Journal of Mathematical Research & Exposition
May, 2009, Vol. 29, No. 3, pp. 477–484
DOI:10.3770/j.issn:1000-341X.2009.03.012
Http://jmre.dlut.edu.cn

The Common Solution for the Question of Fixed Points and the Question of Variational Inclusions

LIU Ying[1], *CHEN Yong Li*[2]
(1. College of Mathematics and Computer, Hebei University, Hebei 071002, China;
2. College of Technology, North China Electric Power University, Hebei 071051, China)
(E-mail: nanbei00@sina.com)

Abstract In this paper, we introduce an iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inclusion for an inverse-strongly monotone mapping and a maximal monotone mapping in a real Hilbert space. Then we show that the sequence converges strongly to a common element of two sets. Using the result, we consider the problem of finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping in a real Hilbert space.

Keywords inverse-strongly monotone mapping; nonexpansive mapping; variational inclusion; strong convergence.

Document code A
MR(2000) Subject Classification 47H05; 47H09
Chinese Library Classification O177.91

1. Introduction

Let H be a real Hilbert space, $A : H \to H$ be a single-valued mapping and $M : H \to 2^H$ be a multivalued mapping. The variational inclusion problem is to find a $u \in H$ such that

$$0 \in A(u) + M(u). \tag{1.1}$$

The set of solutions of the variational inclusion(1.1) is denoted by $VI(H, A, M)$.

Special Cases

(1) When M is a maximal monotone mapping and A is a strongly monotone and Lipschitz continuous mapping, problem (1.1) has been studied by Huang[1].

(2) If $M = \partial\phi$, where $\partial\phi$ denotes the subdifferential of a proper, convex and lower semicontinuous function $\phi : H \to R\bigcup\{+\infty\}$, then problem (1.1) reduces to the following problem: find $u \in H$, such that

$$\langle A(u), v - u\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H, \tag{1.2}$$

which is called a nonlinear variational inequality and has been studied by many authors[2,3].

(3) If $M = \partial\delta_K$, where δ_K is the indicator function of a nonempty, closed and convex subset K of H, then problem (1.1) reduces to the following problem: find $u \in K$, such that

$$\langle A(u), v - u\rangle \geq 0, \quad \forall v \in K, \tag{1.3}$$

Received date: 2007-03-01; **Accepted date**: 2007-10-30

which is the classical variational inequality[4,5].

A single-valued mapping A of H into itself is called α-inverse-strongly monotone if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2,$$

for all $x, y \in H$[5−9]. A mapping S of H into itself is called nonexpansive if

$$\|Sx-Sy\| \leq \|x-y\|, \quad \forall x, y \in H.$$

We denote by $F(S)$ the set of fixed points of S.

In this paper, we introduce a new iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inclusion for an inverse-strongly monotone mapping and a maximal monotone mapping in a Hilbert space. Then we show that the sequence converges strongly to a common element of two sets. The result generalized Theorem 3.1 in [5] from variational inequality (1.3) to variational inclusion (1.1), which not only makes the result of Theorem 3.1 in [5] become a special case of this paper, but also remove a lot of assumptions. Using the result, we consider the problem of finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping in a real Hilbert space.

2. Preliminaries

In what follows, we always let X be a real Banach space with dual space X^*, H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let C be a closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x-P_C x\| \leq \|x-y\|$ for all $y \in C$. P_C is called the metric projection of H into C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C satisfies

$$\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.1}$$

A set-valued mapping $M : H \to 2^H$ is called monotone if for all $x, y \in H, u \in Mx, v \in My$ imply $\langle x-y, u-v\rangle \geq 0$. A monotone mapping $M : H \to 2^H$ is maximal if $(I+\lambda M)H = H$, for all $\lambda > 0$, where I denotes the identity mapping on H. It is known that a monotone mapping M is maximal if and only if for $(x,u) \in H \times H, \langle x-y, u-v\rangle \geq 0$ for every $(y,v) \in G(M)$ (the graph of M) implies $u \in Mx$.

