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Derivations of Certain Linear Lie Algebras over Commutative Rings

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Abstract Let L be the symplectic algebra or the orthogonal algebra over a commutative ring R, h the maximal torus of L consisting of all diagonal matrices in L, and b the standard Borel subalgebra of L containing h. In this paper, we first determine the intermediate algebras between h and b, then for such an intermediate algebra, we give an explicit description on its derivations, provided that R is a commutative ring with identity and 2 is invertible in R.

Keywords Lie algebra; the derivation of linear Lie algebra; a commutative ring.

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1. Introduction

Let R be a commutative ring with identity, R^* the subset of R consisting of all invertible elements in R, $2 \in R^*$, $E^{(m)}$ the $m \times m$ identity matrix over R ($E^{(m)}$ is abbreviated to E), $R^{m\times n}$ the set of all $m \times n$ matrices over R and $\mathrm{gl}(m, R)$ the general linear Lie algebra consisting of all $m \times m$ matrices over R with bracket production: $[X, Y] = XY - YX$. For $A \in R^{m\times n}$, A' denotes the transpose of A. Let $t(m, R)$ (resp., $d(m, R)$) be the subalgebra of $\mathrm{gl}(m, R)$ consisting of all upper triangular(resp., diagonal) matrices. For $\rho = \pm 1$ and $\delta = 0, 1$, we set

$$L_{\rho,\delta} = \left\{ \begin{pmatrix} 0 & \alpha & \beta \\ -\beta' & A & B \\ -\alpha' & 0 & -A' \end{pmatrix} \;\Big|\; \alpha, \beta \in R^{1\times m},\ A \in t(m, R),\ B \in R^{m\times m},\ \text{satisfies} \right.$$

$$\left. \alpha = 2^{-1}\delta(1+\rho)\alpha,\ \beta = 2^{-1}\delta(1+\rho)\beta,\ B' = -\rho B \right\},$$

which is a subalgebra of $\mathrm{gl}(2m + 1, R)$. We see that the symplectic algebra $\mathrm{sp}(2m, R)$ (resp., the orthogonal algebra $o(2m, R)$) is embedded into $L_{-1,0}$ (resp., $L_{1,0}$) and $\mathrm{sp}(2m, R)$ (resp., $o(2m, R)$) is isomorphic to $L_{-1,0}$ (resp., $L_{1,0}$), and $L_{1,1}$ is the orthogonal algebra $o(2m + 1, R)$ (we refer to [1, pp 1-4] for the definitions of the symplectic algebra and the orthogonal algebra). Thus, to determine a problem of the symplectic algebra or the orthogonal algebra one really

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needs to consider the corresponding one of the Lie algebra $L_{\rho,\delta}$ after all. We now give several special subalgebras of $L_{\rho,\delta}$ for later use. Set

$$h_{\rho,\delta} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & -\Lambda' \end{pmatrix} \Big| \Lambda \in d(m,R) \right\};$$

$$b_{\rho,\delta} = \Big\{ \begin{pmatrix} 0 & 0 & \beta \\ -\beta' & A & B \\ 0 & 0 & -A' \end{pmatrix} \Big| \ \beta \in R^{1\times m},\ A \in t(m,R),\ B \in R^{m\times m},\ \text{ satisfies}$$

$$\beta = 2^{-1}\delta(1+\rho)\beta,\ B' = -\rho B \Big\};$$

$$t_{\rho,\delta} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -A' \end{pmatrix} \Big| \ A \in t(m,R) \right\};$$

$$w_{\rho,\delta} = \Big\{ \begin{pmatrix} 0 & 0 & \beta \\ -\beta' & 0 & B \\ 0 & 0 & 0 \end{pmatrix} \Big| \ \beta \in R^{1\times m},\ B \in R^{m\times m}\ \text{ satisfies}$$

$$\beta = 2^{-1}\delta(1+\rho)\beta,\ B' = -\rho B \Big\}.$$

By definition, $h_{\rho,\delta}$ is a maximal torus of $L_{\rho,\delta}$ and $b_{\rho,\delta}$ is a standard Borel subalgebra of $L_{\rho,\delta}$ containing $h_{\rho,\delta}$.

The automorphisms or derivations of linear Lie algebras over commutative rings were recently studied in [2–10]. In this paper, on the basis of main theorem in [3], we give an explicit description on the derivations of each intermediate algebra between $h_{\rho,\delta}$ and $b_{\rho,\delta}$, provided that R is a commutative ring with identity and 2 is invertible in R.

2. The intermediate algebras between $h_{\rho,\delta}$ and $b_{\rho,\delta}$

In the following, we always assume that R is a commutative ring with identity and $2 \in R^*$.

For $1 \le i \le j \le m$, $1 \le t \le m$, let $E_{i,j}$ denote the $(2m+1)\times(2m+1)$ matrix, whose $(i+1, j+1)$-entry is 1, all other entries are 0; $E_{i,-j}$ the $(2m+1)\times(2m+1)$ matrix, whose $(i+1, m+j+1)$-entry is 1, all other entries are 0; $E_{j,-i}$ the $(2m+1)\times(2m+1)$ matrix, whose $(j+1, m+i+1)$-entry is 1, all other entries are 0; $E_{-j,-i}$ the $(2m+1)\times(2m+1)$ matrix, whose $(m+j+1, m+i+1)$-entry is 1, all other entries are 0; $E_{0,-t}$ the $(2m+1)\times(2m+1)$ matrix, whose $(1, m+t+1)$-entry is 1, all other entries are 0; $E_{t,0}$ the $(2m+1)\times(2m+1)$ matrix, whose $(t+1, 1)$-entry is 1, all other entries are 0. Set $T_{i,j} = E_{i,j} - E_{-j,-i}$; $T_{i,-j} = E_{i,-j} - \rho E_{j,-i}$; $T_{0,-t} = \delta(1+\rho)(E_{0,-t} - E_{t,0})$. Let $I(R)$ denote the set of all ideals of R.

Definition 2.1 *Let $\Delta = \{A_{i,j}, A_{k,-l}, A_{0,-t} \in I(R) \mid 1 \le i < j \le m,\ 1 \le k \le l \le m,\ 1 \le t \le m\}$ be a subset of $I(R)$ consisting of ideals of R. We call Δ a flag of ideals of R, if the following conditions are satisfied:*

(1) $A_{i,k}A_{k,j} \subseteq A_{i,j}$ (*if* $1 \le i < k < j \le m$);
(2) $A_{i,k}A_{k,-j} \subseteq A_{i,-j}$ (*if* $1 \le i < k < j \le m$);
(3) $A_{i,k}A_{j,-k} \subseteq A_{i,-j}$ (*if* $1 \le i < j < k \le m$);
(4) $A_{j,k}A_{i,-k} \subseteq A_{i,-j}$ (*if* $1 \le i < j < k \le m$);
(5) $(1-\rho)A_{i,k}A_{k,-k} \subseteq (1-\rho)A_{i,-k}$ (*if* $1 \le i < k \le m$);
(6) $(1-\rho)A_{i,k}A_{i,-k} \subseteq (1-\rho)A_{i,-i}$ (*if* $1 \le i < k \le m$);
(7) $\delta(1+\rho)A_{0,-i}A_{0,-k} \subseteq \delta(1+\rho)A_{i,-k}$ (*if* $1 \le i < k \le m$);
(8) $\delta(1+\rho)A_{i,k}A_{0,-k} \subseteq \delta(1+\rho)A_{0,-i}$ (*if* $1 \le i < k \le m$).

