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Existence and Stability of Periodic Solutions in Delayed Cellular Neural Networks with Impulses

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Abstract By using the continuation theorem of Mawhin's coincidence degree theory, Hölder inequality and some analysis techniques, some effective results are obtained ensuring existence and global exponential stability of periodic solutions in delayed cellular neural networks with impulses. An illustrative example is given to demonstrate the effectiveness of the obtained results.

Keywords neural networks; periodic solution; exponential stability; delay; impulses.

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1. Introduction

It is well known that Hopfield neural networks[1] have been extensively studied in past years and found many applications in different areas such as pattern recognition, associative memory and combinatorial optimization[2−3]. However, the delays in electronic neural networks are inevitably encountered because of finite switching speed of amplifiers.

In recent years, the study of the existence of periodic solutions and almost periodic solutions of the nonautonomous delayed neural networks has received much attention and obtained some interesting results[4−8]. Therefore, many physical systems are also under abrupt changes at certain moments due to instantaneous perturbations, which lead to impulsive effects. So, it is important to investigate the dynamic behavior of delayed neural networks with impulses. Many results about periodic solutions with impulses can be found in [9–11].

In this paper, we consider the following nonautonomous delayed cellular neural networks model with impulses

$$\begin{cases} \dot{x}_i(t) = -a_i x_i(t) + \sum\limits_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum\limits_{j=1}^{n} c_{ij} f_j(x_j(t-\tau_{ij}(t))) + J_i(t),\ t \geq 0, t \neq t_k, \\ \triangle x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = I_k(x_i(t_k)),\ t = t_k, \\ x_i(t) = \phi_i(t),\ t \in [-\tau, 0], i = 1, 2, \ldots, n, k = 1, 2, \ldots \end{cases} \tag{1}$$

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where $\triangle x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$ are the impulses at moments t_k and $t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{k\to\infty} t_k = +\infty$; $x_i(t)$ corresponds to the state of the ith unit at time t, $f_j(x_j(t))$ denotes the output of the jth unit at time t, b_{ij} denotes the strength of the jth unit on the ith unit at time t, c_{ij} denotes the strength of the jth unit on the ith unit at time $t-\tau_{ij}(t)$, $J_i(t)$ is the external bias on the ith unit at time t, $\tau_{ij}(t)$ denotes the transmission delay along the axon of the jth unit, and a_i represents the rate with which the ith unit will reset its potential to the resting state when disconnected from the network and external. The functions $I_k(\cdot) : R \to R$ and $\phi_i : [-\tau, 0] \to R$ both are continuous.

As usual in the theory of impulsive differential equations, at the points of discontinuity t_k of the solution $t \mapsto x_i(t)$ we assume that $x_i(t_k) \equiv x_i'(t_k^-)$. It is clear that, in general, the derivatives $x_i'(t_k)$ do not exist. On the other hand, according to the first equality of (1) there exist the limits $x_i'(t_k^{\mp})$. According to the above convention, we assume $x_i'(t_k) \equiv x_i'(t_k^-)$.

Throughout this paper, we assume that:

(H1) Functions $f_j(u)$ $(j = 1, 2, \ldots, n)$ satisfy the Lipschitz condition, i.e., there are constants $L_j > 0$ such that

$$|f_j(u_1) - f_j(u_2)| \le L_j |u_1 - u_2|, \quad \forall u_1, u_2 \in R, u_1 \ne u_2, f_j(0) = 0.$$

(H2) There exists a positive integer p and constants $\bar{I}_k > 0$ such that $t_{k+p} = t_k + \omega$, $I_{k+p}(x) = I_k(x)$ and $|I_k(x_i(t_k))| \le \bar{I}_k$, $k = 1, 2, \ldots$, $i = 1, 2, \ldots, n$.

(H3) $\tau_{ij} \in C^1(R, R)$, $0 \le \tau_{ij}(t) \le \tau$, $\dot{\tau_{ij}}(t) \in C(R, R)$, $\dot{\tau_{ij}}(t) < 1$, $t \in R$, $a_i > 0$, $J_i(t) \in C(R, R)$, $\tau_{ij}(t + \omega) = \tau_{ij}(t)$, $J_i(t + \omega) = J_i(t)$, $i = 1, 2, \ldots, n$.

Our main aim of this paper is to investigate the stability and existence of periodic solutions of system (1) by using Mawhin's continuation theorem of coincidence degree theory and by constructing Lyapunov functions. As far as we know, there are few papers to deal with such a problem by using Mawhin's continuation theorem of coincidence degree theory and some analysis techniques.

This paper is organized as follows: In Section 2, we introduce some definitions and preliminary results which are needed in later sections. In Section 3, the existence of periodic solutions of system (1) is studied by using the continuation theorem of coincidence degree theory proposed by Gains and Mawhin[12]. In Section 4, the global exponential stability of periodic solutions is discussed based on constructing Lyapunov functions. In Section 5, an illustrative example is given to demonstrate the effectiveness of the obtained results.