If A is an α-inverse-strongly monotone mapping of H into itself, then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous monotone mapping. We also have that for all $x, y \in H$ and $\lambda > 0$,

$$\begin{aligned}\|(I-\lambda A)x-(I-\lambda A)y\|^2 &= \|(x-y)-\lambda(Ax-Ay)\|^2 \\ &= \|x-y\|^2 - 2\lambda\langle x-y, Ax-Ay\rangle + \lambda^2\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2 + \lambda(\lambda-2\alpha)\|Ax-Ay\|^2. \end{aligned}\tag{2.2}$$

So, if $\lambda \leq 2\alpha$, then $I-\lambda A$ is a nonexpansive mapping of H into itself.

A Banach space is said to have the $K-K$ property if a sequence $\{x_n\} \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is well known that if H is a Hilbert space, then H has the $K-K$ property.

Definition 2.1[13] *If M is a maximal monotone operator on H, then the resolvent operator associated with M is defined by*

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}u, \quad \forall u \in H,$$

where $\lambda > 0$ is a constant and I is the identity operator.

Definition 2.2[14] *A single-valued operator $A : H \to H$ is said to be hemi-continuous if for any fixed $x, y, z \in H$, the function $t \to \langle A(x + ty), z\rangle$ is continuous at 0^+. It is well known that a continuous mapping must be hemi-continuous.*

Definition 2.3[14] *A set-valued operator $A : X \to 2^{X^*}$ is said to be bounded if $A(B)$ is bounded for every bounded subset B of X.*

Lemma 2.1[13] *The resolvent operator $J_{M,\lambda}$ is single-valued and nonexpansive, that is,*

$$\|J_{M,\lambda}(u) - J_{M,\lambda}(v)\| \le \|u - v\|, \quad \forall u, v \in H.$$

Lemma 2.2 *The resolvent operator $J_{M,\lambda}$ is inverse-strongly monotone, that is*

$$\langle J_{M,\lambda}u - J_{M,\lambda}v, u - v\rangle \ge \|J_{M,\lambda}u - J_{M,\lambda}v\|^2, \quad \forall u, v \in H. \tag{2.3}$$

Proof Let u, v be any given points in H, let $x = J_{M,\lambda}u, y = J_{M,\lambda}v$. It follows from Definition 2.1 that $u - x \in \lambda Mx$ and $v - y \in \lambda My$. Since M is maximal monotone, we have

$$0 \le \langle (u - x) - (v - y), x - y\rangle = \langle (u - v) - (x - y), x - y\rangle.$$

It follows that

$$\langle u - v, x - y\rangle \ge \|x - y\|^2.$$

Lemma 2.3[15] *There holds the identity in a real Hilbert space H:*

$$\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v\rangle, \quad \forall u, v \in H.$$

Lemma 2.4[15] *Let C be a closed convex subset of a real Hilbert space H and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \neq \emptyset$. If a sequence $\{x_n\}$ in C is such that $x_n \rightharpoonup z$ and $x_n - Sx_n \to 0$, then $z = Sz$.*

Lemma 2.5[15] *Let H be a real Hilbert space. Given a closed convex subset $C \subset H$ and points $x, y, z \in H$, given also a real number $a \in R$. The set*

$$D := \{v \in C : \|y - v\|^2 \le \|x - v\|^2 + \langle w, v\rangle + a\}$$

is convex (and closed).

Lemma 2.6[16] *If $T : X \to 2^{X^*}$ is a maximal monotone mapping and $P : X \to X^*$ is a hemi-continuous bounded monotone operator with $D(P) = X$, then the sum $S = T + P$ is a maximal* *monotone mapping.*

Lemma 2.7[15] *Let K be a closed convex subset of real Hilbert space H and let P_K be the metric projection from H onto K (i.e, for $x \in H, P_K x$ is the only point in K such that $\|x - P_K x\| = \inf\{\|x - z\| : z \in K\}$). Given $x \in H$ and $z \in K$, then $z = P_K x$ if and only if there holds the relation: $\langle x - z, y - z\rangle \le 0$, for all $y \in K$.*

3. Strong convergence theorem

Lemma 3.1 *The function $u \in H$ is a solution of variational inclusion (1.1) if and only if $u \in H$ satisfies the relation*

$$u = J_{M,\lambda}[u - \lambda A u],$$

where $\lambda > 0$ is a constant, M is a maximal monotone mapping and $J_{M,\lambda} = (I + \lambda M)^{-1}$ is the resolvent operator.