Example 2.2 If all $A_{i,j}, A_{k,-l}, A_{0,-t}$ $(1 \le i < j \le m, 1 \le k \le l \le m, 1 \le t \le m)$ are taken to be 0 (resp., R), then $\Delta = \{A_{i,j}, A_{k,-l}, A_{0,-t} \in I(R) | 1 \le i < j \le m,\ 1 \le k \le l \le m,\ 1 \le t \le m\}$ is a flag of ideals of R.

Example 2.3 Let $A_{1,2}, A_{2,3}, \dots, A_{m-1,m}, A_{m,-m}, A_{0,-m}$ be any ideals of R, respectively, and set

$$A_{i,j} = \prod_{1 \le k \le j-i} A_{i+k-1,i+k}, \quad 1 \le i < j \le m;$$

$$A_{k,-m} = A_{k,m}A_{m,-m}, \quad 1 \le k < m;$$

$$A_{k,-l} = A_{l,m}A_{k,-m}, \quad 0 \le k \le l < m.$$

Then $\Delta = \{A_{i,j}, A_{k,-l}, A_{0,-t} \in I(R) |\ 1 \le i < j \le m,\ 1 \le k \le l \le m,\ 1 \le t \le m\}$ is a flag of ideals of R.

Example 2.4 Let $A_{1,2}, A_{2,3}, \dots, A_{m-1,m}, A_{m,-m}, A_{0,-m}$ be any ideals of R, respectively, and set

$$A_{i,j} = \bigcap_{1 \le k \le j-i} A_{i+k-1,i+k}, \quad 1 \le i < j \le m;$$

$$A_{k,-m} = A_{k,m} \bigcap A_{m,-m}, \quad 1 \le k < m;$$

$$A_{k,-l} = A_{l,m} \bigcap A_{k,-m}, \quad 0 \le k \le l < m.$$

Then $\Delta = \{A_{i,j}, A_{k,-l}, A_{0,-t} \in I(R) |\ 1 \le i < j \le m,\ 1 \le k \le l \le m,\ 1 \le t \le m\}$ is a flag of ideals of R.

Theorem 2.5 *Let* $2 \in R^*$. *Then* $\ell_{\rho,\delta}$ *is an intermediate Lie algebra between* $h_{\rho,\delta}$ *and* $b_{\rho,\delta}$ *if and only if there exists a flag* $\Delta = \{A_{i,j}, A_{k,-l}, A_{0,-t} \in I(R) |\ 1 \le i < j \le m,\ 1 \le k \le l \le m,\ 1 \le t \le m\}$ *of ideals of* R *such that*

$$\ell_{\rho,\delta} = h_{\rho,\delta} + \sum_{1 \le i < j \le m} A_{i,j}T_{i,j} + \sum_{1 \le k \le l \le m} A_{k,-l}T_{k,-l} + \sum_{1 \le t \le m} A_{0,-t}T_{0,-t}.$$

Proof Suppose that $\Delta = \{A_{i,j}, A_{k,-l}, A_{0,-t} \in I(R) |\ 1 \le i < j \le m,\ 1 \le k \le l \le m,\ 1 \le t \le m\}$ is a flag of ideals of R and $\ell_{\rho,\delta} = h_{\rho,\delta} + \sum_{1 \le i < j \le m} A_{i,j}T_{i,j} + \sum_{1 \le k \le l \le m} A_{k,-l}T_{k,-l} + \sum_{1 \le t \le m} A_{0,-t}T_{0,-t}$. Let

$$X = \sum_{1 \le i \le j \le m} a_{i,j}T_{i,j} + \sum_{1 \le k \le l \le m} a_{k,-l}T_{k,-l} + \sum_{1 \le t \le m} a_{0,-t}T_{0,-t} \in \ell_{\rho,\delta},$$

$$Y = \sum_{1\le i\le j\le m} b_{i,j}T_{i,j} + \sum_{1\le k\le l\le m} b_{k,-l}T_{k,-l} + \sum_{1\le t\le m} a_{0,-t}T_{0,-t} \in \ell_{\rho,\delta},$$

where $a_{i,i}, b_{i,i}$ lie in R $(i = 1, 2, \ldots, m)$, $a_{i,j}, b_{i,j}$ lie in $A_{i,j}$ $(1 \le i < j \le m)$, $a_{k,-l}, b_{k,-l}$ lie in $A_{k,-l}$ $(1 \le k \le l \le m)$, and $a_{0,-t}, b_{0,-t}$ lie in $A_{0,-t}$ $(1 \le t \le m)$. It is obvious that $rX + sY \in \ell_{\rho,\delta}$ for any $r, s \in R$. Note that

$$[X, Y] = \sum_{1\le i<j\le m} c_{i,j}T_{i,j} + \sum_{1\le k\le l\le m} d_{k,-l}T_{k,-l} + \sum_{1\le t\le m} f_{0,-t}T_{0,-t}, \tag{2.1}$$

$$c_{i,j} = \sum_{i\le n\le j} (a_{i,n}b_{n,j} - b_{i,n}a_{n,j}), \tag{2.2}$$

$$\begin{aligned} d_{k,-l} =& \Big(\sum_{k\le n\le l} a_{k,n}b_{n,-l} - \rho \sum_{l\le n\le m} a_{k,n}b_{l,-n} - \sum_{l\le n\le m} \delta_n a_{k,-n}b_{l,n}\Big) - \\ & \Big(\sum_{k\le n\le l} b_{k,n}a_{n,-l} - \rho \sum_{l\le n\le m} b_{k,n}a_{l,-n} - \sum_{l\le n\le m} \delta_n b_{k,-n}a_{l,n}\Big) + \\ & \delta(1+\rho)(b_{0,-k}a_{0,-l} - a_{0,-k}b_{0,-l}), \end{aligned} \tag{2.3}$$

$$f_{0,-t} = \sum_{t\le n\le m} (a_{t,n}b_{0,-n} - b_{t,n}a_{0,-n}), \tag{2.4}$$

where δ_n is defined to be $1 - \rho$ when $n = k$, otherwise, $\delta_n = 1$. Because Δ is a flag of ideals of R, we know that $c_{i,j} \in A_{i,j}$ $(1 \le i < j \le m)$, $d_{k,-l} \in A_{k,-l}$ $(1 \le k \le l \le m)$, and $f_{0,-t} \in A_{0,-t}$ $(1 \le t \le m)$. Thus $[X, Y] \in \ell_{\rho,\delta}$. Hence $\ell_{\rho,\delta}$ is a subalgebra of $b_{\rho,\delta}$ containing $h_{\rho,\delta}$.

On the other hand, let $\ell_{\rho,\delta}$ be an intermediate Lie algebra between $h_{\rho,\delta}$ and $b_{\rho,\delta}$. For $1 \le i < j \le m, 1 \le k \le l \le m, 1 \le t \le m$, we define

$$A_{i,j} = \{a \in R|\ aT_{i,j} \in \ell_{\rho,\delta}\}, \quad A_{k,-l} = \{a \in R|\ aT_{k,-l} \in \ell_{\rho,\delta}\},$$

$$A_{0,-t} = \{a \in R|\ aT_{0,-t} \in \ell_{\rho,\delta}\}$$

and let

$$\overline{\Delta} = \{A_{i,j}, A_{k,-l}, A_{0,-t} \in I(R) \mid 1 \le i < j \le m,\ 1 \le k \le l \le m,\ 1 \le t \le m\},$$

$$\overline{\ell} = h_{\rho,\delta} + \sum_{1\le i<j\le m} A_{i,j}T_{i,j} + \sum_{1\le k\le l\le m} A_{k,-l}T_{k,-l} + \sum_{1\le t\le m} A_{0,-t}T_{0,-t}.$$

In the following, we will prove that $\overline{\Delta}$ is a flag of ideals of R, and $\overline{\ell} = \ell_{\rho,\delta}$. It is obvious that all $A_{i,j}, A_{k,-l}, A_{0,-t}$ are ideals of R.