2. Preliminaries

We shall introduce some definitions and state some preliminary results.

For any $\phi \in C([-\tau, 0] \to R^n)$, we define $\| \phi \| = \sum_{i=1}^n \max_{t\in[-\tau,0]} |\phi_i(t)|$.

Definition 1 *A piecewise continuous function* $x(t) = (x_1(t), \ldots, x_n(t))^{\mathrm{T}} : [-\tau, +\infty) \to R^n$ *is called a solution of Eq. (1), if*

(I) $x(t)$ *is continuous at* $t \ne t_k, x(t_k) = x(t_k^-)$ *and* $x(t_k^+)$ *exist for* $\forall k \in N$*;*

(II) $x(t)$ *satisfies Eq. (1) for* $t \geq 0$, t_k, $k \in N$, $\{t_k\} \cap [0, \omega] = \{t_1, t_2, \ldots, t_p\}$.

Definition 2 *A piecewise continuous function* $x(t) : [0, \omega] \to R^n$ *is called an* ω*-periodic solution of system (1), if*

(I) $x(t)$ *satisfies (I) and (II) of Definition1 in the interval* $[0, \omega]$*;*

(II) $x(t) = x(t + \omega)$, $t \in R$.

Definition 3 *The periodic solution* $x(t)$ *of system (1) is said to be global exponential stable if there exist some* $\varepsilon > 0$ *and* $M > 1$ *such that*

$$\sum_{i=1}^{n} |x_i(t) - x_i^*(t)| \leq M \parallel \phi - \phi^* \parallel e^{-\varepsilon t}, \quad t > 0,$$

where $\parallel \phi - \phi^* \parallel = \sum_{i=1}^{n} \max_{t \in [-\tau, 0]} |\phi_i - \phi_i^*|$. *Definitions 1, 2 and 3 can be found in [13].*

For the system (1), finding the periodic solutions is equivalent to finding solutions of the following boundary-value problem:

$$\begin{cases} \dot{x}_i(t) = -a_i x_i(t) + \sum\limits_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum\limits_{j=1}^{n} c_{ij} f_j(x_j(t - \tau_{ij}(t))) + J_i(t), \ t \neq t_k, t \in [0, \omega], \\ \triangle x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = I_k(x_i(t_k)), \ t = t_k, x_i(0) = x_i(\omega), \end{cases} \tag{2}$$

where $i = 1, \ldots, n, k = 1, \ldots, p$.

3. Existence of periodic solutions

In this section, we shall study the existence of at least one periodic solution of system (1) based on the Mawhin's continuation theorem. To do so, we shall study the following preparations.

For any nonnegative integer q, let $C^{(q)}[0, \omega; t_1, t_2, \ldots, t_p] = \{x : [0, \omega] \to R^n | x^q(t)$ exist for $t \neq t_1, t_2, \ldots, t_p$; $x^{(q)}(t_k^+)$ and $x^{(q)}(t_k^-)$ exist at $t_1, t_2, \ldots, t_p$; and $x^{(j)}(t_k) = x^{(j)}(t_k^-)$, $k = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q\}$ with the norm $\parallel x \parallel_q = \max\{\sup_{t \in [0, \omega]} \parallel x^{(j)}(t) \parallel\}_{j=1}^{q}$, where $\|.\ \|$ is any norm of R^n. It is easy to show that $C^{(q)}[0, \omega; t_1, t_2, \ldots, t_p]$ is a Banach space and the functions in $C[0, \omega; t_1, t_2, \ldots, t_p]$ are continuous with respect to t different from $t_1, t_2, \ldots, t_p$.

Let X, Y be real Banach spaces, $L : \mathrm{Dom}L \subset X \to Y$ be a linear mapping, and $N : X \to Y$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim\mathrm{Ker}L = \mathrm{codim}\mathrm{Im}L < +\infty$ and $\mathrm{Im}L$ is closed in Y. If L is a Fredfolm mapping of index zero and there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L = \mathrm{Im}(I - Q)$, it follows that mapping $L_{|\mathrm{Dom}L \cap \mathrm{Ker}P} : (I - P)X \to \mathrm{Im}L$ is invertible. We denote the inverse of that mapping by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact. Since $\mathrm{Im}L$ is isomorphic to $\mathrm{Ker}L$, there exists an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$.

Now, we introduce Mawhin's continuation theorem[12] as follows.

Lemma 1[12] *Let* $\Omega \subset X$ *be an open bounded set and let* $N : X \to Y$ *be a continuous operator which is* L*-compact on* $\bar{\Omega}$*. Assume*

(a) for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \mathrm{Dom}L$, $Lx \neq \lambda Nx$*;*

(b) $x \in \partial\Omega \cap \mathrm{Ker}L$, $QNx \neq 0$;

(c) $\deg\{QN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0$.