Theorem 3.1 *Let H be a real Hilbert space. Let A be an α-inverse-strongly monotone mapping of H into itself and $M : H \to 2^H$ be a maximal monotone mapping. Let T be a nonexpansive mapping of H into itself such that $F(T) \bigcap VI(H, A, M) \neq \emptyset$. Assume that $\{t_n\} \subset (0, 1)$, such that $\lim_{n\to\infty} t_n = 0$, let $\{\lambda_n\}_{n=0}^{\infty}$ is a sequence in $[0, 2\alpha]$ such that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in H by the algorithm:*

$$\begin{cases} x_0 \in H, \\ y_n = J_{M,\lambda_n}(x_n - \lambda_n A x_n), \\ z_n = t_n x_0 + (1 - t_n) T y_n, \\ C_n = \{v \in H : \|z_n - v\|^2 \le \|x_n - v\|^2 + t_n(\|x_0\|^2 + 2\langle x_n - x_0, v\rangle)\}, \\ Q_n = \{v \in H : \langle x_n - v, x_n - x_0\rangle \le 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x_0. \end{cases} \tag{3.1}$$

Then $\{x_n\}$ converges strongly to $P_{F(T)\bigcap VI(H,A,M)} x_0$.

Proof It follows from Lemma 3.1 that $VI(H, A, M) = F(J_{M,\lambda}(I - \lambda A))$ (the set of fixed points of $J_{M,\lambda}(I - \lambda A)$). By Lemma 2.1 and formula (2.2), we have $J_{M,\lambda}(I - \lambda A)$ is a nonexpansive mapping of H into itself. Thus, $VI(H, A, M)$ is closed and convex. From the definition of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n \in N \bigcup\{0\}$. By Lemma 2.5, we observe that C_n is also convex. Next, we show that $F(T) \bigcap VI(H, A, M) \subset C_n$ for all n. Indeed, for all $p \in F(T) \bigcap VI(H, A, M)$, we have

$$\begin{aligned} \|z_n - p\|^2 &= \|t_n(x_0 - p) + (1 - t_n)(T y_n - p)\|^2 \\ &\le t_n\|x_0 - p\|^2 + (1 - t_n)\|T y_n - p\|^2 \\ &\le t_n\|x_0 - p\|^2 + (1 - t_n)\|y_n - p\|^2 \\ &\le t_n\|x_0 - p\|^2 + (1 - t_n)\|x_n - p\|^2 \\ &= \|x_n - p\|^2 + t_n(\|x_0 - p\|^2 - \|x_n - p\|^2) \\ &\le \|x_n - p\|^2 + t_n(\|x_0\|^2 + 2\langle x_n - x_0, p\rangle). \end{aligned} \tag{3.2}$$

So, $p \in C_n$ for all n. Next, we show that

$$F(T)\bigcap VI(H,A,M) \subset Q_n \quad \text{for all} \quad n \geq 0. \tag{3.3}$$

We prove this by induction. For $n = 0$, we have $F(T)\bigcap VI(H,A,M) \subset H = Q_0$. Assume that $F(T)\bigcap VI(H,A,M) \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n\bigcap Q_n$, we have

$$\langle x_0 - x_{n+1}, x_{n+1} - z\rangle \geq 0, \quad \forall z \in C_n\bigcap Q_n.$$

As $F(T)\bigcap VI(H,A,M) \subset C_n\bigcap Q_n$ by the induction assumption, the last inequality holds, in particular, for all $z \in F(T)\bigcap VI(H,A,M)$. This together with the definition of Q_{n+1} implies that $F(T)\bigcap VI(H,A,M) \subset Q_{n+1}$. Hence (3.3) holds for all $n \geq 0$.

