It is obvious that H is a linear mapping since $H(0) = 0$ and G is a circled domain. Suppose that $H(z) = zA$, where A is an $n \times n$ matrix with $\mathrm{det} A \ne 0$. According to Montel's normal family theory, we suppose that g_{k_j} converges uniformly on any compact subset of G to g for the sake of convenience. Then g is holomorphic on G and $g(0) = z_0$. According to formula (6), we have

$$|\mathrm{det} g'(0)| = \frac{1}{K_\Omega^G(z_0)}.$$

Next, we will prove $g(G)) \subset \Omega$. Since $f \circ g_{k_j}$ converges uniformly on any compact subset of G to H, $\mathrm{det} f'(g_{k_j}(z))\mathrm{det} g_{k_j}'(z)$ converges uniformly on any compact subset of G to $\mathrm{det} H'(z) = \mathrm{det} A$. Let $a \in G$. Then $\exists \delta > 0$, such that $\overline{B(a, \delta)} \subset G$. There exists a natural number N which is large enough, such that, for $\forall z \in \overline{B(a, \delta)}$, $|\mathrm{det} f'(g_{k_j}(z))\mathrm{det}(g_{k_j}'(z)) - \mathrm{det} A| < \frac{\mathrm{det} A}{2}$ when $j > N$. Hence, for $\forall z \in \overline{B(a, \delta)}, \mathrm{det} g_{k_j}'(z) \ne 0$ when $j > N$. $\mathrm{det} g_{k_j}'(z)(j > N)$ converges uniformly on any compact subset of $B(a, \delta)$ to $\mathrm{det} g'(z)$. According to Hurwitz theorem, $\mathrm{det} g'(z) \equiv 0$ on $B(a, \delta)$, or $\mathrm{det} g'(z) \ne 0$ at every point of $B(a, \delta)$. If $\mathrm{det} g'(z) \equiv 0$ on $B(a, \delta)$, then $\mathrm{det} g'(z) \equiv 0$ on G. This contradicts the fact that $|\mathrm{det} g'(0)| \ne 0$. Therefore, $\mathrm{det} g'(a) \ne 0$. Hence, for $\forall z \in G, \mathrm{det} g'(z) \ne 0$.

According to Theorem 2.9, we get that for $\forall a \in G$, there exists a neighbourhood V of $g(a)$ and a neighbourhood U of a ($U \subset G$), such that, $V \subset g_{k_j}(U) \subset \Omega$ when j is large enough. Hence $g(a) \in \Omega$. Therefore $g(G) \subset \Omega$.

(c) We prove that $f : \Omega \to G$ is a biholomorphic mapping. Consider $\varphi = g \circ f : \Omega \to \Omega, \varphi(z_0) = z_0$. It is obvious that

$$|\mathrm{det}\varphi'(z_0)| = |\mathrm{det} g'(0)||\mathrm{det} f'(z_0)| = \frac{C_\Omega^G(z_0)}{K_\Omega^G(z_0)} = 1.$$

According to Theorem 2.6, we have $\varphi \in \mathrm{Aut}(\Omega)$. Hence f is injective. Consider $\psi = f \circ g : G \to G, \psi(0) = 0$. It is obvious that

$$|\mathrm{det}\psi'(0)| = |\mathrm{det} f'(z_0)||\mathrm{det} g'(0)| = \frac{C_\Omega^G(z_0)}{K_\Omega^G(z_0)} = 1.$$

According to Theorem 2.6, we have $\psi \in \mathrm{Aut}(G)$. Hence, f is surjective. Therefore, $f : \Omega \to G$ is a biholomorphic mapping.

(d) If G is a bounded symmetric domain, then G is biholomorphic to a bounded symmetric and circled domain G_1 with $0 \in G_1$ by Theorem 2.11. Suppose that $\varphi : G \to G_1$ is a biholomorphic mapping. We suppose that $\varphi(0) = 0$ because G_1 is a homogeneous domain. One can get the following result by definition: $C_\Omega^{G_1}(z_0) = C_\Omega^G(z_0)|\mathrm{det}\varphi'(0)|, K_\Omega^{G_1}(z_0) = K_\Omega^G(z_0)|\mathrm{det}\varphi'(0)|$.

Hence

$$1 = \frac{C_\Omega^G(z_0)}{K_\Omega^G(z_0)} = \frac{C_\Omega^{G_1}(z_0)}{K_\Omega^{G_1}(z_0)}.$$

Ω is biholomorphic to G_1 by the previous proof. Therefore, Ω is biholomorphic to G.

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