Then $Lx = Nx$ has at least one ω-periodic solution in $\bar{\Omega} \cap \mathrm{Dom}L$.

For the sake of convenience, let

$$\| x \|_2 = \Big(\int_0^\omega |x(t)|^2 \mathrm{d}t \Big)^{\frac{1}{2}}, \quad x \in C(R, R),$$

$$k_{ij} = \Big(\max_{0 \le t \le \omega} \frac{1}{1 - \dot{\tau}_{ij}(t)} \Big)^{\frac{1}{2}}, \quad \bar{J}_i = \frac{1}{\omega} \int_0^\omega J_i(t) \mathrm{d}t.$$

We are now in a position to state and prove the existence of periodic solutions of system (1).

Theorem 1 *Assume (H1)–(H3) holds and*

$$a_i - \frac{1}{2} \sum_{j=1}^n \big((|b_{ij}| + |c_{ij}|) L_j + (|b_{ji}| + |c_{ji}| k_{ji}^2) L_i \big) > 0, \quad i = 1, 2, \ldots, n.$$

Then system (1) has at least one ω-periodic solution.

Proof According to the discussion in Section 2, we need to prove that boundary value problem (2) has a solution. In order to use the continuation theorem of coincidence degree theorem to establish the existence of an ω-periodic solution of (1), we take

$$X = \big\{ x \in C[0, \omega; t_1, t_2, \ldots, t_p] : x(0) = x(\omega) \big\}, \quad Y = X \times R^{n \times (p+1)}.$$

Then X is a Banach space with the norm $\| x \|_0 = \sup_{t \in [0,\omega]} \sum_{i=1}^n |x_i(t)|$ and Y is also a Banach space with the norm $\| z \| = \| x \|_0 + \| y \|$, $x \in X$, $y \in R^{n \times (p+1)}$.

Let

$$\begin{aligned}
&\mathrm{Dom}L = \big\{ x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in C^1[0, \omega; t_1, t_2, \ldots, t_p] : x(0) = x(\omega) \big\}, \\
&L : \mathrm{Dom}L \to Y, x \to \big(x', \triangle x(t_1), \triangle x(t_2), \ldots, \triangle x(t_p), 0 \big), \\
&N : X \to Y, \\
&Nx = \big(A, \triangle x_i(t_1), \triangle x_i(t_2), \ldots, \triangle x_i(t_p), 0 \big)_{n \times (p+2)},
\end{aligned}$$

where $A = -a_i x_i(t) + \sum_{j=1}^n b_{ij} f_j(x_j(t)) + \sum_{j=1}^n c_{ij} f_j(x_j(t - \tau_{ij}(t))) + J_i(t)$. Obviously

$$\mathrm{Ker}L = \big\{ x \in R^n, t \in [0, \omega] \big\},$$

$$\mathrm{Im}L = \Big\{ z = (f, c_1, c_2, \ldots, c_n, d) \in Y : \int_0^\omega f(s) \mathrm{d}s + \sum_{k=1}^p c_k + d = 0 \Big\} = X \times R^{n \times p} \times \{0\}$$

and

$$\mathrm{dimKer}L = \mathrm{codimIm}L = n < +\infty.$$

So, $\mathrm{Im}L$ is closed in Y, L is a Fredholm mapping of index zero. Define two projectors

$$Px = \frac{1}{\omega} \int_0^\omega x(t) \mathrm{d}t$$

and

$$Qy = Q(f, c_1, c_2, \ldots, c_n, d) = \Big(\frac{1}{\omega}[\int_0^\omega f(s)\mathrm{d}s + \sum_{k=1}^{p} c_k + d], 0, 0, \ldots, 0, 0\Big).$$

It is easy to show that P and Q are continuous and satisfy

$$\mathrm{Im}P = \mathrm{Ker}L, \quad \mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I - Q).$$

Further, through an easy computation, we can find the inverse $K_P : ImL \to \mathrm{Ker}P \cap \mathrm{Dom}L$ of L_P has the form

$$K_P(z) = \int_0^t f(s)\mathrm{d}s + \sum_{t>t_k} c_k - \frac{1}{\omega}\int_0^\omega \int_0^t f(s)\mathrm{d}s\mathrm{d}t - \sum_{k=1}^{p} c_k + \frac{1}{\omega}\sum_{k=1}^{p} t_k c_k.$$