Now since $x_n = P_{Q_n}x_0$ (by the definition of Q_n) and $F(T)\bigcap VI(H,A,M) \subset Q_n$, we have $\|x_n - x_0\| \leq \|p - x_0\|$ for all $p \in F(T)\bigcap VI(H,A,M)$. In particular, $\{x_n\}$ is bounded and

$$\|x_n - x_0\| \leq \|q - x_0\|, \tag{3.4}$$

where $q = P_{F(T)\bigcap VI(H,A,M)}x_0$. Hence $\{z_n\}, \{y_n\}$ are also bounded. The fact that $x_{n+1} \in Q_n$ implies that $\langle x_{n+1} - x_n, x_n - x_0\rangle \geq 0$. This together with Lemma 2.3 implies

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\
&\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.
\end{aligned}$$

It follows that

$$\|x_{n+1} - x_n\| \to 0. \tag{3.5}$$

That $x_{n+1} \in C_n$ implies that

$$\|z_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + t_n(\|x_0\|^2 + 2\langle x_n - x_0, x_{n+1}\rangle) \to 0. \tag{3.6}$$

Therefore, we have

$$\|z_n - x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \to 0. \tag{3.7}$$

$$\|z_n - Ty_n\| = t_n\|x_0 - Ty_n\| \to 0. \tag{3.8}$$

For $p \in F(T)\bigcap VI(H,A,M)$, by formula (2.2), we obtain

$$\begin{aligned}
\|z_n - p\|^2 &\leq t_n\|x_0 - p\|^2 + (1 - t_n)\|Ty_n - p\|^2 \\
&\leq t_n\|x_0 - p\|^2 + (1 - t_n)\|y_n - p\|^2 \\
&\leq t_n\|x_0 - p\|^2 + (1 - t_n)\|x_n - \lambda_n Ax_n - (p - \lambda_n Ap)\|^2 \\
&\leq t_n\|x_0 - p\|^2 + (1 - t_n)[\|x_n - p\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Ap\|^2] \\
&\leq t_n\|x_0 - p\|^2 + (1 - t_n)[\|x_n - p\|^2 + a(b - 2\alpha)\|Ax_n - Ap\|^2].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
-(1 - t_n)a(b - 2\alpha)\|Ax_n - Ap\|^2 \leq &t_n\|x_0 - p\|^2 - \|z_n - p\|^2 + (1 - t_n)\|x_n - p\|^2 \\
= &t_n(\|x_0 - p\|^2 - \|z_n - p\|^2) + (1 - t_n)(\|x_n - p\|^2 - \|z_n - p\|^2)
\end{aligned}$$

$$
\begin{aligned}
&= t_n(\|x_0-p\|^2-\|z_n-p\|^2)+\\
&\quad (1-t_n)(\|x_n-p\|+\|z_n-p\|)(\|x_n-p\|-\|z_n-p\|)\\
&\le t_n(\|x_0-p\|^2-\|z_n-p\|^2)+\\
&\quad (1-t_n)(\|x_n-p\|+\|z_n-p\|)\|x_n-z_n\|.
\end{aligned}
$$

Since $\{x_n\},\{z_n\}$ are bounded, $t_n \to 0$ and $\|x_n - z_n\| \to 0$, we obtain that $\|Ax_n - Ap\| \to 0$. By formula (2.3), we have

$$
\begin{aligned}
\|y_n-p\|^2 =& \|J_{M,\lambda_n}(x_n-\lambda_n Ax_n)-J_{M,\lambda_n}(p-\lambda_n Ap)\|^2\\
\le& \langle (x_n-\lambda_n Ax_n)-(p-\lambda_n Ap), y_n-p\rangle\\
=& \frac{1}{2}\{\|(x_n-\lambda_n Ax_n)-(p-\lambda_n Ap)\|^2+\|y_n-p\|^2-\\
& \|(x_n-\lambda_n Ax_n)-(p-\lambda_n Ap)-(y_n-p)\|^2\}\\
\le& \frac{1}{2}\{\|x_n-p\|^2+\|y_n-p\|^2-\|x_n-y_n-\lambda_n(Ax_n-Ap)\|^2\}.
\end{aligned}
$$