If $1 \le i < k < j \le m$ and $a_{i,k} \in A_{i,k}, a_{k,j} \in A_{k,j}, a_{k,-j} \in A_{k,-j}$, then by

$$[a_{i,k}T_{i,k}, a_{k,j}T_{k,j}] = a_{i,k}a_{k,j}T_{i,j} \in \ell_{\rho,\delta},$$

$$[a_{i,k}T_{i,k}, a_{k,-j}T_{k,-j}] = a_{i,k}a_{k,-j}T_{i,-j} \in \ell_{\rho,\delta},$$

we have that $a_{i,k}a_{k,j} \in A_{i,j}$ and $a_{i,k}a_{k,-j} \in A_{i,-j}$, which lead to $A_{i,k}A_{k,j} \subseteq A_{i,j}$ and $A_{i,k}A_{k,-j} \subseteq A_{i,-j}$, respectively.

If $1 \le i < j < k \le m$ and $a_{i,k} \in A_{i,k}, a_{j,-k} \in A_{j,-k}$, $a_{j,k} \in A_{j,k}, a_{i,-k} \in A_{i,-k}$, then by

$$[a_{i,k}T_{i,k}, a_{j,-k}T_{j,-k}] = -a_{i,k}a_{j,-k}T_{i,-j} \in \ell_{\rho,\delta},$$

$$[a_{j,k}T_{j,k}, a_{i,-k}T_{i,-k}] = a_{j,k}a_{i,-k}T_{i,-j} \in \ell_{\rho,\delta},$$

we have that $a_{i,k}a_{j,-k} \in A_{i,-j}$ and $a_{j,k}a_{i,-k} \in A_{i,-j}$, which lead to $A_{i,k}A_{j,-k} \subseteq A_{i,-j}$ and $A_{j,k}A_{i,-k} \subseteq A_{i,-j}$, respectively.

If $\rho = 1$, the conditions (5) and (6) in Definition 2.1 obviously hold. If $\rho = -1$, suppose that $1 \le i < k \le m$, $a_{i,k} \in A_{i,k}$, $a_{k,-k} \in A_{k,-k}$, $a_{i,-k} \in A_{i,-k}$. Then by

$$[a_{i,k}T_{i,k}, a_{k,-k}T_{k,-k}] = 2a_{i,k}a_{k,-k}T_{i,-k} \in \ell_{\rho,\delta},$$

$$[a_{i,k}T_{i,k}, a_{i,-k}T_{i,-k}] = -a_{i,k}a_{i,-k}T_{i,-i} \in \ell_{\rho,\delta},$$

we have that $A_{i,k}A_{k,-k} \subseteq A_{i,-k}$ and $A_{i,k}A_{i,-k} \subseteq A_{i,-i}$, respectively. We see that (5) and (6) hold for $\rho = \pm 1$.

If $\rho = -1$ or $\delta = 0$, the conditions (7) and (8) in Definition 2.1 obviously hold. If $\rho = \delta = 1$, suppose that $1 \le i < k \le m$, and $a_{0,-i} \in A_{0,-i}$, $a_{0,-k} \in A_{0,-k}, a_{i,k} \in A_{i,k}$. Then by

$$[a_{0,-i}T_{0,-i}, a_{0,-k}T_{0,-k}] = -2a_{0,-i}a_{0,-k}T_{i,-k} \in \ell_{\rho,\delta},$$

$$[a_{i,k}T_{i,k}, a_{0,-k}T_{0,-k}] = a_{i,k}a_{0,-k}T_{0,-i} \in \ell_{\rho,\delta},$$

we have that $A_{0,-i}A_{0,-k} \subseteq A_{i,-k}$ and $A_{i,k}A_{0,-k} \subseteq A_{0,-i}$, respectively. We see that (7) and (8) hold for $\rho = \pm 1$.

These imply that $\overline{\Delta}$ is a flag of ideals of R and $\overline{\ell}$ is an intermediate Lie algebra between $h_{\rho,\delta}$ and $b_{\rho,\delta}$. It is obvious that $\overline{\ell} \subseteq \ell_{\rho,\delta}$. Let $X = H + \sum_{1\le i<j\le m} a_{i,j}T_{i,j} + \sum_{1\le k\le l\le m} a_{k,-l}T_{k,-l} + \sum_{1\le t\le m} a_{0,-t}T_{0,-t} \in \ell_{\rho,\delta}$, where $H \in h_{\rho,\delta}$. For any $1 \le i < j \le m$, by

$$[T_{j,j}, [T_{i,i}, X]] = -a_{i,j}T_{i,j} + a_{i,-j}T_{i,-j} \in \ell_{\rho,\delta},$$

$$[T_{j,j} + T_{i,i}, [T_{j,j}, [T_{i,i}, X]]] = 2a_{i,-j}T_{i,-j} \in \ell_{\rho,\delta},$$

we have that $a_{i,-j} \in A_{i,-j}$ and $a_{i,j} \in A_{i,j}$. It follows that $\sum_{1\le k\le m}(a_{k,-k}T_{k,-k} + a_{0,-k}T_{0,-k}) \in \ell_{\rho,\delta}$, leading to $[T_{t,t}, \sum_{1\le k\le m}(a_{k,-k}T_{k,-k} + a_{0,-k}T_{0,-k})] = 2a_{t,-t}T_{t,-t} + a_{0,-t}T_{0,-t} \in \ell_{\rho,\delta}$ for all $1 \le t \le m$. Since either $T_{t,-t} = 0$ (if $\rho = 1$) or $T_{0,-t} = 0$ (if $\rho = -1$), we know that $a_{t,-t}T_{t,-t}$, $a_{0,-t}T_{0,-t} \in \ell_{\rho,\delta}$, which implies $a_{t,-t} \in A_{t,-t}$, $a_{0,-t} \in A_{0,-t}$, $1 \le t \le m$. We see that $X \in \overline{\ell}$. Thus $\ell_{\rho,\delta} \subseteq \overline{\ell}$, which leads to $\ell_{\rho,\delta} = \overline{\ell}$. This completes the proof. □

3. Derivations of a subalgebra of $\ell_{\rho,\delta}$

For R-modules M and K, we denote by $\mathrm{Hom}_R(M, K)$ the set of all homomorphisms from M to K. $\mathrm{Hom}_R(M, M)$ is abbreviated to $\mathrm{End}_R(M)$. For $1 \le i \le m$, $\chi_i : d(m, R) \to R$, defined by $\chi_i(\mathrm{diag}(d_1, d_2, \ldots, d_m)) = d_i$, is a standard homomorphism from $d(m, R)$ to R. It is easy to see that $\mathrm{Hom}_R(d(m, R), R)$ is a free R-module of rank m with a basis $\{\chi_i |\ i = 1, 2, \ldots, m\}$.