Thus, the expression of QNx is

$$\Big(\frac{1}{\omega}\int_0^\omega A\mathrm{d}t - \frac{1}{\omega}\sum_{k=1}^{p} I_i(x_i(t_k)), 0, 0, \ldots, 0\Big)_{n\times 1},$$

where $A = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} f_j(x_j(t - \tau_{ij}(t))) + J_i(t)$ and then

$$\begin{aligned} K_P(I-Q)Nx = \Big(&\big(\int_0^t A\mathrm{d}s - \sum_{t>t_k} I_i(x_i(t_k))\big) - \big(\frac{1}{\omega}\int_0^\omega \int_0^t \big(-a_i x_i(s) + \sum_{j=1}^{n} b_{ij} f_j(x_j(s)) + \\ &\sum_{j=1}^{n} c_{ij} f_j(x_j(s - \tau_{ij}(s))) + J_i(s)\big)\mathrm{d}s\mathrm{d}t + \big(\frac{t}{\omega} - \frac{1}{2}\big)\big(\int_0^t \big(-a_i x_i(s) + \sum_{j=1}^{n} b_{ij} f_j(x_j(s)) + \\ &\sum_{j=1}^{n} c_{ij} f_j(x_j(s - \tau_{ij}(s))) + J_i(s)\big)\mathrm{d}s - \big(\sum_{k=1}^{p} I_i(x_i(t_k))\big)\Big)_{n\times 1}, \quad i = 1, 2, \ldots, n, \end{aligned}$$

where $A = -a_i x_i(s) + \sum_{j=1}^{n} b_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij} f_j(x_j(s - \tau_{ij}(s))) + J_i(s)$. Clearly, QN and $K_P(I-Q)N$ are continuous. Using the Arzela-Ascoli theorem, it is not difficult to show that $QN(\bar{\Omega})$, $K_P(I-Q)(\bar{\Omega})$ are relatively compact for any open bounded set $\Omega \subset X$. So, N is L-compact on $\bar{\Omega}$ for any open bounded set $\Omega \subset X$.

Now we reach the position to search for an appropriate open, bounded Ω for the application of the continuation theorem. Corresponding to operator equation $Lx = \lambda Nx$, $\lambda \in (0, 1)$, we have

$$\begin{cases} \dot{x}_i(t) = \lambda\big\{-a_i x_i(t) + \sum\limits_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum\limits_{j=1}^{n} c_{ij} f_j(x_j(t - \tau_{ij}(t))) + J_i(t)\big\}, \ t > 0, t \neq t_k, \\ \triangle x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = \lambda I_k(x_i(t_k)), \ i = 1, 2, \ldots, n, k = 1, 2, \ldots \end{cases} \tag{3}$$

Suppose that $x(t) = \big(x_1(t), x_2(t), \ldots, x_n(t)\big)^{\mathrm{T}} \in X$ is a solution of system (3) for a certain $\lambda \in (0, 1)$. Integrating (3) over the interval $[0, \omega]$, we obtain

$$\int_0^\omega \big(-a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} f_j(x_j(t - \tau_{ij}(t))) + J_i(t)\big)\mathrm{d}t + \sum_{k=1}^{p} I_k(x_i(t_k)) = 0.$$

Hence

$$\int_0^\omega a_i x_i(s)\mathrm{d}s = \int_0^\omega \big(\sum_{j=1}^{n} b_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij} f_j(x_j(s - \tau_{ij}(s))) + J_i(s)\big)\mathrm{d}s + \sum_{k=1}^{p} I_k(x_i(t_k)). \tag{4}$$

Let $t_0=t_0^+=0$, $t_{p+1}=\omega$. From (2), (4) and Hölder inequality, we have

$$\begin{aligned}\int_0^\omega |x_i'(t)|\mathrm{d}t=&\sum_{k=1}^p\int_{t_{k-1+0}}^{t_k}|x_i'(t)|\mathrm{d}t+\sum_{k=1}^p|x_i(t_k^+)-x_i(t_k)|\\ \le&\int_0^\omega a_i|x_i(t)|\mathrm{d}t+\int_0^\omega\sum_{j=1}^n|b_{ij}||f_j(x_j(t))|\mathrm{d}t+\int_0^\omega\sum_{j=1}^n|c_{ij}||f_j(x_j(t-\tau_{ij}(t)))|\mathrm{d}t+\\ &\int_0^\omega|J_i(t)|\mathrm{d}t+\sum_{k=1}^p|I_k(x_i(t_k))|\\ \le& a_i\sqrt{\omega}\parallel x_i\parallel_2+\sum_{j=1}^n|b_{ij}|L_j\sqrt{\omega}\parallel x_j\parallel_2+\sum_{j=1}^n|c_{ij}|L_jk_{ij}\sqrt{\omega}\parallel x_j\parallel_2+\\ &\sqrt{\omega}\parallel J_i\parallel_2+\sum_{k=1}^p\bar{I}_k.\end{aligned}\tag{5}$$