Therefore, we obtain

$$
\|y_n-p\|^2 \le \|x_n-p\|^2-\|x_n-y_n-\lambda_n(Ax_n-Ap)\|^2.
$$

Hence

$$
\begin{aligned}
\|z_n-p\|^2 \le& t_n\|x_0-p\|^2+(1-t_n)\|Ty_n-p\|^2\\
\le& t_n\|x_0-p\|^2+(1-t_n)\|y_n-p\|^2\\
\le& t_n\|x_0-p\|^2+(1-t_n)\|x_n-p\|^2-(1-t_n)\|x_n-y_n-\lambda_n(Ax_n-Ap)\|^2\\
=& t_n\|x_0-p\|^2+(1-t_n)\|x_n-p\|^2-(1-t_n)\|x_n-y_n\|^2-\\
& (1-t_n)\lambda_n^2\|Ax_n-Ap\|^2+2\lambda_n(1-t_n)\langle x_n-y_n, Ax_n-Ap\rangle.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
(1-t_n)\|x_n-y_n\|^2 \le& t_n\|x_0-p\|^2-\|z_n-p\|^2+(1-t_n)\|x_n-p\|^2-\\
& (1-t_n)\lambda_n^2\|Ax_n-Ap\|^2+2\lambda_n(1-t_n)\langle x_n-y_n, Ax_n-Ap\rangle\\
\le& t_n(\|x_0-p\|^2-\|z_n-p\|^2)+(1-t_n)(\|x_n-p\|+\|z_n-p\|)\|x_n-z_n\|-\\
& (1-t_n)\lambda_n^2\|Ax_n-Ap\|^2+2\lambda_n(1-t_n)\langle x_n-y_n, Ax_n-Ap\rangle.
\end{aligned}
$$

Since $\{x_n\},\{y_n\},\{z_n\}$ are bounded and $t_n \to 0, \|x_n - z_n\| \to 0$ and $\|Ax_n - Ap\| \to 0$, we obtain

$$
\|x_n-y_n\| \to 0. \tag{3.9}
$$

By (3.7), (3.8) and (3.9), we have

$$
\begin{aligned}
\|x_n-Tx_n\| &\le \|x_n-z_n\|+\|z_n-Ty_n\|+\|Ty_n-Tx_n\|\\
&\le \|x_n-z_n\|+\|z_n-Ty_n\|+\|y_n-x_n\| \to 0.
\end{aligned} \tag{3.10}
$$

Assume $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w$. By Lemma 2.4, $w \in F(T)$. As $\|x_n - y_n\| \to 0$, we obtain $y_{n_i} \rightharpoonup w$. We now prove that $w \in VI(H, A, M)$. Since A is a $\frac{1}{\alpha}$-Lipschitz continuous monotone mapping and $D(A) = H$, by Lemma 2.6, $M + A$ is a maximal monotone mapping. Let $(v,f)\in G(M+A)$. Since $f-Av\in Mv$ and $\frac{1}{\lambda_{n_i}}(x_{n_i}-y_{n_i}-\lambda_{n_i}Ax_{n_i})\in My_{n_i}$, we have

$$\langle v-y_{n_i},(f-Av)-\frac{1}{\lambda_{n_i}}(x_{n_i}-y_{n_i}-\lambda_{n_i}Ax_{n_i})\rangle\geq 0.$$