Let

$$\ell_{\rho,\delta} = h_{\rho,\delta} + \sum_{1\le i<j\le m} A_{i,j}T_{i,j} + \sum_{1\le k\le l\le m} A_{k,-l}T_{k,-l} + \sum_{1\le t\le m} A_{0,-t}T_{0,-t}$$

be any given intermediate Lie algebra between $h_{\rho,\delta}$ and $b_{\rho,\delta}$, with $\Delta = \{A_{i,j}, A_{k,-l}, A_{0,-t} \in I(R) \mid 1 \le i < j \le m,\ 1 \le k \le l \le m,\ 1 \le t \le m\}$ a flag of ideals of R. In the following, we shall determine its derivation algebra. As a start, we first consider the derivation algebra of a subalgebra of $\ell_{\rho,\delta}$. Let $p = \ell_{\rho,\delta} \cap t_{\rho,\delta}$. Then $p = h_{\rho,\delta} + \sum_{1\le i<j\le m} A_{i,j}T_{i,j}$. In fact, p is an intermediate algebra between $h_{\rho,\delta}$ and $t_{\rho,\delta}$. It is easy to see that the map $\varphi : t_{\rho,\delta} \to t(m,R)$, defined by $\begin{pmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -A' \end{pmatrix} \mapsto A$ (where $A \in t(m,R)$), is an isomorphism of Lie algebras, under which the image of p is $d(m,R) + \sum_{1\le i<j\le m} A_{i,j}E_{i,j}^{(m)}$ (where $E_{i,j}^{(m)}$ denotes the $m\times m$ matrix, whose (i,j)-entry is 1, all other entries are 0). In [3], the derivation algebra of $d(m,R) + \sum_{1\le i<j\le m} A_{i,j}E_{i,j}^{(m)}$ was determined. We now transfer it to p for later use.

The standard derivations of p are as follows.

(A) **Inner derivations of** p

Let $X \in p$. Then $ad_p\, X : p \to p$, defined by $Y \mapsto [X,Y]$, is a derivation of p, called the inner derivation of p induced by X.

(B) **Extremal derivations of** p

Definition 3.1 *Let $\phi = \{\phi_{i,j} \in \mathrm{End}_R(A_{i,j}),\ |1\le i<j\le m\}$ be a set consisting of homomorphisms of R-modules. ϕ is called suitable for extremal derivations of p if*

$$\phi_{i,j}(a_{i,k}a_{k,j}) = \phi_{i,k}(a_{i,k})a_{k,j} + a_{i,k}\phi_{k,j}(a_{k,j})$$

for any $1\le i<k<j\le m$ (if exists), any $a_{i,k}\in A_{i,k}$ and any $a_{k,j}\in A_{k,j}$.

Using $\phi = \{\phi_{i,j} \in \mathrm{End}_R(A_{i,j})|1\le i<j\le m\}$ which is suitable for extremal derivations, we define $\eta_{p,\phi} : p \to p$ by

$$\eta_{p,\phi}\Big(\sum_{1\le i\le j\le m} a_{i,j}T_{i,j}\Big) = \sum_{1\le i<j\le m} \phi_{i,j}(a_{i,j})T_{i,j},$$

where $a_{i,i}\in R$ and $a_{i,j}\in A_{i,j}$ if $i<j$.

Lemma 3.2 *$\eta_{p,\phi}$ is a derivation of p, provided that $\phi = \{\phi_{i,j}\in \mathrm{End}_R(A_{i,j})|1\le i<j\le m\}$ is suitable for extremal derivations (we call $\eta_{p,\phi}$ an extremal derivation of p).*

(C) **Central derivations of** p

If $1\le i<j\le m$, let $B_{i,j}$ be the annihilator of $A_{i,j}$ in R: $B_{i,j} = \{r\in R|rA_{i,j}=0\}$.

Definition 3.3 *Let $\sigma : h_{\rho,\delta} \to h_{\rho,\delta}$ be a homomorphism of R-modules. σ is called suitable for central derivations of p, if $\chi_i(\varphi(\sigma(H))) - \chi_j(\varphi(\sigma(H))) \in B_{i,j}$ for all $1\le i<j\le m$ and all $H\in h_{\rho,\delta}$.*

Using the homomorphism $\sigma : h_{\rho,\delta}\to h_{\rho,\delta}$ which is suitable for central derivations of p, we define $\tau_{p,\sigma} :\ p\to p$ by $\tau_{p,\sigma}(X) = \sigma(H_X)$, $X\in p$, where H_X denotes the projection of X to $h_{\rho,\delta}$ (if $X = \sum_{1\le i\le j\le m} a_{i,j}T_{i,j}\in p$. Then $H_X = \sum_{1\le i\le m} a_{i,i}T_{i,i}$).

Lemma 3.4[3] *$\tau_{p,\sigma}$ is a derivation of p, provided that σ is suitable for central derivations of p (we call $\tau_{p,\sigma}$ a central derivation of p).*

Theorem 3.5[3] *Let $m > 1$, R an arbitrary commutative ring with identity, and $p = h_{\rho,\delta} + \sum_{1\le i<j\le m} A_{i,j}T_{i,j}$ an intermediate Lie algebra between $h_{\rho,\delta}$ and $t_{\rho,\delta}$ with $\Delta = \{A_{i,j} \in I(R) | 1 \le i < j \le m\}$ a subset of $I(R)$ satisfying $A_{i,k}A_{k,j} \subseteq A_{i,j}$ (if k exists such that $1 \le i < k < j \le m$). Then any derivation ψ_p of p may be written as the sum of an inner derivation, an extremal derivation and a central derivation.*

4. Standard derivations of $\ell_{\rho,\delta}$

Let

$$\ell_{\rho,\delta} = h_{\rho,\delta} + \sum_{1\le i<j\le m} A_{i,j}T_{i,j} + \sum_{1\le k\le l\le m} A_{k,-l}T_{k,-l} + \sum_{1\le t\le m} A_{0,-t}T_{0,-t}$$

be any given intermediate Lie algebra between $h_{\rho,\delta}$ and $b_{\rho,\delta}$, with $\Delta = \{A_{i,j}, A_{k,-l}, A_{0,-t} \in I(R) \mid 1 \le i < j \le m,\ 1 \le k \le l \le m,\ 1 \le t \le m\}$ a flag of ideals of R. In this section, we will define some standard derivations of $\ell_{\rho,\delta}$.

(A) Inner derivations of $\ell_{\rho,\delta}$

Let $X \in \ell_{\rho,\delta}$. Then $adX : \ell_{\rho,\delta} \to \ell_{\rho,\delta}$, sending Y to $[X, Y]$, is a derivation of $\ell_{\rho,\delta}$, called the inner derivation of $\ell_{\rho,\delta}$ induced by X.