Since

$$\begin{aligned}\int_0^\omega x_i(t)x_i'(t)\mathrm{d}t=&\frac{\lambda}{2}\big\{x_i(t_1)-x_i(0)+\sum_{l=2}^p[x_i^2(t_l)-x_i^2(t_{l-1}^+)]+x_i^2(\omega)-x_i^2(t_p^+)\big\}\\ =&\frac{\lambda}{2}\sum_{l=1}^p[x_i^2(t_l)-x_i^2(t_{l-1}^+)]\\ =&-\lambda\sum_{k=1}^p[x_i(t_k)+\frac{1}{2}I_k(x_i(t_k))]I_k(x_i(t_k)),\end{aligned}$$

multiplying both sides of (3) by $x_i(t)$ and integrating over interval $[0,\omega]$, we obtain

$$\begin{aligned}0=&\int_0^\omega x_i(t)x_i'(t)\mathrm{d}t=\lambda\Big(\int_0^\omega\big(-a_ix_i(t)+\sum_{j=1}^n b_{ij}f_j(x_j(t))+\sum_{j=1}^n c_{ij}f_j(x_j(t-\tau_{ij}(t)))+\\ &J_i(t)\big)x_i(t)\mathrm{d}t\Big)+\lambda\sum_{k=1}^p[x_i(t_k)+\frac{1}{2}I_k(x_i(t_k))]I_k(x_i(t_k)),\end{aligned}$$

$$\begin{aligned}a_i\int_0^\omega|x_i(t)|^2\mathrm{d}t\le&\sum_{j=1}^n|b_{ij}|\int_0^\omega|f_j(x_j(t))||x_i(t)|\mathrm{d}t+\sum_{j=1}^n|c_{ij}|\int_0^\omega|f_j(x_j(t-\tau_{ij}(t)))||x_i(t)|\mathrm{d}t+\\ &\int_0^\omega|J_i(t)||x_i(t)|\mathrm{d}t+\sum_{k=1}^p[x_i(t_k)+\frac{1}{2}I_k(x_i(t_k))]I_k(x_i(t_k))\\ \le&\sum_{j=1}^n|b_{ij}|L_j\int_0^\omega|x_j(t)||x_i(t)|\mathrm{d}t+\sum_{j=1}^n|c_{ij}|L_j\int_0^\omega|x_j(t-\tau_{ij}(t))||x_i(t)|\mathrm{d}t\\ &\int_0^\omega|J_i(t)||x_i(t)|\mathrm{d}t+\sum_{k=1}^p[x_i(t_k)+\frac{1}{2}I_k(x_i(t_k))]I_k(x_i(t_k))\\ \le&\frac{1}{2}\sum_{j=1}^n|b_{ij}|L_j\int_0^\omega|x_j(t)|^2\mathrm{d}t+\\ &\frac{1}{2}\sum_{j=1}^n|b_{ij}|L_j\int_0^\omega|x_i(t)|^2\mathrm{d}t+\frac{1}{2}\sum_{j=1}^n|c_{ij}|L_j\int_0^\omega|x_i(t)|^2\mathrm{d}t+\end{aligned}$$

$$\frac{1}{2}\sum_{j=1}^{n}|c_{ij}|L_j\int_0^{\omega}|x_j(t-\tau_{ij}(t))|^2\mathrm{d}t+\big(\int_0^{\omega}|J_i(t)|^2\mathrm{d}t\big)^{\frac{1}{2}}\big(\int_0^{\omega}|x_i(t)|^2\mathrm{d}t\big)^{\frac{1}{2}}+$$
$$\sum_{k=1}^{p}[x_i(t_k)+\frac{1}{2}I_k(x_i(t_k))]I_k(x_i(t_k))$$
$$\leq\frac{1}{2}\sum_{j=1}^{n}|b_{ij}|L_j\parallel x_j\parallel_2^2+\frac{1}{2}\sum_{j=1}^{n}|b_{ij}|L_j\parallel x_i\parallel_2^2+\frac{1}{2}\sum_{j=1}^{n}|c_{ij}|L_jk_{ij}^2\parallel x_j\parallel_2^2+$$
$$\frac{1}{2}\sum_{j=1}^{n}|c_{ij}|L_j\parallel x_i\parallel_2^2+\parallel J_i\parallel_2\parallel x_i\parallel_2+\sum_{k=1}^{p}[x_i(t_k)+$$
$$\frac{1}{2}I_k(x_i(t_k))]I_k(x_i(t_k)). \tag{6}$$

From (6) it follows that

$$\sum_{i=1}^{n}\big(a_i-\frac{1}{2}\sum_{j=1}^{n}[(|b_{ij}|+|c_{ij}|)L_j+(|b_{ji}|+|c_{ji}k_{ji}^2)L_i]\big)\parallel x_i\parallel_2^2+$$
$$\parallel J_i\parallel_2\parallel x_i\parallel_2+\sum_{k=1}^{p}[x_i(t_k)+\frac{1}{2}I_k(x_i(t_k))]I_k(x_i(t_k))\leq 0.$$

Obviously, $E\parallel x_i\parallel_2^2-F\parallel x_i\parallel_2-G\leq 0$, where

$$E=\max_{1\leq i\leq n}a_i-\frac{1}{2}\sum_{j=1}^{n}[(|b_{ij}|+|c_{ij}|)L_j+(|b_{ji}|+|c_{ji}|k_{ji}^2)L_i],\quad F=\min_{1\leq i\leq n}\parallel J_i\parallel_2,$$

$$G=\min_{1\leq i\leq n}\sum_{k=1}^{p}[x_i(t_k)+\frac{1}{2}I_k(x_i(t_k))]I_k(x_i(t_k)).$$