Therefore, we obtain

$$\begin{aligned}\langle v-y_{n_i},f\rangle \geq&\langle v-y_{n_i},Av+\frac{1}{\lambda_{n_i}}(x_{n_i}-y_{n_i}-\lambda_{n_i}Ax_{n_i})\rangle\\ =&\langle v-y_{n_i},Av-Ax_{n_i}\rangle+\langle v-y_{n_i},\frac{1}{\lambda_{n_i}}(x_{n_i}-y_{n_i})\rangle\\ =&\langle v-y_{n_i},Av-Ay_{n_i}\rangle+\langle v-y_{n_i},Ay_{n_i}-Ax_{n_i}\rangle+\langle v-y_{n_i},\frac{1}{\lambda_{n_i}}(x_{n_i}-y_{n_i})\rangle\\ \geq&\langle v-y_{n_i},Ay_{n_i}-Ax_{n_i}\rangle+\langle v-y_{n_i},\frac{1}{\lambda_{n_i}}(x_{n_i}-y_{n_i})\rangle.\end{aligned}$$

Let $i\to\infty$, we obtain $\langle v-w,f\rangle\geq 0$. Since $A+M$ is maximal monotone, we have $0\in Aw+Mw$ and hence $w\in F(T)\bigcap VI(H,A,M)$. We now show that $w=P_{F(T)\bigcap VI(H,A,M)}x_0$ and $x_n\to w$. Put $\tilde{w}=P_{F(T)\bigcap VI(H,A,M)}x_0$ and consider the sequence $\{x_0-x_{n_i}\}$. Then we have $x_0-x_{n_i}\rightharpoonup x_0-w$. By the weak lower semicontinuity of the norm and the fact that $\|x_0-x_{n+1}\|\leq\|x_0-\tilde{w}\|$ for all $n\geq 0$ which is implied by the fact that $x_{n+1}=P_{C_n\bigcap Q_n}x_0$ and $F(T)\bigcap VI(H,A,M)\subset C_n\bigcap Q_n$, we obtain

$$\|x_0-\tilde{w}\|\leq\|x_0-w\|\leq\liminf_{i\to\infty}\|x_0-x_{n_i}\|\leq\limsup_{i\to\infty}\|x_0-x_{n_i}\|\leq\|x_0-\tilde{w}\|.$$

This implies that $\|x_0-\tilde{w}\|=\|x_0-w\|$ (hence $\tilde{w}=w$ by the uniqueness of the nearest point of x_0 onto $F(T)\bigcap VI(H,A,M)$.) and that $\|x_0-x_{n_i}\|\to\|x_0-w\|$. Using the $K-K$ property of H, we obtain $x_0-x_{n_i}\to x_0-w$, hence, $x_{n_i}\to w$. Since $\{x_{n_i}\}$ is an arbitrary (weakly convergent) subsequence of $\{x_n\}$, we conclude that $x_n\to w$ and $w=P_{F(T)\bigcap VI(H,A,M)}x_0$.

4. Applications

In this section, we prove a strong convergence theorem for finding a common fixed point of a nonexpansive mapping and a strictly pseudocontractive mapping in a real Hilbert space H by using Theorem 3.1. A mapping $T:H\to H$ is called k-strictly pseudocontractive if there exists k with $0\leq k<1$ such that

$$\|Tx-Ty\|^2\leq\|x-y\|^2+k\|(I-T)x-(I-T)y\|^2$$

for all $x,y\in H$. If $k=0$, then T is nonexpansive. Put $A=I-T$, where $T:H\to H$ is a k-strictly pseudocontractive mapping. Then A is $\frac{1-k}{2}$-inverse-strongly monotone[5].

Theorem 4.1 *Let H be a real Hilbert space. Let S be a nonexpansive mapping of H into itself and let T be a k-strictly pseudocontractive mapping of H into itself such that $F(S)\bigcap F(T)\neq\emptyset$. Assume that $\{t_n\}\subset(0,1)$, such that $\lim_{n\to\infty}t_n=0$. Let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence in $[0,1-k]$ such that $\lambda_n\in[a,b]$ for some a,b with $0<a<b<1-k$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in H by* *the algorithm:*

$$\begin{cases} x_0 \in H, \\ y_n = (1-\lambda_n)x_n + \lambda_n T x_n, \\ z_n = t_n x_0 + (1-t_n) S y_n, \\ C_n = \{v \in H : \|z_n - v\|^2 \le \|x_n - v\|^2 + t_n(\|x_0\|^2 + 2\langle x_n - x_0, v\rangle)\}, \\ Q_n = \{v \in H : \langle x_n - v, x_n - x_0\rangle \le 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x_0. \end{cases} \tag{4.1}$$

Then $\{x_n\}$ *converges strongly to* $P_{F(S) \bigcap F(T)} x_0$.