(B) Extremal derivations of $\ell_{\rho,\delta}$

Definition 4.1 *Let $\widetilde{\phi} = \{\phi_{i,j} \in \mathrm{End}_R(A_{i,j}), \phi_{k,-l} \in \mathrm{End}_R(A_{k,-l}), \phi_{0,-t} \in \mathrm{End}_R(A_{0,-t}) | 1 \le i < j \le m, 1 \le k \le l \le m,\ 1 \le t \le m\}$ be a set consisting of homomorphisms of R-modules. We call $\widetilde{\phi}$ suitable for extremal derivations of $\ell_{\rho,\delta}$ if the following conditions are satisfied:*

(1) $\phi_{i,j}(a_{i,k}a_{k,j}) = \phi_{i,k}(a_{i,k})a_{k,j} + a_{i,k}\phi_{k,j}(a_{k,j})$ *(if $1 \le i < k < j \le m$),*

(2) $\phi_{i,-j}(a_{i,k}a_{k,-j}) = \phi_{i,k}(a_{i,k})a_{k,-j} + a_{i,k}\phi_{k,-j}(a_{k,-j})$ *(if $1 \le i < k < j \le m$),*

(3) $\phi_{i,-j}(a_{i,k}a_{j,-k}) = \phi_{i,k}(a_{i,k})a_{j,-k} + a_{i,k}\phi_{j,-k}(a_{j,-k})$ *(if $1 \le i < j < k \le m$),*

(4) $\phi_{i,-j}(a_{j,k}a_{i,-k}) = \phi_{j,k}(a_{j,k})a_{i,-k} + a_{j,k}\phi_{i,-k}(a_{i,-k})$ *(if $1 \le i < j < k \le m$),*

(5) $(1-\rho)\phi_{i,-k}(a_{i,k}a_{k,-k}) = (1-\rho)[\phi_{i,k}(a_{i,k})a_{k,-k} + a_{i,k}\phi_{k,-k}(a_{k,-k})]$ *(if $1 \le i < k \le m$),*

(6) $(1-\rho)\phi_{i,-i}(a_{i,k}a_{i,-k}) = (1-\rho)[\phi_{i,k}(a_{i,k})a_{i,-k} + a_{i,k}\phi_{i,-k}(a_{i,-k})]$ *(if $1 \le i < k \le m$),*

(7) $\delta(1-\rho)\phi_{i,-k}(a_{0,-i}a_{0,-k}) = \delta(1-\rho)[\phi_{0,-i}(a_{0,-i})a_{0,-k} + a_{0,-i}\phi_{0,-k}(a_{0,-k})]$ *(if $1 \le i < k \le m$),*

(8) $\delta(1-\rho)\phi_{0,-i}(a_{i,k}a_{0,-k}) = \delta(1-\rho)[\phi_{i,k}(a_{i,k})a_{0,-k} + a_{i,k}\phi_{0,-k}(a_{0,-k})]$ *(if $1 \le i < k \le m$),*

where $a_{i,k} \in A_{i,k}, \ldots, a_{k,-j} \in A_{k,-j}, \ldots$.

Using $\widetilde{\phi} = \{\phi_{i,j} \in \mathrm{End}_R(A_{i,j}), \phi_{k,-l} \in \mathrm{End}_R(A_{k,-l}), \phi_{0,-t} \in \mathrm{End}_R(A_{0,-t}) | 1 \le i < j \le m, 1 \le k \le l \le m, 1 \le t \le m\}$ which is suitable for extremal derivations, we define $\eta_{\widetilde{\phi}} : \ell_{\rho,\delta} \to \ell_{\rho,\delta}$ by

$$\begin{aligned}
&\eta_{\widetilde{\phi}}\Big(\sum_{1\le i\le j\le m} a_{i,j}T_{i,j} + \sum_{1\le k\le l\le m} a_{k,-l}T_{k,-l} + \sum_{1\le t\le m} a_{0,-t}T_{0,-t}\Big)\\
&= \sum_{1\le i<j\le m} \phi_{i,j}(a_{i,j})T_{i,j} + \sum_{1\le k\le l\le m} \phi_{k,-l}(a_{k,-l})T_{k,-l} + \sum_{1\le t\le m} \phi_{0,-t}(a_{0,-t})T_{0,-t},
\end{aligned}$$

where $a_{i,i} \in R$, $a_{i,j} \in A_{i,j}$, $a_{k,-l} \in A_{k,-l}$, $a_{0,-t} \in A_{0,-t}$.

Lemma 4.2 *$\eta_{\widetilde{\phi}}$ is a derivation of $\ell_{\rho,\delta}$, provided that $\widetilde{\phi}$ is suitable for extremal derivations of $\ell_{\rho,\delta}$.*

Proof Let $\widetilde{\phi} = \{\phi_{i,j} \in \mathrm{End}_R(A_{i,j}), \phi_{k,-l} \in \mathrm{End}_R(A_{k,-l}), \phi_{0,-t} \in \mathrm{End}_R(A_{0,-t}) | 1 \le i < j \le m, 1 \le k \le l \le m,\ 1 \le t \le m\}$ be suitable for extremal derivations of $\ell_{\rho,\delta}$, and set

$$X = \sum_{1\le i\le j\le m} a_{i,j}T_{i,j} + \sum_{1\le k\le l\le m} a_{k,-l}T_{k,-l} + \sum_{1\le t\le m} a_{0,-t}T_{0,-t} \in \ell_{\rho,\delta},$$
$$Y = \sum_{1\le i\le j\le m} b_{i,j}T_{i,j} + \sum_{1\le k\le l\le m} b_{k,-l}T_{k,-l} + \sum_{1\le t\le m} a_{0,-t}T_{0,-t} \in \ell_{\rho,\delta},$$

where $a_{i,i}, b_{i,i}$ lie in R, $a_{i,j}, b_{i,j}$ lie in $A_{i,j}$ $(1 \le i < j \le m)$, $a_{k,-l}, b_{k,-l}$ lie in $A_{k,-l}$ $(1 \le k \le l \le m)$, and $a_{0,-t}, b_{0,-t}$ lie in $A_{0,-t}$ $(1 \le t \le m)$. It is obvious that $\eta_{\widetilde{\phi}}(rX + sY) = r\eta_{\widetilde{\phi}}(X) + s\eta_{\widetilde{\phi}}(Y)$ for any $r, s \in R$. Note that the equalities (2.1)–(2.4) hold. Because $\widetilde{\phi}$ is suitable for extremal derivations of $\ell_{\rho,\delta}$, we know (by calculation) that

$$\eta_{\widetilde{\phi}}([X,Y]) = [\eta_{\widetilde{\phi}}(X), Y] + [X, \eta_{\widetilde{\phi}}(Y)].$$

So $\eta_{\widetilde{\phi}}$ is a derivation of $\ell_{\rho,\delta}$.

Definition 4.3 *The above $\eta_{\widetilde{\phi}}$ is called an extremal derivation of $\ell_{\rho,\delta}$.*

Remark 4.4 The restriction of $\eta_{\widetilde{\phi}}$ to p exactly is $\eta_{p,\phi}$.

(C) **Central derivations of $\ell_{\rho,\delta}$**

For $1 \le i < j \le m, 1 \le k \le l \le m, 1 \le t \le m$, let $B_{i,j}$ (resp., $B_{k,-l}$, $B_{0,-t}$) be the annihilator of $A_{i,j}$ (resp., $A_{k,-l}$, $A_{0,-t}$) in R:

$$B_{i,j} = \{r \in R \mid rA_{i,j} = 0\}, \quad B_{k,-l} = \{r \in R \mid rA_{k,-l} = 0\}, \quad B_{0,-t} = \{r \in R \mid rA_{0,-t} = 0\}.$$

Definition 4.5 *Let $\widetilde{\sigma} : h_{\rho,\delta} \to h_{\rho,\delta}$ be a homomorphism of R-modules. We call $\widetilde{\sigma}$ suitable for central derivations of $\ell_{\rho,\delta}$, if $\chi_i(\varphi(\widetilde{\sigma}(H))) - \chi_j(\varphi(\widetilde{\sigma}(H))) \in B_{i,j}$, $\chi_k(\varphi(\widetilde{\sigma}(H))) + \chi_l(\varphi(\widetilde{\sigma}(H))) \in B_{k,-l}$, and $\chi_t(\varphi(\widetilde{\sigma}(H))) \in B_{0,-t}$ for all $1 \le i < j \le m$, all $1 \le k \le l \le m$, all $1 \le t \le m$ and all $H \in h_{\rho,\delta}$.*

Using the homomorphism $\widetilde{\sigma} : h_{\rho,\delta} \to h_{\rho,\delta}$ which is suitable for central derivations of $\ell_{\rho,\delta}$, we define $\tau_{\widetilde{\sigma}} :\ \ell_{\rho,\delta} \to \ell_{\rho,\delta}$ by

$$\tau_{\widetilde{\sigma}}(X) = \widetilde{\sigma}(H_X), \quad X \in \ell_{\rho,\delta},$$

where H_X denotes the projection of X to $h_{\rho,\delta}$ (if $X = \sum_{1\le i\le j\le m} a_{i,j}T_{i,j} + \sum_{1\le k\le l\le m} a_{k,-l}T_{k,-l} + \sum_{1\le t\le m} a_{0,-t}T_{0,-t} \in \ell_{\rho,\delta}$, then $H_X = \sum_{1\le i\le m} a_{i,i}T_{i,i}$).