From which it follows that

$$\parallel x_i\parallel_2\leq\frac{F+\sqrt{F^2+4EG}}{2E}:=R_i,\quad i=1,2,\ldots,n. \tag{7}$$

Let $\xi\in[0,\omega](t\neq t_k)$, $k=1,2,\ldots,p$, such that $x_i(\xi)=\inf_{t\in[0,\omega]}x_i(t)$, $i=1,2,\ldots,n$. From (7), we get

$$|\,x_i(\xi)\,|\leq\frac{R_i}{\sqrt{\omega}},\quad i=1,2,\ldots,n. \tag{8}$$

Substituting (7) into (5), we obtain

$$\int_0^{\omega}|x_i'(t)|\mathrm{d}t\leq a_i\sqrt{\omega}R_i+\sum_{j=1}^{n}|b_{ij}|L_j\sqrt{\omega}R_j+\sum_{j=1}^{n}|c_{ij}|L_jk_{ij}\sqrt{\omega}R_j+\sqrt{\omega}\parallel J_i\parallel_2+\sum_{k=1}^{p}\bar{I}_k. \tag{9}$$

Since for $t\in[0,\omega]$,

$$|x_i(t)|\leq|x_i(\xi)|+\int_0^{\omega}|x_i'(s)|\mathrm{d}s, \tag{10}$$

from (8), (9) and (10), there exist positive constants D_i $(i=1,2,\ldots,n)$ such that for $t\in[0.\omega]$,

$$|x_i(t)|\leq D_i,\quad i=1,2,\ldots,n.$$

Clearly, D_i $(i=1,2,\ldots,n)$ are independent of λ. Denote $R=\sum_{i=1}^n D_i+H$, where $H>0$ is taken sufficiently large so that

$$\min_{1\le i\le n}\big(a_i-\sum_{j=1}^n L_i|b_{ji}+c_{ji}|+\frac{1}{\omega}\sum_{k=1}^p \bar{I}_k\big)R>\sum_{i=1}^n \bar{J}_i.$$

Now, we take $\Omega=\big\{x(t)=(x_1(t),x_2(t),\ldots,x_n(t))^{\mathrm{T}}\in X|\ \|x\|\le R, x(t_k+0)\in\Omega,\ k=1,2,\ldots,p\big\}$, it is clear that Ω satisfies the condition (a) of Lemma 1. When $x\in \mathrm{Ker}L\cap\partial\Omega$, x is a constant vector in R^n with $\|x\|=\sum_{i=1}^n|x_i|=R$. Then

$$QNx=\big(-a_ix_i+\sum_{j=1}^n b_{ij}f_j(x_j)+\sum_{j=1}^n c_{ij}f_j(x_j)+\frac{1}{\omega}\int_0^\omega J_i(t)\mathrm{d}t-\frac{1}{\omega}\sum_{k=1}^p I_k(x_i(t_k))\big)_{n\times 1}.$$

Therefore

$$\begin{aligned}\|QNx\|&=\sum_{j=1}^n\Big|a_ix_i+\frac{n}{\omega}\sum_{k=1}^p I_k(x_i)-\sum_{j=1}^n b_{ij}f_j(x_j)-\sum_{j=1}^n c_{ij}f_j(x_j)-\bar{J}_i\Big|\\
&\ge\sum_{i=1}^n a_i|x_i|+\frac{n}{\omega}\sum_{k=1}^p\bar{I}_k-\sum_{i=1}^n\sum_{j=1}^n L_j|b_{ij}+c_{ij}|-\sum_{i=1}^n|\bar{J}_i|\\
&=\sum_{i=1}^n\big(a_i-\sum_{j=1}^n L_j|b_{ij}+c_{ij}|+\frac{1}{\omega}\sum_{k=1}^p\bar{I}_k\big)|x_i|-\sum_{i=1}^n|\bar{J}_i|\\
&\ge\min_{1\le i\le n}\big(a_i-\sum_{j=1}^n L_i|b_{ji}+c_{ji}|+\frac{1}{\omega}\sum_{k=1}^p\bar{I}_k\big)\sum_{i=1}^n|x_i|-\sum_{i=1}^n|\bar{J}_i|\\
&=\min_{1\le i\le n}\big(a_i-\sum_{j=1}^n L_i|b_{ji}+c_{ji}|+\frac{1}{\omega}\sum_{k=1}^p\bar{I}_k\big)R-\sum_{i=1}^n|\bar{J}_i|>0.\end{aligned}$$

Consequently,

$$QN(x_1,x_2,\ldots,x_n)^{\mathrm{T}}\ne 0^{\mathrm{T}}\ \text{ for } (x_1,x_2,\ldots,x_n)^{\mathrm{T}}\in\mathrm{Ker}L\cap\partial\Omega.$$