Proof Put $A = I - T$ and $M = 0$, then A is $\frac{1-k}{2}$-inverse-strongly monotone mapping. We have $F(T) = VI(H, A, M)$ and $J_{M,\lambda_n}(x_n - \lambda_n A x_n) = (1-\lambda_n)x_n + \lambda_n T x_n$. So, by Theorem 3.1, we obtain the desired result.

References

[1] HUANG Nanjing. *A new completely general class of variational inclusions with noncompact valued mappings* [J]. Comput. Math. Appl., 1998, **35**(10): 9–14.

[2] ZENG Luchuan. *Perturbed proximal point algorithm for generalized nonlinear set-valued mixed quasi-variational inclusions* [J]. Acta Math. Sinica (Chin. Ser.), 2004, **47**(1): 11–18.

[3] ZENG Luchuan. *Existence and algorithm of solutions for general multivalued mixed implicit quasi-variational inequalities* [J]. Appl. Math. Mech. (English Ed.), 2003, **24**(11): 1324–1333.

[4] HASSOUNI A, MOUDAFI A. *A perturbed algorithm for variational inclusions* [J]. J. Math. Anal. Appl., 1994, **185**(3): 706–712.

[5] IIDUKA H, TAKAHASHI W. *Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings* [J]. Nonlinear Anal., 2005, **61**(3): 341–350.

[6] BROWDER F E, PETRYSHYN W V. *Construction of fixed points of nonlinear mappings in Hilbert space* [J]. J. Math. Anal. Appl., 1967, **20**: 197–228.

[7] IIDUKA H, TAKAHASHI W, TOYODA M. *Approximation of solutions of variational inequalities for monotone mappings* [J]. Panamer. Math. J., 2004, **14**(2): 49–61.

[8] LIU Fengshan, NASHED M Z. *Regularization of nonlinear ill-posed variational inequalities and convergence rates* [J]. Set-Valued Anal., 1998, **6**(4): 313–344.

[9] NAKAJO K, TAKAHASHI W. *Strong and weak convergence theorems by an improved splitting method* [J]. Comm. Appl. Nonlinear Anal., 2002, **9**(2): 99–107.

[10] TAKAHASHI W. *Nonlinear variational inequalities and fixed point theorems* [J]. J. Math. Soc. Japan, 1976, **28**(1): 168–181.

[11] TAKAHASHI W. *Convex Analysis and Approximation of Fixed Points* [M]. Yokohama Publishers, Yokohama, 2000. (in Japanese)

[12] TAKAHASHI W. *Nonlinear Functional Analysis* [M]. Yokohama Publishers, Yokohama, 2000.

[13] NOOR M A. *Equivalence of variational inclusions with resolvent equations* [J]. Nonlinear Anal., 2000, **41**(7-8): 963–970.

[14] FANG Yaping, HUANG Nanjing. *H-accretive operators and resolvent operator technique for solving variational inclusions in Banach spaces* [J]. Appl. Math. Lett., 2004, **17**(6): 647–653.

[15] MARTINEZ-YANES C, XU Hongkun. *Strong convergence of the CQ method for fixed point iteration processes* [J]. Nonlinear Anal., 2006, **64**(11): 2400–2411.

[16] PASCALI D, SBURLAN S. *Nonlinear Mappings of Monotone Type* [M]. Martinus Nijhoff Publishers, The Hague; Sijthoff & Noordhoff International Publishers, Alphen aan den Rijn, 1978.