Lemma 4.6 *$\tau_{\widetilde{\sigma}}$ is a derivation of $\ell_{\rho,\delta}$, provided that $\widetilde{\sigma}$ is suitable for central derivations of $\ell_{\rho,\delta}$.*

Proof By definition, $\tau_{\widetilde{\sigma}}([X,Y]) = 0$ for any $X, Y \in \ell_{\rho,\delta}$. On the other hand, $[\tau_{\widetilde{\sigma}}(X), Y] + [X, \tau_{\widetilde{\sigma}}(Y)] = 0$, because $\tau_{\widetilde{\sigma}}$ sends each element in $\ell_{\rho,\delta}$ to its center $Z(\ell_{\rho,\delta})$. This shows that $\tau_{\widetilde{\sigma}}$ is a derivation of $\ell_{\rho,\delta}$.

Definition 4.7 *We call $\tau_{\widetilde{\sigma}}$ a central derivation of $\ell_{\rho,\delta}$.*

5. Derivations of $\ell_{\rho,\delta}$

When $m=1$, the derivation algebra of $\ell_{\rho,\delta}$ has been studied in [3] (the result is more trivial). In this paper, we only consider the case when $m>1$.

Theorem 5.1 *Let $m>1$, R an arbitrary commutative ring with identity, and $\ell_{\rho,\delta}$ any given intermediate Lie algebra between $h_{\rho,\delta}$ and $b_{\rho,\delta}$. If $2\in R^*$, then any derivation ψ of $\ell_{\rho,\delta}$ may be written in the form:*

$$\psi=\operatorname{ad}X+\eta_{\tilde{\phi}}+\tau_{\tilde{\sigma}},$$

where $\operatorname{ad}X$, $\eta_{\tilde{\phi}}$, and $\tau_{\tilde{\sigma}}$ are the inner, extremal and central derivations of $\ell_{\rho,\delta}$, respectively.

Proof Let $\ell_{\rho,\delta}=h_{\rho,\delta}+\sum_{1\le i<j\le m}A_{i,j}T_{i,j}+\sum_{1\le k\le l\le m}A_{k,-l}T_{k,-l}+\sum_{1\le t\le m}A_{0,-t}T_{0,-t}$ with $\Delta=\{A_{i,j},A_{k,-l},A_{0,-t}\in I(R)|1\le i<j\le m,\ 1\le k\le l\le m,\ 1\le t\le m\}$ a flag of ideals of R. Let ψ be any derivation of $\ell_{\rho,\delta}$. Set $z=\ell_{\rho,\delta}\bigcap w_{\rho,\delta}$ and $p=\ell_{\rho,\delta}\bigcap t_{\rho,\delta}$. Then $z=\sum_{1\le k\le l\le m}A_{k,-l}T_{k,-l}+\sum_{1\le t\le m}A_{0,-t}T_{0,-t}$, $p=\sum_{1\le i\le j\le m}A_{i,j}T_{i,j}$. Denote

$$J=\begin{pmatrix}0&0&0\\0&E&0\\0&0&-E\end{pmatrix}.$$

In the following, we will give the proof by steps.

Step 1 There exists $Z_0\in z$ such that $(\psi+\operatorname{ad}Z_0)(h_{\rho,\delta})\subseteq p$.

For any $\Lambda=\operatorname{diag}(d_1,d_2,\dots,d_m)\in d(m,R)$, we suppose that

$$\psi\begin{pmatrix}0&0&0\\0&\Lambda&0\\0&0&-\Lambda\end{pmatrix}\equiv\sum_{1\le k\le l\le m}r_{k,-l}T_{k,-l}+\sum_{1\le t\le m}r_{0,-t}T_{0,-t}\ (\operatorname{mod}p),$$

and suppose that

$$\psi(J)\equiv\sum_{1\le k\le l\le m}a_{k,-l}T_{k,-l}+\sum_{1\le t\le m}a_{0,-t}T_{0,-t}\ (\operatorname{mod}p),$$

where $r_{k,-l},a_{k,-l}$ lie in $A_{k,-l}$, and $r_{0,-t},a_{0,-t}$ lie in $A_{0,-t}$. By applying ψ on

$$\left[\begin{pmatrix}0&0&0\\0&\Lambda&0\\0&0&-\Lambda\end{pmatrix},J\right]=0,$$

we get that $r_{k,-l}=2^{-1}(d_k+d_l)a_{k,-l}$ and $r_{0,-t}=d_ta_{0,-t}$. Choose $Z_0=2^{-1}\sum_{1\le k\le l\le m}a_{k,-l}T_{k,-l}+\sum_{1\le t\le m}a_{0,-t}T_{0,-t}\in z$. Then we have that

$$(\psi+\operatorname{ad}Z_0)\begin{pmatrix}0&0&0\\0&\Lambda&0\\0&0&-\Lambda\end{pmatrix}\equiv 0\ (\operatorname{mod}p).$$

Hence $(\psi+\operatorname{ad}Z_0)(h_{\rho,\delta})\subseteq p$. Now we denote $\psi+\operatorname{ad}Z_0$ by ψ_1.

Step 2 z is stable under ψ_1.

For any $Z_1 \in \sum_{1\le t\le m} A_{0,-t}T_{0,-t}$, we first prove that $\psi_1(Z_1) \in z$. By applying ψ_1 on $[J, Z_1] = Z_1$, we see that $[\psi_1(J), Z_1] + [J, \psi_1(Z_1)] = \psi_1(Z_1)$. It is easy to know that $[\psi_1(J), Z_1] \in z$, $[J, \psi_1(Z_1)] \in z$. Then we get $\psi_1(Z_1) \subseteq z$.

For any $Z_2 \in \sum_{1\le k\le l\le m} A_{k,-l}T_{k,-l}$, we next prove that $\psi_1(Z_2) \in z$. By applying ψ_1 on $[J, Z_2] = 2Z_2$, we see that

$$[\psi_1(J), Z_2] + [J, \psi_1(Z_2)] = 2\psi_1(Z_2).$$

It is easy to see that the left hand side lies in z, which leads to $\psi_1(Z_2) \in z$.

Step 3 p is stable under ψ_1.