This satisfies condition (b) of Lemma 1. Furthermore, let $\psi(\mu,x)=-\mu x+(1-\mu)QNx$. Then for any $x\in\mathrm{Ker}L\cap\partial\Omega$, $x^{\mathrm{T}}\psi(\mu,x)<0$, we get

$$\deg\{QN,\Omega\cap\mathrm{Ker}L,0\}=\deg\{-x,\Omega\cap\mathrm{Ker}L,0\}\ne 0,$$

where $x=(x_1,x_2,\ldots,x_n)^{\mathrm{T}}$. So condition (c) of Lemma 1 is also satisfied. Thus, by Lemma 1 we know that $Lx=Nx$ has at least one solution in X, that is, system (3) has at least one ω-periodic solution. The proof is completed. □

4. Global exponential stability of the periodic solution

In this section, we will discuss the stability of system (1) by constructing suitable Lyapunov function.

Let $z_i(t)=x_i(t)-x_i^*(t)$, where $x_i(t)=\phi_i(t)$ and $x_i^*(t)=\phi_i^*(t)$ are solutions of system (1) when $t\in[-\tau,0]$.

Theorem 2 *Assume that (H1)–(H3) holds. Furthermore, assume that the following inequalities hold*

(H4) $a_* - \sum_{j=1}^n |b_{ji}|L_i - \sum_{j=1}^n \frac{|c_{ji}|L_i e^{\tau}}{1-\dot{\tau_{ji}}(\eta_{ji}^{-1}(t))} > 0$, *where* η_{ij}^{-1} *is inverse function of* $\eta_{ij}(t) = t - \tau_{ij}(t)$, $a_* = \min\{a_1, a_2, \dots, a_n\}$.

(H5) The impulsive operators $I_i(x_i(t))$ *satisfy* $I_i(x_i(t)) = -\gamma_{ik} x_i(t_k)$, $0 < \gamma_{ik} < 1$, $i = 1, 2, \dots, n$, $k = 1, 2, \dots$.

Then the ω*-periodic solution of the system (1) is of globally exponential stability.*

Proof Let $a_* = \min\{a_1, a_2, \dots, a_n\}$. We define a Lyapunov function $V(t)$ by

$$V(t) = \sum_{i=1}^n \Big[e^{\varepsilon t}|z_i(t)| + \sum_{j=1}^n |c_{ij}|L_j \int_{t-\tau_{ij}(t)}^t \frac{e^{\varepsilon(s+\tau_{ij}(\eta_{ij}^{-1}(s)))}}{1-\dot{\tau_{ij}}(\eta_{ij}^{-1}(s))}|z_j(s)|\mathrm{d}s \Big].$$

Then

$$\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t}\Big|_{(1)} =& \sum_{i=1}^n \Big[\varepsilon e^{\varepsilon t}|z_i(t)| + e^{\varepsilon t}\mathrm{sgn} z_i(t)\big(-a_i x_i(t) + a_i x_i^*(t) + \sum_{j=1}^n \big(b_{ij} f_j(x_j(t)) - b_{ij} f_j(x_j^*(t))\big) + \\
& \sum_{j=1}^n \big(c_{ij} f_j(x_j(t-\tau_{ij}(t))) - c_{ij} f_j(x_j^*(t-\tau_{ij}(t)))\big) + \\
& \sum_{j=1}^n |c_{ij}|L_j \big(\frac{e^{\varepsilon(t+\tau_{ij}(\eta_{ij}^{-1}(t)))}}{1-\dot{\tau_{ij}}(\eta_{ij}^{-1}(t))}|z_j(t)| - e^{\varepsilon t}|z_j(t-\tau_{ij}(t))|\big) \Big] \\
\le & \sum_{i=1}^n e^{\varepsilon t}\Big[\varepsilon - a_* + \sum_{j=1}^n |b_{ji}|L_i + \sum_{j=1}^n \frac{|c_{ji}|L_i e^{\tau}}{1-\dot{\tau_{ji}}(\eta_{ji}^{-1}(t))} \Big] |z_i(t)|, \quad t \ne t_k.
\end{aligned}$$

From condition (H4), we conclude that there exists $\varepsilon > 0$, such that

$$\varepsilon - a_* + \sum_{j=1}^n |b_{ji}|L_i + \sum_{j=1}^n \frac{|c_{ji}|L_i e^{\tau}}{1-\dot{\tau_{ji}}(\eta_{ji}^{-1}(t))} \le 0.$$

Hence

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t}\Big|_{(1)} \le \sum_{i=1}^n e^{\varepsilon t}\Big[\varepsilon - a_* + \sum_{j=1}^n |b_{ji}|L_i + \sum_{j=1}^n \frac{|c_{ji}|L_i e^{\tau}}{1-\dot{\tau_{ji}}(\eta_{ji}^{-1}(t))} \Big] |z_i(t)| \le 0, \quad t \ne t_k.$$