For any $P \in p$, we suppose that $\psi_1(P) = \begin{pmatrix} 0 & 0 & \beta \\ -\beta & A & B \\ 0 & 0 & -A' \end{pmatrix} \in \ell_{\rho,\delta}$. By applying ψ_1 on $[J, P] = 0$, we have $[\psi_1(J), P] + [J, \psi_1(P)] = 0$. Because $\psi_1(h_{\rho,\delta}) \subseteq p$, we get $[\psi_1(J), P] \in p$. Set $[\psi_1(J), P] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & -A_1' \end{pmatrix}$. On the other hand, $[J, \psi_1(P)] = \begin{pmatrix} 0 & 0 & \beta \\ -\beta & 0 & 2B \\ 0 & 0 & 0 \end{pmatrix}$. Then we see that $\begin{pmatrix} 0 & 0 & \beta \\ -\beta & A_1 & 2B \\ 0 & 0 & -A_1' \end{pmatrix} = 0$. This implies that $A_1 = 0$, $B = 0$, $\beta = 0$. Thus $\psi_1(P) \in p$ leads to $\psi_1(p) \subseteq p$.

Step 4 There exists $P_0 \in p$ and there exists $\phi = \{\phi_{i,j} \in \mathrm{End}_R(A_{i,j}), |1 \le i < j \le m\}$ which is suitable for extremal derivations of p, such that for any $\sum_{1\le i<j\le m} a_{i,j}T_{i,j} \in p$,

$$(\psi_1 - \mathrm{ad}\, P_0)\Big(\sum_{1\le i<j\le m} a_{i,j}T_{i,j}\Big) = \sum_{1\le i<j\le m} \phi_{i,j}(a_{i,j})T_{i,j}.$$

Since p is stable under ψ_1, ψ_1 may induce a derivation $\psi_1|_p$ of p by restricting ψ_1 to p. Thus by Theorem 3.5, $\psi_1|_p$ can be written in the form:

$$\psi_1|_p = \mathrm{ad}_p\ P_0 + \eta_{p,\phi} + \tau_{p,\sigma},$$

where $P_0 \in p$, $\phi = \{\phi_{i,j} \in \mathrm{End}_R(A_{i,j})|1 \le i < j \le m\}$ is suitable for extremal derivations of p (satisfies the condition that $\phi_{i,j}(a_{i,k}a_{k,j}) = \phi_{i,k}(a_{i,k})a_{k,j} + a_{i,k}\phi_{k,j}(a_{k,j})$), and $\sigma : h_{\rho,\delta} \to h_{\rho,\delta}$ is suitable for central derivations of p satisfying the condition that $\chi_i(\varphi(\sigma(H))) - \chi_j(\varphi(\sigma(H))) \in B_{i,j}$ for all $1 \le i < j \le m$ and all $H \in h_{\rho,\delta}$. It is obvious that the restriction of adP_0 to p is $\mathrm{ad}_p P_0$. Then $(\psi_1 - \mathrm{ad}\, P_0)|_p = \eta_{p,\phi} + \tau_{p,\sigma}$. We denote $\psi_1 - \mathrm{ad}\, P_0$ by ψ_2. Then

$$\psi_2\Big(\sum_{1\le i<j\le m} a_{i,j}T_{i,j}\Big) = \sum_{1\le i<j\le m} \phi_{i,j}(a_{i,j})T_{i,j},$$

for any $\sum_{1\le i<j\le m} a_{i,j}T_{i,j} \in p$.

Step 5 $\psi_2(A_{k,-l}T_{k,-l}) \subseteq A_{k,-l}T_{k,-l}$ for all $1 \le k \le l \le m$ and $\psi_2(A_{0,-l}T_{0,-l}) \subseteq A_{0,-l}T_{0,-l}$ for all $1 \le l \le m$.

For any $0 \le k \le l \le m$ ($l \neq 0$) and $a_{k,-l} \in A_{k,-l}$, assume that

$$\psi_2(a_{k,-l}T_{k,-l}) = \sum_{1\le i\le j\le m} r_{i,-j}T_{i,-j} + \sum_{1\le t\le m} r_{0,-t}T_{0,-t}, \tag{5.1}$$

with $r_{i,-j} \in A_{i,-j}$, $r_{0,-t} \in A_{0,-t}$. By applying ψ_2 on $[T_{l,l}, a_{k,-l}T_{k,-l}] = a_{k,-l}T_{k,-l}$, we have that

$$[\psi_2(T_{l,l}), a_{k,-l}T_{k,-l}] + [T_{l,l}, \psi_2(a_{k,-l}T_{k,-l})] = \psi_2(a_{k,-l}T_{k,-l}).$$

Since $\psi_2(T_{l,l}) \in p$, we have $[\psi_2(T_{l,l}), a_{k,-l}T_{k,-l}] \in A_{i,-l}T_{i,-l}$. On the other hand, we have

$$[T_{l,l}, \psi_2(a_{k,-l}T_{k,-l})] = \sum_{l\le j\le m} r_{l,-j}T_{l,-j} + \sum_{1\le i\le l-1} r_{i,-l}T_{i,-l} + r_{0,-l}T_{0,-l}.$$

These show that $r_{i,-j} = 0$ when $i \neq l$ and $j \neq l$, and $r_{0,-t} = 0$ when $t \neq l$ in (5.1).

If $k \neq l$, similarly, by applying ψ_2 on $[T_{k,k}, a_{k,-l}T_{k,-l}] = a_{k,-l}T_{k,-l}$, we can get that $r_{i,-j} = 0$ when $i \neq k$ and $j \neq k$, and $r_{0,-t} = 0$ when $t \neq k$ in (5.1). Thus, $\psi_2(a_{k,-l}T_{k,-l}) \in A_{k,-l}T_{k,-l}$, which shows that $\psi_2(A_{k,-l}T_{k,-l}) \in A_{k,-l}T_{k,-l}$ for any $0 \le k < l \le m$.

Now we consider the condition of $k = l$. For $\rho = 1$, which leads to $T_{l,-l} = 0$, $\psi_2(a_{l,-l}T_{l,-l}) \in A_{l,-l}T_{l,-l}$ obviously holds. For $\rho = -1$, we see that $T_{0,-t} = 0$ in (5.1), $1 \le t \le m$. Set $s \neq l$ ($1 \le s \le m$). By applying ψ_2 on $[T_{s,s}, a_{l,-l}T_{l,-l}] = 0$, we have that

$$[\psi_2(T_{s,s}), a_{l,-l}T_{l,-l}] + \Big[T_{s,s}, \sum_{l\le j\le m} r_{l,-j}T_{l,-j} + \sum_{1\le i\le l-1} r_{i,-l}T_{i,-l}\Big] = 0.$$

This shows that $r_{l,-j} = 0$ when $j = s$, and $r_{i,-l} = 0$ when $i = s$. Since $s \neq l$ is chosen arbitrarily, we see that $\psi_2(a_{l,-l}T_{l,-l}) \in A_{l,-l}T_{l,-l}$. Hence $\psi_2(A_{l,-l}T_{l,-l}) \subseteq A_{l,-l}T_{l,-l}$.

Now for any $1 \le k \le l \le m$, $1 \le t \le m$, we define $\phi_{k,-l} : A_{k,-l} \to A_{k,-l}$ and $\phi_{0,-t} : A_{0,-t} \to A_{0,-t}$ such that $\psi_2(a_{k,-l}T_{k,-l}) = \phi_{k,-l}(a_{k,-l})T_{k,-l}$ and $\psi_2(a_{0,-t}T_{0,-t}) = \phi_{0,-t}(a_{0,-t})T_{0,-t}$. Then $\phi_{k,-l}$ and $\phi_{0,-t}$ are endomorphisms of the R-modulars $A_{k,-l}$ and $A_{0,-t}$, respectively. Let

$$\widetilde{\phi} = \{\phi_{i,j}, \phi_{k,-l}, \phi_{0,-t} | 1 \le i < j \le m, 1 \le k \le l \le m, 1 \le t \le m\}.$$

Step 6 $\widetilde{\phi}$ is suitable for extremal derivations of $\ell_{\rho,\delta}$.

We know in Step 4 that $\phi_{i,j}(a_{i,k}a_{k,j}) = \phi_{i,k}(a_{i,k})a_{k,j} + a_{i,k}\phi_{k,j}(a_{k,j})$, where $1 \le i < k < j \le m$ and $a_{i,k} \in A_{i,k}, a_{k,j} \in A_{k,j}$.

For $1 \le i < k < j \le m$ and $a_{i,k} \in A_{i,k}, a_{k,-j} \in A_{k,-j}$, by applying ψ_2 on

$$a_{i,k}a_{k,-j}T_{i,-j} = [a_{i,k}T_{i,k}, a_{k,-j}T_{k,-j}],$$

we have that $\phi_{i,-j}(a_{i,k}a_{k,-j}) = \phi_{i,k}(a_{i,k})a_{k,-j} + a_{i,k}\phi_{k,-j}(a_{k,-j})$.

For $1 \le i < j < k \le m$ and $a_{i,k} \in A_{i,k}, a_{j,-k} \in A_{j,-k}$, $a_{j,k} \in A_{j,k}$, $a_{i,-k} \in A_{i,-k}$, by applying ψ_2 on

$$a_{i,k}a_{j,-k}T_{i,-j} = -[a_{i,k}T_{i,k}, a_{j,-k}T_{j,-k}], \quad a_{j,k}a_{i,-k}T_{i,-j} = [a_{j,k}T_{j,k}, a_{i,-k}T_{i,-k}],$$

we have that

$$\phi_{i,-j}(a_{i,k}a_{j,-k}) = \phi_{i,k}(a_{i,k})a_{j,-k} + a_{i,k}\phi_{j,-k}(a_{j,-k}),$$

$$\phi_{i,-j}(a_{j,k}a_{i,-k}) = \phi_{j,k}(a_{j,k})a_{i,-k} + a_{j,k}\phi_{i,-k}(a_{i,-k}).$$

For $1 \leq i < k \leq m$, and $a_{i,k} \in A_{i,k}$, $a_{k,-k} \in A_{k,-k}$, $a_{i,-k} \in A_{i,-k}$, $a_{0,-i} \in A_{0,-i}$, $a_{0,-k} \in A_{0,-k}$, by applying ψ_2 on

$$\begin{aligned}
&(1-\rho)a_{i,k}a_{k,-k}T_{i,-k} = [a_{i,k}T_{i,k}, a_{k,-k}T_{k,-k}],\\
&(1-\rho)a_{i,k}a_{i,-k}T_{i,-i} = 2[a_{i,k}T_{i,k}, a_{i,-k}T_{i,-k}],\\
&\delta(1+\rho)a_{0,-i}a_{0,-k}T_{i,-k} = -[a_{0,-i}T_{0,-i}, a_{0,-k}T_{0,-k}],\\
&\delta(1+\rho)a_{i,k}a_{0,-k}T_{0,-i} = 2[a_{i,k}T_{i,k}, a_{0,-k}T_{0,-k}],
\end{aligned}$$

we have that

$$\begin{aligned}
&(1-\rho)\phi_{i,-k}(a_{i,k}a_{k,-k}) = (1-\rho)[\phi_{i,k}(a_{i,k})a_{k,-k} + a_{i,k}\phi_{k,-k}(a_{k,-k})],\\
&(1-\rho)\phi_{i,-i}(a_{i,k}a_{i,-k}) = (1-\rho)[\phi_{i,k}(a_{i,k})a_{i,-k} + a_{i,k}\phi_{i,-k}(a_{i,-k})],\\
&\delta(1+\rho)\phi_{i,-k}(a_{0,-i}a_{0,-k}) = \delta(1+\rho)[\phi_{0,-i}(a_{0,-i})a_{0,-k} + a_{0,-i}\phi_{0,-k}(a_{0,-k})],\\
&\delta(1+\rho)\phi_{0,-i}(a_{i,k}a_{0,-k}) = \delta(1+\rho)[\phi_{i,k}(a_{i,k})a_{0,-k} + a_{i,k}\phi_{0,-k}(a_{0,-k})].
\end{aligned}$$

Hence $\widetilde{\phi}$ is suitable for extremal derivations of $\ell_{\rho,\delta}$. Using $\widetilde{\phi}$, we construct the extremal derivation $\eta_{\widetilde{\phi}}$ of $\ell_{\rho,\delta}$ by

$$\begin{aligned}
&\eta_{\widetilde{\phi}}\Big(H + \sum_{1\leq i<j\leq m} a_{i,j}T_{i,j} + \sum_{1\leq k\leq l\leq m} a_{k,-l}T_{k,-l} + \sum_{1\leq t\leq m} a_{0,-t}T_{0,-t}\Big)\\
&= \sum_{1\leq i<j\leq m} \phi_{i,j}(a_{i,j})T_{i,j} + \sum_{1\leq k\leq l\leq m} \phi_{k,-l}(a_{k,-l})T_{k,-l} + \sum_{1\leq t\leq m} \phi_{0,-t}(a_{0,-t})T_{0,-t},
\end{aligned}$$

where $H \in h_{\rho,\delta}$, $a_{i,j} \in A_{i,j}$, $a_{k,-l} \in A_{k,-l}$ and $a_{0,-t} \in A_{0,-t}$. Let ψ_3 denote $\psi_2 - \eta_{\widetilde{\phi}}$. Then $\psi_3(Z) = 0$ for any $Z \in z$ and $\psi_3(\sum_{1\leq i<j\leq m} a_{i,j}T_{i,j}) = 0$ for $\sum_{1\leq i<j\leq m} a_{i,j}T_{i,j} \in p$.

Step 7 ψ_3 is a central derivation of $\ell_{\rho,\delta}$.

For any $\Lambda = \mathrm{diag}(d_1, d_2, \ldots, d_m) \in d(m, R)$, let $H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & -\Lambda \end{pmatrix} \in h_{\rho,\delta}$. For any $a_{i,j} \in A_{i,j}$, $a_{k,-l} \in A_{k,-l}$, $a_{0,-t} \in A_{0,-t}$, by applying ψ_3 on

$$\begin{aligned}
&[H, a_{i,j}T_{i,j}] = (d_i - d_j)a_{i,j}T_{i,j},\\
&[H, a_{k,-l}T_{k,-l}] = (d_k + d_l)a_{k,-l}T_{k,-l},\\
&[H, a_{0,-t}T_{0,-t}] = d_t a_{0,-t}T_{0,-t},
\end{aligned}$$

we have that

$$[\psi_3(H), a_{i,j}T_{i,j}] = 0, \quad [\psi_3(H), a_{k,-l}T_{k,-l}] = 0, \quad [\psi_3(H), a_{0,-t}T_{0,-t}] = 0,$$

which implies that $(\chi_i - \chi_j)(\varphi(\psi_3(H)))\cdot a_{i,j} = 0$, $(\chi_k + \chi_l)(\varphi(\psi_3(H)))\cdot a_{k,-l} = 0$ and $\chi_t(\varphi(\psi_3(H)))\cdot a_{0,-t} = 0$. It is easy to see that ψ_3 is exactly a central derivation of $\ell_{\rho,\delta}$.

Now we see that ψ is the sum of an inner derivation, an extremal derivation and a central derivation of $\ell_{\rho,\delta}$. The proof is completed. $\square$

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