On the other hand

$$V(t_{k+0}) \le \max_{1\le i\le n} (1-\gamma_{ik}) V(t_k) \le V(t_k) \quad \text{for} \quad k \in N.$$

So $V(t) \le V(0)$ for $t \in (t_k, t_{k+1}]$. By the definition of $V(t)$, we obtain

$$\begin{aligned}
V(0) \le & \sum_{i=1}^n \Big[|z_i(0)| + \sum_{j=1}^n |c_{ij}|L_j \int_{0-\tau_{ij}(0)}^0 \frac{e^{\varepsilon(s+\tau_{ij}(\eta_{ij}^{-1}(s)))}}{1-\dot{\tau_{ij}}(\eta_{ij}^{-1}(s))}|z_i(s)|\mathrm{d}s \Big] \\
\le & \sum_{i=1}^n \Big[|z_i(0)| + \sum_{j=1}^n \frac{|c_{ij}|L_j e^{\varepsilon\tau}}{1-\sup\limits_{s\in[-\tau,0]} \dot{\tau_{ij}}(\eta^{-1}(s))} \int_{-\tau}^0 |z_i(s)|\mathrm{d}s \Big] \\
\le & \parallel \phi - \phi^* \parallel + \sum_{j=1}^n \frac{|c_{ij}|L_j e^{\varepsilon\tau}\tau}{1-\dot{\tau_{ij}}} \parallel \phi - \phi^* \parallel
\end{aligned}$$

$$= \Big(1 + \sum_{j=1}^{n} \frac{|c_{ij}|L_j e^{\varepsilon\tau}\tau}{1 - \dot{\tau_{ij}}}\Big) \parallel \phi - \phi^* \parallel .$$

On the other hand, $V(t) \geq \sum\limits_{i=1}^{n} e^{\varepsilon t}|z_i(t)|$. So, we obtain

$$\sum_{i=1}^{n} e^{\varepsilon t}|z_i(t)| \leq \Big(1 + \sum_{j=1}^{n} \frac{|c_{ij}|L_j e^{\varepsilon\tau}\tau}{1 - \dot{\tau_{ij}}}\Big) \parallel \phi - \phi^* \parallel,$$

i.e.,

$$\sum_{i=1}^{n} |x_i(t) - x_i^*(t)| \leq M \parallel \phi - \phi^* \parallel e^{-\varepsilon t},$$

where $M = \big(1 + \sum_{j=1}^{n} \frac{|c_{ij}|L_j e^{\varepsilon\tau}\tau}{1-\dot{\tau_{ij}}}\big)$ for all $t > 0$. The proof is completed. $\square$

5. An illustrative example

Consider the following delayed neural networks with impulses:

$$\begin{cases} \dot{x_1}(t) = -50x_1(t) + \frac{1}{8}\sin((x_1(t)) + \frac{1}{4}\sin(x_2(t)) + \frac{1}{6}\sin((x_1(t-1+\frac{1}{2}\sin(t)))+ \\ \qquad\quad \frac{1}{10}\sin(x_2(t-1+\frac{1}{2}\cos(t))) + \sin(t), \\ \triangle x_1(t_k) = x_1(t_k^+) - x_1(t_k^-) = I_k(x_1(t_k)) = -\frac{3}{4}(1 - \frac{1}{2}\cos\frac{2k\pi}{p}), \\ \dot{x_2}(t) = -100x_2(t) + \frac{1}{12}\sin((x_1(t)) + \frac{1}{5}\sin(x_2(t)) + \frac{1}{14}\sin((x_1(t-1+\frac{1}{2}\cos(t)))+ \\ \qquad\quad \frac{1}{15}\sin(x_2(t-1+\frac{1}{2}\sin(t))) + \cos(t), \\ \triangle x_2(t_k) = x_2(t_k^+) - x_2(t_k^-) = I_k(x_2(t_k)) = -\frac{1}{4}(1 - \frac{1}{2}\sin\frac{2k\pi}{p}), \end{cases} \tag{11}$$

in which $t_{k+p} = t_k + 2\pi$, $[0, 2\pi] \cap \{t_k\} = \{t_1, t_2, \ldots, t_p\}$. Through simple computations, we can find

$$a_i - \frac{1}{2}\sum_{j=1}^{2}((|b_{ij}| + |c_{ij}|)L_j + (|b_{ji}| + |c_{ji}|k_{ji}^2)L_i) > 0, \quad i = 1, 2.$$

Then conditions (4) and (5) are satisfied. So system (11) has at least one 2π-periodic solution, which is exponentially stable.

6. Conclusion

In this paper, we use the continuation theorem of coincidence degree theory and Lyapunov functions to study the existence and global stability of periodic solution for delayed neural network model with impulses. A set of easily verifiable sufficient conditions are obtained for the existence and global stability of periodic solution. The method of this paper may be extended to study some other systems.

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